GENERAL AND PARTICULAR SOLUTION

Solving or integrating a differential equation involves determining the dependent variable in terms of the independent variable. The solution or integral of a differential equation is a relationship between the dependent and independent variables that is free from derivatives and satisfies the given differential equation.

Therefore, obtaining the solution of $\frac{dy}{dx} = e^x$ involves integrating both sides.

i.e., $y = e^x + c$ and that of, $\frac{dy}{dx} = px + q$ is $y = \frac{px^2}{2} + qx + c$, where c is arbitrary constant.

The primitive of the differential equation is also referred to as its solution, as the differential equation can be seen as a relation derived from it.
There can be three types of solution of a differential equation:

(i) **General solution (or complete integral or complete primitive)**

A relation in x and y that satisfies a given differential equation and involves exactly the same number of arbitrary constants as the order of the differential equation.

For example, a general solution of the differential equation $\frac{d^2x}{dt^2} = -4x$ is

$x = A \cos 2t + B \sin 2t$ where A and B are the arbitrary constants.

(ii) **Particular solution or particular integral**

Is the solution to the differential equation derived by substituting specific values for the arbitrary constant in the general solution?
For example, A specific solution to the differential equation is given by $x = 10 \cos 2t + 5 \sin 2t$.

$$\frac{d^2x}{dt^2} = -4x.$$

The general solution of a differential equation can be represented in various forms that are equivalent to each other.
For example $\log x - \log (y + 2) = k$ ....(i)
The general solution of the differential equation $xy' = y + 2$, where k is an arbitrary constant, is expressed as equation (i). This solution, as given in equation (i), can also be reformulated as follows:

$$\log\left(\frac{x}{y+2}\right) = k \text{ or } \frac{x}{y+2} = e^k = c_1 \qquad .....(ii)$$

Or $x = c_1 (y + 2)$ .....(iii)

Where $c_1 = e^k$ is an additional arbitrary constant, the solution (iii) can also be expressed as:

$$y + 2 = c_2 x$$

Where $c_2 = 1/c_1$ is another arbitrary constant.

Every differential equation we encounter possesses either a unique solution or a family of solutions. For instance, the differential equation $\left|\frac{dy}{dx}\right| + |y| =$ has solely the trivial solution, wherein y equals 0.

The differential equation $\left|\frac{dy}{dx}\right| + |y| + c = 0, c > 0$ has no solution.

(iii) **Singular Solution**

It cannot be derived from the general solution. Geometrically, the general solution serves as the envelope for the singular solution.

METHODS OF SOLVING ORDINARY DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DERESS

Differential Equation Of First Order And First Degree

A first-order and first-degree differential equation belongs to the type:

$$\frac{dy}{dx}+f(x,y)=0$$

This expression can also be represented as: Mdx + Ndy = 0, where M and N are functions of x and y.

Solution methods of First Order and First Degree Differential Equations

Variables Separation

Certain differential equations can be solved using the separation of variables method. This approach is applicable when the differential equation can be expressed in the form A(x) dx + B(y) dy = 0.

Where A (x) is solely a function of 'x' and B(y) is solely a function of 'y'.

A general solution for this is provided by,

$$\int A(x)\,dx + \int B(y)dy = c$$

Where 'c' is the arbitrary constant.

Ex. Find the solution to the differential equation (1 + x) y, dx = (y – 1) x , dy.

Sol. The equation can be written as –

$$\left(\frac{1+x}{x}\right)dx=\left(\frac{y-1}{y}\right)dy$$

$$\int\left(\frac{1}{x}+1\right)dx=\int\left(1-\frac{1}{y}\right)dy2$$

In x + x = y – In y + c

In y + In x = y – x + c

$$xy=e^{y-x}$$

Ex. Solve the differential equation $xy\frac{dy}{dx}=\frac{1+y^2}{1+x^2}\left(1+x+x^2\right)$.

Sol. The differential equation can be expressed as:

$$xy\frac{dy}{dx}=\left(1+y^2\right)\left(1+\frac{x}{1+x^2}\right)$$

$$\frac{y}{1+y^2}dy=\left(\frac{1}{x}+\frac{1}{1+x^2}\right)dx$$

Integrating,

$$\frac{1}{2}\ln\left(1+y^2\right)=\ln x+\tan^{-1}x+c$$

$$\sqrt{1+y^2}=cxe^{\tan^{-1}x}$$

Ex. Solve $\frac{dy}{dx}=\left(e^x+1\right)\left(1+y^2\right)$

Sol. The equation can be formulated as:

$$\frac{dy}{1+y^2}=\left(e^x+1\right)dx$$

Integrating both sides, $\tan^{-1}y = e^x + x + c$.

Ex. Solve the differential equation $\left(x^3 - y^2x^3\right)\frac{dy}{dx} + y^3 + x^2y^3 = 0$

Sol. $\left(x^3 - y^2x^3\right)\frac{dy}{dx} + y^3 + x^2y^3 = 0$

$$\frac{1-y^2}{y^3}dy + \frac{1+x^2}{x^3}dx = 0$$

$$\int\left(\frac{1}{y^3} - \frac{1}{y}\right)dy + \int\left(\frac{1}{x^3} + \frac{1}{x}\right)dx = 0$$

$$\log\left(\frac{x}{y}\right) = \frac{1}{2}\left(\frac{1}{y^2} + \frac{1}{x^2}\right) + c$$

Polar Coordinates Transformations

At times, converting to polar coordinates makes the separation of variables more convenient. In this context, it is helpful to recall the following differentials:

(A) If $x = r\cos\theta$; $y = r\sin\theta$ then,

(i) $xdx + ydy = rdr$

(ii) $dx^2 + dy^2 = dr^2 + r^2d\theta^2$

(iii) $xdy - ydx = r^2d\theta$

(B) If $x = r\sec\theta$ & $y = r\tan\theta$ then

(i) $xdx - ydy = rdr$

(ii) $xdy - ydx = r^2\sec\theta\, d\theta$.

Ex. Solve the differential equation $xdx + ydy = x(xdy - ydx)$

Sol. Taking $x = r\cos\theta$, $y = r\sin\theta$

$$x^2 + y^2 = r^2$$

$$2xdx + 2ydy = 2rdr$$

$$xdx + ydy = rdr \qquad \text{......... (i)}$$

$$\frac{y}{x} = \tan\theta$$

$$\frac{x\frac{dy}{dx} - y}{x^2} = \sec^2\theta\cdot\frac{d\theta}{dx}$$

$$xdy - ydx = x^2\sec^2\theta\cdot d\theta$$

$$xdy - ydx = r^2d\theta \qquad \text{........ (ii)}$$

By employing (i) and (ii) in the provided differential equation, it transforms into:

$$rdr = r\cos\theta\cdot r^2d\theta$$

$$\frac{dr}{r^2} = \cos\theta d\theta$$

$$\frac{1}{r} = \sin\theta + \lambda$$

$$-\frac{1}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} + \lambda$$

$$\frac{y+1}{\sqrt{x^2+y^2}} = c \;\Rightarrow (y+1)^2 = c\left(x^2+y^2\right)$$

Ex. Solve : $\dfrac{x+y\dfrac{dy}{dx}}{y-x\dfrac{dy}{dx}}=x^2+2y^2+\dfrac{y^4}{x^2}$

Sol. $\dfrac{x+y\dfrac{dy}{dx}}{y-x\dfrac{dy}{dx}}=x^2+2y^2+\dfrac{y^4}{x^2}$

$$\frac{x^4+2x^2y^2+y^4}{x^2}$$

$$\frac{xdx+ydy}{ydx-xdy}=\frac{\left(x^2+y^2\right)^2}{x^2}$$

$$\frac{xdx+ydx}{\left(x^2+y^2\right)^2}=\frac{-\left(xdy-ydx\right)}{x^2}$$

$$\frac{1}{2}\frac{2xdx+2ydy}{\left(x^2+y^2\right)^2}=-d\left(\frac{y}{x}\right)$$

$$\frac{1}{2}\frac{d\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2}=-d\left(\frac{y}{x}\right)$$

Now integrating both sides

$$-\frac{1}{2}\frac{1}{x^2+y^2}=-\frac{y}{x}+c$$

$$\frac{y}{x}-\frac{1}{2\left(x^2+y^2\right)}=c$$

Reducible to the Variables Separable form

If a differential equation can be transformed into a separable variables form through an appropriate substitution, it is termed "Reducible to the variables separable type." Its general form is. $\dfrac{dy}{dx}=f(ax+by+c)a,b\neq 0$. To solve this, put $ax + by + c = t$.

Ex. Solve $\dfrac{dy}{dx}=\cos(x+y)-\sin(x+y)$.

Sol. $\dfrac{dy}{dx}=\cos(x+y)-\sin(x+y)$

$$x + y = t$$

$$\frac{dy}{dx}=\frac{dt}{dx}-1$$

$$\frac{dt}{dx}-1=\cos t-\sin t$$

$$\int\frac{dt}{1+\cos t-\sin t}=\int dx$$

$$\int \frac{\sec^2 \frac{t}{2} dt}{2\left(1 - \tan\frac{t}{2}\right)} = \int dx$$

$$-\ln\left|1 - \tan\frac{x+y}{2}\right| = x + c$$

Ex. Solve $\frac{dy}{dy} = (4x + y + 1)^2$

Sol. Putting $4x + y + 1 = t$

$$4 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 4$$

Given equation becomes $\frac{dt}{dx} - 4 = t^2$

$$\frac{dt}{t^2 + 4} = dx$$

(Variables are separated)
Integrating both sides,

$$\int \frac{dt}{4 + t^2} = \int dx$$

$$\frac{1}{2}\tan^{-1}\frac{t}{2} = x + c$$

$$\frac{1}{2}\tan^{-1}\left(\frac{4x + y + 1}{2}\right) = x + c$$

Ex. Solve: $y' = (x + y + 1)^2$

Sol. $y' = (x + y + 1)^2$ ....(i)

Let $t = x + y + 1$

$$\frac{dt}{dx} = 1 + \frac{dy}{dx}$$

Substituting in equation (i)

We get, $\frac{dt}{dx} = t^2 + 1$

$$\int \frac{dt}{1 + t^2} = \int dx$$

$$\tan^{-1} t = x + C$$

$$t = \tan(x + C)$$

$$x + y + 1 = \tan(x + C)$$

$$y = \tan(x + C) - x - 1$$

Equation of the Form

$\Rightarrow$ $yf(xy)dx + xg(xy)dy = 0$ .... (i)

The substitution $xy = z$ transforms the differential equation of this structure into a form where the variables are separable.

Let $xy = z$ .... (ii)

$$dy = \left[\frac{xdz - zdx}{x^2}\right]$$

using equation (ii) & (iii), equation (i) becomes

$$\frac{z}{x}f(z)dx + xg(z)\left[\frac{xdz - zdx}{x^2}\right] = 0$$

$$\frac{z}{x}f(z)dx + g(z)dz - \frac{z}{x}g(z)dx = 0$$

$$\frac{z}{x}\{f(z) - g(z)\}dx + g(z)dz = 0a$$

$$\frac{1}{x}dx + \frac{g(z)dz}{z\{f(z) - g(z)\}} = 0$$

Solution by Inspection

At times, by recognizing a certain group of terms as being part of an exact differential, we can solve the differential equation in which they occur by inspection. The following list will be of help in finding a perfect differential made up of group of terms.

$$d(xy) = xdy + ydx$$

$$d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$$

$$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$

$$d\left(\ln\frac{y}{x}\right) = \frac{xdy - ydx}{xy}$$

$$d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$$

$$d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydx - xdy}{x^2 + y^2}$$

$$d\left(\frac{y^2}{x}\right) = \frac{2xydy - y^2dx}{x^2}$$

$$d\left(\frac{x^2}{y}\right) = \frac{2xydx - x^2dy}{y^2}$$

$$d\left(\frac{1}{xy^2}\right) = -\frac{y^2dx + 2xydy}{x^2y^4}$$

$$d\{\sin^{-1}(xy)\} = \frac{xdy + ydx}{\sqrt{1 - x^2y^2}}$$

Homogeneous Equations

A function f(x, y) is considered a homogeneous function of degree n if the substitution $x = \lambda x$, $y = \lambda y$, , $\lambda > 0$ results in the equality:

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

The degree of homogeneity, denoted by 'n', can take any real number.

Ex. Find the degree of homogeneity for the given function.

(i) $f(x,y) = x^2 + y^2$ (ii) $f(x,y) = \left(\dfrac{x^{\frac{3}{2}} + y^{\frac{3}{2}}}{(x+y)}\right)$ (iii) $f(x,y) = \sin\left(\dfrac{x}{y}\right)$

Sol. (i) $f(\lambda x, \lambda y) = \lambda^2 x^2 + \lambda^2 y^2$

$= \lambda^2\left(x^2 + y^2\right)$

$= \lambda^2 f(x,y)$

Degree of homogeneity $\rightarrow 2$

(ii) $f(\lambda x, \lambda y) = \dfrac{\lambda^{3/2}x^{3/2} + \lambda^{3/2}y^{3/2}}{\lambda x + \lambda y}$

$f(\lambda x, \lambda y) = \lambda^{1/2} f(x,y)$

Degree of homogeneity $\rightarrow$ ½

(iii) $f(\lambda x, \lambda y) = \sin\left(\dfrac{\lambda x}{\lambda y}\right)$

$= \lambda^{\circ} \sin\left(\dfrac{x}{y}\right)$

$= \lambda^{\circ} f(x,y)$

Degree of homogeneity $\rightarrow 0$

Ex. Ascertain whether each of the following functions is homogeneous or not.

(i) $f(x,y) = x^2 - xy$ (ii) $f(x,y) = \dfrac{xy}{x + y^2}$ (iii) $f(x,y) = \sin xy$

Sol. (i) $= \lambda^2 x^2 - \lambda^2 xy$

$= \lambda^2\left(x^2 - xy\right) = \lambda^2 f(x,y)$ Homogeneous.

(ii) $f(\lambda x, \lambda y) = \dfrac{\lambda^2 xy}{\lambda x + \lambda^2 y^2} \neq \lambda^n f(x,y)$ Not homogeneous.

(i) $f(\lambda x, \lambda y) = \sin\left(\lambda^2 xy\right) \neq \lambda^{n_f}(x,y)$ Not homogeneous.

Homogeneous first order differential equation

A differential equation of the form $\dfrac{dy}{dx} = \dfrac{f(x,y)}{g(x,y)}$

Where f (x,y) and g(x,y) are homogeneous functions of x,y and of the same degree, is said to be homogeneous. Such equations can be solved by substituting $y = vx$, A function F(x, y) where f(x, y) and g (x, y) are homogeneous functions of x and y and of the same degree, is considered homogeneous. Such equations can be solved by substituting y = vx.

(i) So that the dependent variable y is changed to another variable v.

As f(x, y) and g(x, y) are homogeneous functions of the same degree, denoted as n, they can be expressed as:

$$f(x,y) = x^n f_1\left(\frac{y}{x}\right)$$

$$g(x,y) = x^n g_1\left(\frac{y}{x}\right)$$

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

The given differential equation, therefore, becomes

v + x

$$v + x\frac{dv}{dx} = \frac{f_1(v)}{g_1(v)}$$

$$\frac{g_1(v)dv}{f_1(v) - vg_1(v)} = \frac{dx}{x},$$

So that the variables v and x are now separable.

In certain cases, homogeneous equations can be resolved by substituting (x = vy) or by employing polar coordinate substitution.

Ex. Find the solution for the differential equation $(x^2 - y^2)\,dx + 2xydy = 0$, with the initial condition y = 1 when x = 1.

Sol.

$$\frac{dy}{dx} = -\frac{x^2 - y^2}{2xy}$$

$$y = ux$$

$$\frac{dy}{dx} = v + \frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = -\frac{1 - v^2}{2v}$$

$$\int \frac{2v}{1+v^2}dv = -\int \frac{dx}{x}$$

$$\text{In}\left(1+v^2\right) = -\text{In}x + c$$

$$x=1,\ y=1,\quad v=1$$

$$\text{In } 2 = c$$

$$\text{In}\left\{\left(1+\frac{y^2}{x^2}\right).x\right\} = \text{In}2$$

$$x^2 + y^2 = 2x$$

Ex. Find the solution to the differential equation: $\left(1+2e^{\frac{x}{y}}\right)dx+2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$

Sol. The equation is homogeneous with a degree of 0.

Put $x=vy, \quad dx=vdy+ydv$

Subsequently, the differential equation transforms into

$$\left(1+2e^{v}\right)(vdy+ydv)+2e^{v}(1-v)dy=0$$

$$\left(v+2e^{v}\right)dy+y\left(1+2e^{v}\right)dv=0$$

$$\frac{dy}{y}+\frac{1+2e^{v}}{v+2e^{v}}dv=0$$

Integrating and replacing v by $\frac{x}{y}$,

We get $\ln y+\ln\left(v+2e^{V}\right)=\ln c$

And $x+2ye^{\frac{x}{y}}=c$

Non-homogeneous Equation of First Degree in X and Y

Equations reducible to homogeneous form

An equation of the form where $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$ can be reduced to homogeneous form by changing the variables x, y to u, v as $x=u+h, y=v+k$

Where h, k are constants chosen to make the given equation homogeneous.

We have $\frac{dy}{dx}=\frac{dv}{du}$

The equation becomes,

$$\frac{dv}{du}=\frac{a_1u+b_1v+\left(a_1h+b_1k+c_1\right)}{a_2u+b_2v+\left(a_2h+b_2k+c_2\right)}$$

Let, h and k be chosen so as to satisfy the equation

$$a_1\ h+b_1k+c_1=0 \quad \text{... (i)}$$

$$a_2\ h+b_2k+c_2=0 \quad \text{... (ii)}$$

Solve for h and k from (i) and (ii)

Now, $\frac{du}{dv}=\frac{a_1u+b_1v}{a_2u+b_2v}$

Is a homogeneous equation and can be solved by substituting v = ut.

Ex. Solve the differential equation $\frac{dy}{dx}=\frac{x+2y-5}{2x+y-4}$

Sol. Let $x=x+h, y=Y+k$

$$\frac{dy}{dX}=\frac{d}{dX}(Y+k)$$

$$\frac{dy}{dX}=\frac{dY}{dX}$$

$$\frac{dx}{dX}=1+0$$

On dividing (i) by (ii)

$$\frac{dy}{dx}=\frac{dY}{dx}$$

$$\frac{dY}{dX}=\frac{X+h+2(Y+k)-5}{2X+2h+Y+k-4}$$

$$=\frac{X+2Y+(h+2k-5)}{2X+Y+(2h+k-4)}$$

h & k are such that

$h + 2k – 5 = 0$

$2h + k – 4 = 0$

$h = 1, k = 2$

$\frac{dY}{dX}=\frac{X+2Y}{2X+Y}$ Which is homogeneous differential equation.

Now, substituting $Y = vX$

$$\frac{dY}{dX}=v+X\frac{dv}{dX}$$

$$X\frac{dv}{dX}=\frac{1+2v}{2+v}-v$$

$$\int\frac{2+v}{1-v^2}dv=\int\frac{dX}{X}$$

$$\int\left(\frac{1}{2(v+1)}+\frac{3}{2(1-v)}\right)dv=\text{In}X+c$$

$$\frac{1}{2}\ln(v+1)-\frac{3}{2}\ln(1-v)=\ln X+c$$

$$\ln\left|\frac{v+1}{(1-v)^3}\right|=\ln X^2+2c$$

$$\frac{(Y+X)}{(X-Y)^3}\frac{X^2}{X^2}=e^{2c}$$

$$X+Y=c'(X-Y)^3$$

$$e^{2c}=c^1$$

$$x-1+y-2=c'(x-1-y+2)^3$$

$$x+y-3=c'(x-y+1)^3$$

If $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$ then the substitution $ax + by = v$ will reduce it to the form in which variables are separable.

Ex. Solve: $(x + y)dx + (3x + 3y – 4)\, dy = 0$

Sol. Let $t = x + y$

$dy = dt – dx$

$tdx + (3t – 4)(dt – dx) = 0$

$$2dx + \left(\frac{3t-4}{2-t}\right)dt = 0$$

$$2dx - 3dt + \frac{2}{2-t}dt = 0$$

Integrating and replacing t by x + y, we get

$$2x - 3t - 2[\ln|(2-t)|] = c_1$$

$$2x - 3(x+y) - 2[\ln|(2-x-y)|] = c_1$$

$$x + 3y + 2\ln|(2-x-y)| = c$$

Ex. Evaluate $\frac{dy}{dx} = \frac{2x+3y-1}{4x+6y-5}$

Sol. Putting $u = 2x + 3y$

$$\frac{1}{3}\left(\frac{du}{dx} - 2\right) = \frac{u-1}{2u-5}$$

$$\frac{du}{dx} = \frac{3u-3+4u-10}{2u-5}$$

$$\int \frac{2u-5}{7u-13}dx = \int dx$$

$$\frac{2}{7}\int 1 \cdot du - \frac{9}{7}\int \frac{1}{7u-13} \cdot du = x + c$$

$$\frac{2}{7}u - \frac{9}{7} \cdot \frac{1}{7}\ln(7u-13) = x + c$$

$$4x + 6y - \frac{9}{7}\ln(14x + 21y - 13) = 7x + 7c$$

$$-3x + 6y - \frac{9}{7}\ln(14x + 21y - 13) = c'$$

If $a_2 + b_1 = 0$, then a simple cross multiplication and substituting d (xy) for xdy + ydx and integrating term by term, yield the results easily.

Ex. Evaluate $\frac{dy}{dx} = \frac{x-2y+1}{2x+2y+3}$

Sol.

$$\frac{dy}{dx} = \frac{x-2y+1}{2x+2y+3}$$

$$2xdy + 2ydy + 3dy = xdx - 2ydx + dx$$

$$(2y+3)dy = (x+1)dx - 2(xdy + ydx)$$

On integrating,

$$\int(2y+3)dy = \int(x+1)dx - \int 2\, d(xy)$$

$$y^2 + 3y = \frac{x^2}{2} + x - 2xy + c$$

Ex. Find $\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1}$

Sol. Cross multiplying,

$$2xdy + ydy - dy = xdx - 2ydx + 5dx$$

$$2(xdy + ydx) + ydy - dy = xdx + 5dx$$

$$2d(xy) + ydy - dy = xdx + 5dx$$

On integrating,

$$2xy+\frac{y^2}{2}-y=\frac{x^2}{2}+5x+c$$

$$x^2-4xy-y^2+10x+2y=c'$$

$$c'=-2c$$

Ex. Solve $y(xy+1)dx+x\left(1+xy+x^2y^2\right)dy=0a$

Sol. Let $xy=v$

$$y=\frac{v}{x}$$

$$dy=\frac{xdv-vdx}{x^2}$$

Now, differential equation becomes

$$\frac{v}{x}(v+1)dx+x\left(1+v+v^2\right)\left(\frac{xdv-vdx}{x^2}\right)=0$$

On solving, we get

$$v^3dx-x\,(1+v+v^2)\,dv=0$$

Separating the variables & integrating

We get, $$\int\frac{dx}{x}-\int\left(\frac{1}{v^3}+\frac{1}{v^2}+\frac{1}{v}\right)dv=0$$

$$\ln x+\frac{1}{2v^2}+\frac{1}{v}-\ln v=c$$

$$2v^2\ln\left(\frac{v}{x}\right)-2v-1=-2cv^2$$

$$2x^2y^2\ln n-2xy-1=Kxx^2y^2$$

where $K=-2c$

Linear Differential Equation (Lagrange's Linear Differential Equation)

A linear differential equation has the following characteristics:

1. The dependent variable and its derivative are of the first degree and not multiplied together.
2. All derivatives should be in polynomial form.
3. The order of the derivatives may be more than one.

The m^{th} order linear differential equation is of the form.

$$P_0(x)\frac{d^m y}{dx^m}+P_1(x)\frac{d^{m-1}y}{dx^{min-1}}+\ldots\ldots\ldots\ldots\ldots+P_{m-1}(x)\frac{dy}{dx}+P_m(x)y=\phi(x)$$

Where $P_0(x)$, $P_1(x)$ ....$P_m(x)$ are called the coefficients of the differential equation.

The coefficients of the differential equation are denoted as $P_0(x)$, $P_1(x)$ ....$P_m(x)$.

Note: While a linear differential equation is always of the first degree, it's important to note that not every differential equation of the first degree is linear.

E.g. the differential equation $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+y^2=$ is not linear, though its degree is 1.

$\frac{dy}{dx}+y^2\sin x=\ln x$ is not a Linear differential equation.

Ex. Which of the following equations is linear and which one is nonlinear?

(A) $\frac{dy}{dx}+xy^2=1$ (B) $x^2\frac{dy}{dx}+y=e^x$ (C) $\frac{dy}{dx}+3y=xy^2$

(D) $x\frac{dy}{dx}+y^2=\sin x$ (E) $\frac{dy}{dx}=\cos x$ (F) $\frac{d^2y}{dx^2}+y=0$

(G) $dx+dy=0$ (H) $x\left(\frac{dy}{dx}\right)+\frac{3}{\left(\frac{dy}{dx}\right)}=y^2$

Sol. None Linear (A), (C), (D) , (H)

Linear (B), (E), (F),(G)

Linear differential equations of first order

The differential equation $\frac{dy}{dx}+Py=Q$ is linear in y.

(Where P and Q are functions solely dependent on x.)

Integrating Factor (I.F.) :

It is an expression that, when multiplied by a differential equation, transforms it into an exact form.

The integrating factor for a linear differential equation is $= g(x)=e^{\int Pdx}$ After multiplying the above equation by (ignoring the constant of integration),

I.F it becomes;

$$\frac{dy}{dx}\cdot e^{\int Pdx}+Py\cdot e^{\int Pdx}=Q\cdot e^{\int Pdx}$$

$$\frac{d}{dx}\left(y\cdot e^{\int Pdx}\right)=Q\cdot e^{\int Pdx}$$

$$e^{\int Pdx}=\int Q\cdot e^{\int Pdx}+C$$

At times, the differential equation becomes linear when x is considered the dependent variable and y as the independent variable. In such cases, the differential equation takes the following form:

$$\frac{dx}{dy} + P_1 x = Q_1$$

Here, P_1 and Q_1 are functions of y. The I.F. now is $e^{\int P_1 dy}$

Ex. Find $\left(1+y^2\right)+\left(x-e^{\tan^{-1}y}\right)\frac{dy}{dx}=0$

Sol. The differential equation can be expressed as:

$$\left(1+y^2\right)\frac{dx}{dy}+x=e^{\tan^{-1}y}$$

$$\frac{dx}{dy}+\frac{1}{1+y^2}\cdot x=\frac{e^{\tan^{-1}y}}{1+y^2}$$

So, solution is $\quad xe^{\tan^{-1}y}=\int\frac{e^{\tan-1}ye^{\tan^{-1}y}}{1+y^2}dy$

Let, $\quad e^{\tan-1}y=t$

$$\frac{e^{\tan-1}y}{1+y^2}dy=dt$$

$$xe^{\tan^{-1}y}=\int tdt$$

Putting $\quad e^{\tan-1}y=t$

Or $\quad xe^{\tan-1}y=\frac{t^2}{2}+\frac{c}{2}$

$$2xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+c$$

Ex. Solve $\frac{dy}{dx}+\frac{3x^2}{1+x^3}y=\frac{\sin^2 x}{1+x^3}$

Sol. $\frac{dy}{dx}+Py=Q$

$$P=\frac{3x^2}{1+x^3}$$

$$F=e^{\int P\cdot dx}=e^{\int\frac{3x^2}{1+x^3}dx}=e^{\ln\left(1+x^3\right)}=1+x^3$$

General solution is

$$y(\ IF)=\int Q(\ IF)\cdot dx+c$$

$$y\left(1+x^3\right)=\int\frac{\sin^2 x}{1+x^3}\left(1+x^3\right)dx+c$$

$$y\left(1+x^3\right)=\int\frac{1-\cos 2x}{2}dx+c$$

$$y\left(1+x^3\right)=\frac{1}{2}x-\frac{\sin 2x}{4}+c$$

Ex. Evaluate $x\ln x\frac{dy}{dx}+y=2\ln x$

Sol. $\frac{dy}{dx}+y=\frac{2}{x}$

$$P = \frac{1}{x / \ln x}, Q = \frac{2}{x}$$

$$IF = e^{\int P.dx} = e^{\int \frac{1}{x \ln x} dx} = e^{\ln(\ln x)} = \ln x$$

General solution is

$$y \cdot (\ln x) = \int \frac{2}{y} \ln x \cdot dx + c$$

$$y(\ln x) = (\ln x)^2 + c$$

Equations reducible to linear form

By change of variable.

Frequently, a differential equation can be transformed into a linear form through a suitable substitution of the non-linear term.

Ex. Solve: $\frac{dy}{dx} = \cos x\left(\sin x - y^2\right)$

Sol. The provided differential equation can be transformed into a linear form through a change of variable by an appropriate substitution.

Substituting $y^2 = z$

$$2y\frac{dy}{dx} = \frac{dz}{dx}$$

differential equation becomes

$$\frac{\sin x}{2}\frac{dz}{dx} + \cos x \cdot z = \sin x \cos x$$

$\frac{dz}{dx} + 2\cot x \cdot z = 2\cos x$ which is linear in $\frac{dz}{dx}$

$$IF = e^{\int 2\cos x dx} = e^{2\min x} = \sin^2 x$$

General solution is

$$z.\sin^2 x = \int 2\cos x \cdot \sin^2 x \cdot dx + c$$

$$y^2 \sin^2 x = \frac{2}{3}\sin^3 x + c$$

Bernoulli's equation

Equations of the form $\frac{dy}{dx} + Py = Q \cdot y^n, n \neq 0$ and $n \neq 1$ where P and Q are functions of x, is called Bernoulli's equation.

Equations in the form $\frac{dy}{dx} + Py = Q \cdot y^n, n \neq 0$ and $n \neq 1$ where P and Q are functions of x, are referred to as Bernoulli's equations.

e.g. $2\sin x\frac{dy}{dy} - y\cos x = xy^3e^x$

On dividing by y^n

We get $y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q$

Let $y^{n-1} = t,$

So that $(-n+1)y^{-1}\frac{dy}{dx} = \frac{dt}{dx}$

Then equation becomes $\frac{dt}{dx} + P(1-n)t = Q(1-n)$ linear with t as a dependent variable.

Ex. Find $\frac{dy}{dx}-\frac{y}{x}=\frac{y^2}{x^2}$ (Bernoulli's equation)

Sol. Dividing both sides by y^2

$$\frac{1}{y^2}\frac{dy}{dx}-\frac{1}{xy}=\frac{1}{x^2}2$$

Putting $\frac{1}{y}=t$

$$-\frac{1}{y^2}\frac{dy}{dx}=\frac{dt}{dx}$$

Differential equation (i) becomes,

$$-\frac{dt}{dx}-\frac{t}{x}=\frac{1}{x^2}$$

$\frac{dt}{dx}+\frac{t}{x}=-\frac{1}{x^2}$ which is linear differential equation in $\frac{dt}{dx}$

$$=e^{\int\frac{1}{x}dx}=e^{\ln x}=x$$

$$t\cdot x=\int-\frac{1}{x^2}\cdot x\,dx+c$$

$$tx=-\ln x+c$$

$$\frac{x}{y}=-\ln x+c$$

Ex. Determine the solution for the given differential equation: $\frac{dy}{dx}-y\tan x=-y^2\sec x$

Sol. $\frac{1}{y^2}\frac{dy}{dx}-\frac{1}{y}\tan x=-\sec x$

$$\frac{1}{y}=v;\frac{-1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$$

$$\frac{-dv}{dx}-v\tan x=-\sec x$$

$$\frac{dv}{dx}+v\tan x=\sec x,$$

Here, $P=\tan x, Q=\sec x$

$\text{I.F.}=e^{\int \text{sanxdx}}=|\sec x|\ v\,|\sec x|=\int\sec^2 x\,dx+c$

Hence the solution is $y^{-1}|\sec x|=\tan x+c$

GEOMETRICAL AND PHYSICAL APPLICATIONS

The following results will often be used to solve problems having geometrical flavour.
For the curve $y = f(x)$ or $f(x, y) = 0$

i. Slope of tangent to the curve at (x, y) is $\frac{dy}{dx}$.

ii. Slope of normal to the curve at (x, y) is $-\frac{dx}{dy} = -\frac{1}{(\frac{dy}{dx})}$.

iii. The equation of tangent to the curve at (x, y) is $Y - y = \frac{dy}{dx}(X - x)X, Y$ being the current co-ordinates

iv. The equation of the normal to the curve at (x, y) is $(X - x) + \frac{dy}{dx}(Y - y) = 0$

v. The intercepts of the tangent on the x-axis and y-axis are $x - y\frac{dx}{dy}$ and $y - x\frac{dy}{dx}$ respectively.

vi. The lengths of tangent and normal are respectively $y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}$ and $y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

vii. The lengths of sub-tangent and sub-normal are respectively $y\frac{dx}{dy}$ and $y\frac{dy}{dx}$

ORTHOGONAL TRAJECTORY

Let a family of curves be represented by the equation

$$f(x, y. c) = 0 \qquad ...(1)$$

And we want to determine the family of curves each member of which cuts members of the given family (i) at right angles. Then the family of curves we are seeking constitute, the orthogonal trajectory of family (i).

The procedure can be systematically described as thus:

(i) We first form the differential equation of the family by differentiating (i) and eliminating the arbitrary constants.

(ii) As products of slopes of tangent at a point of intersection of (i) and its orthogonal trajectory is -1, we replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$ in the differential equation of family (i).

(iii) Now we solve the resulting differential equation to obtain the family of orthogonal trajectory.

EXERCISE LEVEL –I

EL– I

Q.1 Which of the following functions satisfies the differential equation $\frac{dy}{dx}+2y=0$?

(a) $y=-2e^{-x}$ (b) $y=2e^{x}$ (c) $y=e^{-2x}$ (d) $y=e^{2x}$

Q.2 The function $y = 8\sin^2 x$ is a solution to the differential equation $\frac{d^2y}{dx^2}+4y=0$.

(a) True (b) False

Q.3 Which of the following functions satisfies the differential equation $xy' - y = 0$?

(a) $y = 4x$ (b) $y=x^2$ (c) $y=-4x$ (d) $y=2x$

Q.4 Which of the following differential equations is satisfied by the solution $y = 3x^2$?

(a) $\frac{d^2y}{dx^2}-6x=0$ (b) $\frac{dy}{dx}-3x=0$

(c) $x\frac{d^2y}{dx^2}-\frac{dy}{dx}=0$ (d) $\frac{d^2y}{dx^2}-\frac{3dy}{dx}=0$

Q.5 Which of the following functions satisfies the differential equation $y'' + 6y = 0$?

(a) $y = 5\cos 3x$ (b) $y = 5\tan 3x$

(c) $y = \cos 3x$ (d) $y = 6\cos 3x$

Q.6 Which function among the following is a solution to the differential equation $\frac{dy}{dx} - 14x = 0$?

(a) $y = 7x^2$ (b) $y = 7x^3$ (c) $y = x^7$ (d) $y = 14x$

Q.7 Which of the following given differential equations has $y = \log x$ as a solution?

(a) $\frac{d^2y}{dx^2}-x=0$ (b) $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$

(c) $\frac{d^2y}{dx^2}-\frac{dy}{dx}=0$ (d) $x\frac{d^2y}{dx^2}-\log x=0$

Q.8 How many arbitrary constants will be present in the general solution of a second-order differential equation?

(a) 3 (b) 4 (c) 2 (d) 1

Q.9 The count of arbitrary constants in a specific solution of a fourth-order differential equation is ___

(a) 1 (b) 0 (c) 4 (d) 3

Q.10 The function $y = 3 \cos x$ is a solution to the equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 0$

(a) True (b) False

Q.11 Degree of the differential equation $e^{dy/dx} = x$ is

(A) 1 (B) 2

(C) 3 (D) Zero

Q.12 The order of the differential equation whose solution is given by
$y = c_1x + (c_2 + c_3)e^{\log x} + c_4\cos(x + c_5)$
where c_1, c_2, c_3, c_4 and c_5 are arbitrary constants, is

(A) 2 (B) 3

(C) 4 (D) 5

Q.13 The differential equation of all circles at fixed centre (α, β) is

(A) $\frac{dy}{dx} = (x - \alpha)(y - \beta)$ (B) $\frac{dy}{dx} = \frac{x-\alpha}{y-\beta}$

(C) $\frac{dy}{dx} = \frac{\alpha-x}{y-\beta}$ (D) $\frac{dy}{dx} = \frac{x-\alpha}{\beta-y}$

Q.14 The order of the differential equation of ellipse whose minor and minor axes are along x-axis and y-axis respectively, is

(A) 1 (B) 2

(C) 3 (D) 4

Q.15 The differential equation satisfying all the curves $y = ae^{2x} + be^{-3x}$ where a and b are arbitrary constants, is

(A) $6y = y_1 + y_2$ (B) $y = y_1 + y_2$

(C) $6y = 2y_1 + 2y_2$ (D) $6y = y_1 - y_2$

Q.16 The degree of the differential equation $\frac{d^2y}{dx^2} + \sqrt{1 + (\frac{dy}{dx})^3} = 0$ is

(A) 1 (B) 2

(C) 3 (D) 6

Q.17 Which of the following differential equation is linear?

(A) $\sqrt{1 - x^2}dx + \sqrt{1 - y^2}dy = 0$ (B) $(\frac{ds}{dt})^4 + 3s\frac{d^2s}{dt^2} = 0$

(C) $\frac{1}{x}\frac{d^2y}{dx^2} = e^x$ (D) $(xy^2 + x)dx + (y - x^2y)dy = 0$

Q.18 The differential equation of all the non-vertical lines in the xy-plane is

(A) $\frac{dy}{dx} - x = 0$ (B) $\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$

(C) $\frac{d^2y}{dx^2} = 0$ (D) $\frac{d^2y}{dx^2} + x = 0$

Q.19 The differential equation of the family of curves represented by the equation $(x - a)^2 + y^2 = a^2$ is

(A) $2xy\frac{dy}{dx} + x^2 = y^2$ (B) $2xy\frac{dy}{dx} + x^2 + y^2 = 0$

(C) $xy\frac{dy}{dx} + x^2 = y^2$ (D) $x\frac{dy}{dx} + x^2 - y^2 = 0$

Q.20 The differential equation of the family of curves $v = \frac{A}{r} + B$, where A and B are arbitrary constants, is

(A) $\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} = 0$ (B) $\frac{d^2v}{dr^2} - \frac{1}{r}\frac{dv}{dr} = 0$

(C) $\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0$ (D) $\frac{d^2v}{dr^2} + r\frac{dv}{dr} = 0$

Q.21 The general solution of differential equation $\frac{dy}{dx} = \frac{x+y}{x-y}$ is

(A) $\sqrt{x^2 + y^2} = c$ (B) $\tan^{-1}\left(\frac{y}{x}\right) = \log(c\sqrt{x^2 + y^2})$

(C) $x\tan^{-1}(\frac{y}{x}) = \log\sqrt{x^2 + y^2} + c$ (D) $y = x + c$

Q.22 If $x\frac{dy}{dx} = y(\log y - \log x + 1)$, then the solution of the equation is

(A) $\log\left(\frac{x}{y}\right) = cy$ (B) $\log\left(\frac{y}{x}\right) = cy$

(C) $\log\left(\frac{x}{y}\right) = cx^2$ (D) $\log\left(\frac{y}{x}\right) = cx$

Q.23 The slope of the tangent at (x, y) to a curve passing through (2,1) is $\frac{x^2+y^2}{2xy}$, then the equation of the curve is

(A) $2(x^2 - y^2) = 3x$ (B) $2(x^2 - y^2) = 3x$

(C) $x(x^2 - y^2) = 6$ (D) $x(x^2 + y^2) = 10$

Q.24 Solution of the differential equation : $(2x\cos y + y^2\cos x)dx + (2y\sin x - x^2\sin y)dy = 0$ is

(A) $x^2\sin x + y^2\cos x = c$ (B) $x^2\sin y + y^2\cos x = c$

(C) $x^2\cos y + y^2\sin x = c$ (D) $x^2\sin y - y^2\cos x = c$

Q.25 Solution of the differential equation $(3xy^2 + x\sin(xy))dy + (y^3 + y\sin(xy))dx = 0$ is

(A) $xy^3 - \cos xy = c$ (B) $xy^3 + \cos xy = c$

(C) $xy^2 - \cos xy = c$ (D) $xy^2 + \sin xy = c$

Q.26 The solution of the differential equation $(1 + x^2)(1 + y)dy + (1 + x)(1 + y^2)dx = 0$ is

(A) $\tan^{-1}x + \log(1 + x^2) + \tan^{-1}y + \log(1 + y^2) = c$

(B) $\tan^{-1}x - \frac{1}{2}\log(1 + x^2) + \tan^{-1}y - \frac{1}{2}\log(1 + y^2) = c$

(C) $\tan^{-1}x + \frac{1}{2}\log(1 + x^2) + \tan^{-1}y + \frac{1}{2}\log(1 + y^2) = c$

(D) $\tan^{-1}x + \frac{1}{2}\log(1 + x^2) - \tan^{-1}y + \frac{1}{2}\log(1 + y^2) = c$

Q.27 The solution of $\frac{dy}{dx} = (4x + y + 1)^2$ is

(A) $4x - y + 1 = 2\tan(2x + 2c)$ (B) $4x - y - 1 = 2\tan(2x + 2c)$

(C) $4x + y + 1 = 2\tan(2x + 2c)$ (D) $y = 3x + c$

Q.28 The solution of $ydx - xdy + 3x^2y^2e^{x^3}dx = 0$ is

(A) $\frac{x}{y} + e^{x^3} = c$ (B) $\frac{x}{y} - e^{x^3} = c$

(C) $-\frac{x}{y} + e^{x^3} = c$ (D) $y = x^2 + c$

Q.29 The solution of the differential equation $3e^x\tan y dx + (1 + e^x)\sec^2 y dy = 0$ is

(A) $(1 + e^x)^3\tan y = c$ (B) $(1 - e^x)^3\tan y = -c$

(C) $(-1 + e^x)^3\tan y = -c$ (D) $y = x^3 + c$

Q.30 The slope of the tangent at (x, y) to a curve passing through $\left(1, \frac{\pi}{4}\right)$ is given by $\frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$ then the equation of the curve is

(A) $y = \tan^{-1}\left(\log\left(\frac{e}{x}\right)\right)$
(B) $y = x\tan^{-1}\left(\log\left(\frac{x}{e}\right)\right)$
(C) $y = x\tan^{-1}\left(\log\left(\frac{e}{x}\right)\right)$
(D) $y = \log\left(\frac{x}{y}\right)$

Q.31 Integrating factor of the differential equation $\frac{dy}{dx} + \left(\frac{x}{1-x^2}\right)y = \sin x$ is

(A) $\sqrt{1-x^2}$
(B) $\frac{1}{1-x^2}$
(C) $\frac{1}{\sqrt{1-x^2}}$
(D) $y = \cos x$

Q.32 The solution of the differential equation $x\frac{dy}{dx} = -\frac{y}{2} - \frac{\sin 2x}{2y}$ is given by

(A) $xy^2 = \cos^2 x + c$
(B) $xy^2 = \sin^2 x + c$
(C) $yx^2 = \cos^2 x + c$
(D) $xy = \sin x + c$

Q.33 A spherical rain drop evaporates at a rate proportional to its surface area at any instant t. The rate of change of the radius of the rain droo is

(A) Proportional to radius
(B) Proportional to surface area
(C) Proportional to volume
(D) Constant

Q.34 If $y(t)$ is a solution of $(1+t)\frac{dy}{dt} - ty = 1$ and $y(0) = 1$, then $y(1)$ is equal to

(A) $\frac{e^2}{2}$
(B) $e + \frac{1}{2}$
(C) $e - \frac{1}{2}$
(D) $e^2 - \frac{1}{2}$

Q.35 The orthogonal trajectories of the family of curves $a^{n-1}y = x^n$ are given by

(A) $x^n + n^2y =$ constant
(B) $ny^2 + x^2 =$ constant
(C) $n^2x + y^n =$ Constant
(D) $y = x$

Q.36 One of the solution of the differential equation, $(\frac{dy}{dx})^2 - x\frac{dy}{dx} + y = 0$ may be

(A) $y = 2$
(B) $y = 2x$
(C) $y = 2x - 4$
(D) $y = 2x^2 - 4$

Q.37 The equation of the curve passing through the origin and satisfying the differential equation $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$ is

(A) $(1+x^2)y = x^3$
(B) $2(1+x^2)y = 3x^3$
(C) $3(1+x^2)y = 4x^3$
(D) $x + y = x^2$

Q.38 The solution of the differential equation $y' = \frac{y}{x} + \frac{\phi(\frac{y}{x})}{\phi'(\frac{y}{x})}$ is

(A) $x\phi(\frac{y}{x}) = k$
(B) $\phi(\frac{y}{x}) = kx$
(C) $y\phi(\frac{y}{x}) = k$
(D) $\phi(\frac{y}{x}) = ky$

Q.39 The differential equation $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \sin y + x^2 = 0$, is which of the following types question mark

(A) Linear
(B) Homogeneous
(C) Order two
(D) Degree two

Q.40 If $y = y(x)$ satisfies $\frac{2+\text{si}}{1+y}\left(\frac{dy}{dx}\right) = -\cos x$, such that $y(0) = 1$, then $y\left(\frac{\pi}{2}\right)$ is equal to

(A) $\frac{3}{2}$ (B) $\frac{5}{2}$

(C) $\frac{1}{3}$ (D) 1

Q.41 If m, n are order and degree of differential equation $y\frac{dy}{dx} + x^3\left(\frac{d^2y}{dx^2}\right) - xy = \cos x$, then

(A) $m < n$ (B) $m = n$

(C) $m > n$ (D) $m - n = 3$

Q.42 The degree of the differential equation $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = \frac{d^2y}{dx^2}$ is

(A) 4 (B) $\frac{3}{2}$

(C) Not defined (D) 2

Q.43 The order and degree of the differential equation whose general equation is $y = c(x - c)^2$ are

(A) 1, 1 (B) 1, 2

(C) 1, 3 (D) 2, 1

Q.44 The second order differential equation is

(A) $(y')^2 = y^2 - x$ (B) $y'y'' + y = \sin x$

(C) $y''' + y'' + y = 0$ (D) $y' = y$

Q.45 The order of differential equation of family of all concentric circles centered at (h, k) is

(A) 1 (B) 2

(C) 3 (D) 4

Q.46 Integrating factor of the differential equation $\cos x\frac{dy}{dx} + y\sin x = 1$ is

(A) $\cos x$ (B) $\tan x$

(C) $\sec x$ (D) $\sin x$

Q.47 Integrating factor of $x\frac{dy}{dx} - y = x^4 - 3x$ is

(A) x (B) $\log x$

(C) $\frac{1}{x}$ (D) $-x$

Q.48 The solution of differential equation $\cos x \cdot \sin y dx + \sin x \cdot \cos y dy = 0$ is

(A) $\frac{\sin x}{\sin y} = c$ (B) $\sin x \cdot \sin y = c$

(C) $\sin x + \sin y = c$ (D) $\cos x \cdot \cos y = c$

Q.49 If $\frac{dP}{dy} = 3^{\cos y}\sin y$, then P is equal to

(A) $\sin y + c$ (B) $3^{\cos y} + c$

(C) $\frac{-3^{\cos y}}{\ln 3} + c$ (D) $3^{\sin y} + c$

Q.50 The integration factor of equation $(x^2 + 1)\frac{dy}{dx} + 2xy = x^2 - 1$, is

(A) $x^2 + 1$ (B) $\frac{2x}{x^2+1}$

(C) $\frac{x^2-1}{x^2+1}$ (D) $1 - x^2$

Q.51 Which of the following differential equation has $y = x$ as one of its particular solution question mark

(A) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = x$
(B) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0$
(C) $\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + xy = x$
(D) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = 0$

Q.52 Family $y = Ax + A^3$ of curves will correspond to a differential equation of order

(A) 3
(B) 2
(C) 1
(D) Not defined

Q.53 The differential equation representing the family of curves $y^2 = 2c(x + \sqrt{c})$ (where c is a positive parameter), is of

(A) Order 1, degree 3
(B) Order 1, degree 2
(C) Order 2, degree 3
(D) Order 2, degree 2

Q.54 The solution of $\frac{dy}{dx} + y = e^{-x}, y(0) = 0$ is

(A) $y = e^{-x}(x - 1)$
(B) $y = xe^{x}$
(C) $y = xe^{-x} + 1$
(D) $y = xe^{-x}$

Q.55 The general solution of differential equation $(e^x + 1)ydy = (y + 1)e^x dx$ is

(A) $y + 1 = k(e^x + 1)$
(B) $y + 1 = e^x + k$
(C) $y = \log\{k(y + 1)(e^x + 1)\}$
(D) $y = \log\left(\frac{e^x+1}{y+1}\right) + k$

Q.56 For solving $\frac{dy}{dx} = 4x + y + 1$, suitable substitution is

(A) $y = vx$
(B) $y = 4x$
(C) $y = 4x + v$
(D) $y + 4x + 1 = v$

Q.57 If $y(t)$ is a solution of $(1 + t)\frac{dy}{dt} - ty = 1$ and $y(0) = -1$ then $y(1)$ equals,

(A) $-\frac{1}{2}$
(B) $e + \frac{1}{2}$
(C) $e - \frac{1}{2}$
(D) $\frac{1}{2}$

Q.58 STATEMENT-1: The degree of the differential equation, $e^{y''} - xy'' + y = 0$ is not defined.
And
STATEMENT - 2 : The differential equation mentioned in statement - 1 can't be written as a polynomial in derivatives.

(A)Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

Q.59 STATEMENT-1: The differential equation $\frac{dy}{dx} = \frac{2xy}{x^2+y^2}$ can't be solved by the substitution $x = vy$
And
STATEMENT-2: When the differential equation is homogeneous of first order and first degree, then the substitution that solves the equation is $y = vx$

(A)Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

Q.60 STATEMENT-1: The orthogonal trajectory of a family of circles touching x-axis at origin and whose center the on y-axis is self-orthogonal.

And

STATEMENT-2: In order to find the orthogonal trajectory of a family of curves we put $-\frac{dx}{dy}$ in place of $\frac{dy}{dx}$ in the differential equation of the given family of curves.

(A)Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True

A tangent to a curve at P(x, y) intersects x-axis and y-axis at A and B respectively. Let the point of contact divides AB in the ratio $y^2 : x^2$. **(For questions 61 to 65)**

Q.61 The differential equation of family of curves is

(A) $x^2 + y^2 = c$ (B) $x^2 + y^2 - 2x = c$

(C) $x^2 + y^2 = cx^2y^2$ (D) $xy = c$

Q.62 If a member of this family passes through point (5,12) then area of this curve in square units is

(A) 25π (B) 144π

(C) 169π (D) 225π

Q.63 The centre of each member of this family is

(A) (0, 0) (B) (–2, 0)

(C) (0, 2) (D) (2, 2)

Q.64 If a member of this family passes through (3,4), then its equation is

(A) $x^2 + y^2 = 25$ (B) $x^2 + y^2 - 2x = 19$

(C) $x^2 + y^2 = 25x^2y^2$ (D) $x^2 + y^2 = 7$

Q.65 If a member of this family is passes through (3,4) then area bounded by this curve in square units is

(A) $\left(\frac{25}{4} + \frac{2\pi}{3}\right)$ (B) $2(\pi + 4)$

(C) 25π (D) $\left(16\pi + \frac{4}{5}\right)$

Q.66 The degree of the differential equation $\left(\frac{d^3y}{dx^3}\right)^{\frac{2}{3}} + 4 - 3\frac{d^2y}{dx^2} + 5\frac{d^3y}{dx^3} = 0$ is

(A) 1 (B) 2

(C) 3 (D) Not defined

Q.67 The differential equation for which $y = a\cos x + b\sin x$ is a solution is

(A) $\frac{d^2y}{dx^2} + y = 0$ (B) $\frac{d^2y}{dx^2} - y = 0$

(C) $\frac{d^2y}{dx^2} + (a + b)y = 0$ (D) $\frac{d^2y}{dx^2} + (a - b)y = 0$

Q.68 The degree of the differential equation of the curve $(x - a)^2 + y^2 = 16$ will be

(A) 0 (B) 2

(C) 3 (D) 1

Q.69 The differential equation of all parabolas with axis parallel to the axis of y is

(A) $y_2 = 2y_1 + x$ (B) $y_3 = 2y_1$

(C) ${y_2}^3 = y_1$ (D) $y_3 = 0$

Q.70 The differential equation of all circles passing through the origin and having their centers on the x - axis is

(A) $y^2 = x^2 - 2xy\frac{dy}{dx}$ (B) $x^2 = y^2 + xy\frac{dy}{dx}$

(C) $x^2 = y^2 + 3xy\frac{dy}{dx}$ (D) $y^2 = x^2 + 2xy\frac{dy}{dx}$

Q.71 Solution of the differential equation $xdy - ydx = 0$ represents

(A) A rectangular hyperbola

(B) Parabola whose vertex is at origin

(C) Straight line passing through origin

(D) A circle whose center is at origin

Q.72 Solution of the differential equation$\tan y \cdot \sec^2 xdx + \tan x \cdot \sec^2 ydy = 0$ is

(A) $\tan x + \tan y = k$ (B) $\tan x - \tan y = k$

(C) $\frac{\tan x}{\tan y} = k$ (D) $\tan x \cdot \tan y = k$

Q.73 The solution of differential equation $\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{1}{(1+x^2)^2}$ is

(A) $y(1 + x^2) = c + \tan^{-1} x$

(B) $\frac{y}{1+x^2} = c + \tan^{-1} x$

(C) $y\log(1 + x^2) = c + \tan^{-1} x$

(D) $y(1 + x^2) = c + \cos^{-1} x$

Q.74 The solution of the equation $(2y - 1)dx - (2x + 3)dy = 0$ is $(x > -\frac{3}{2}, y > \frac{1}{2})$

(A) $\left(\frac{2x-1}{2y+3}\right) = k$ (B) $\frac{2y+1}{2x-3} = k$

(C) $\frac{2x+3}{2y-1} = k$ (D) $\frac{2x-1}{2y-1} = k$

Q.75 $\frac{dy}{dx} = \frac{xy+y}{xy+x}$, then the solution of differential equation is

(A) $y = xe^x + c$ (B) $y = e^x + c$

(C) $y = cxe^{x-y}$ (D) $y = x + c$

Q.76 The differential equation $y\frac{dy}{dx} + x = c$ represents

(A) Family of hyperbolas (B) Family of parabolas

(C) Family of ellipse (D) Family of circles

Q.77 The differential equation of family of curves$x^2 + y^2 - 2ay = 0$, where a. is an arbitrary constant is

(A) $(x^2 - y^2)\frac{dy}{dx} = 2xy$ (B) $2(x^2 + y^2)\frac{dy}{dx} = xy$

(C) $(x^2 - y^2)\frac{dy}{dx} = xy$ (D) $(x^2 + y^2)\frac{dy}{dx} = 2xy$

Q.78 The solution of differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$ is

(A) $y = \tan^{-1} x + c$ (B) $\tan^{-1} y = x + c$

(C) $(y - x) = c(1 + xy)$ (D) $\tan xy = c$

Q.79 Let $f(x) = \sec x. f(x), f(0) = 1$, then $f\left(\frac{\pi}{6}\right)$ equal to

(A) $\frac{1}{\sqrt{e}}$ (B) $\sqrt{e}$

(C) $e^{\frac{3}{2}}$ (D) $\frac{1}{2\sqrt{e}}$

Q.80 The solution of differential equation $\frac{dy}{dx} = \cos(x - y)$

(A) $y + \cot\left(\frac{x-y}{2}\right) = c$ (B) $x + \cot\left(\frac{x-y}{2}\right) = c$

(C) $x - \tan\left(\frac{x-y}{2}\right) = c$ (D) $x + \tan\left(\frac{x+y}{2}\right) = c$

Q.81 The general solution of differential equation $\frac{dy}{dx} = e^{\frac{x^2}{2}} + xy$ is

(A) $y = ce^{-\frac{x^2}{2}}$ (B) $y = ce^{\frac{x^2}{2}}$

(C) $y = (x + c)e^{\frac{x^2}{2}}$ (D) $y = (c - x)e^{\frac{x^2}{2}}$

Q.82 The differential equation vdy + xdy = dy renresents

(A) A set of circles with center on x axis

(B) A set of circles with center on y-axis

(C) A set of ellipse

(D) A set of circles with center on y axis

Q.83 The general solution of $\frac{dy}{dx} = 2xe^{x^2-y}$ is

(A) $e^{x^2-y} = c$ (B) $e^{-y} + e^{x^2} = c$

(C) $e^{y} = e^{x^2} + c$ (D) $e^{x^2+y} = c$

Q.84 The solution of the equation $2xy' - y = 3$ represents a family of

(A) Circle (B) Straight line

(C) Ellipse (D) Parabola

Q.85 The solution of $y' - y = 1, y(0) = 1$, is given by $y(x) =$

(A) e^x (B) $-e^{-x}$

(C) 1 (D) $2e^x - 1$

Q.86 The solution of differential equation $(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1} x}$ is

(A) $ye^{\tan^{-1} x} = \frac{1}{2}e^{2\tan^{-1} x} + c$ (B) $y = \frac{1}{2}e^{2\tan^{-1} x} + c$

(C) $ye^{\tan^{-1} x} = 2e^{2\tan^{-1} x} + c$ (D) $y \cdot \tan^{-1} x = \frac{1}{2}e^{2ta\ ^{-1} x} + c$

Q.87 The solution of differential equation $x^2y^2dy = (1 - xy^3)dx$ is

(A) $x^3y^3 = x^2 + c$ (B) $2x^3y^3 = 3x^2 + c$

(C) $x^3y^3 = x^2 + x + c$ (D) $x^3y^3 = 3x^2 + c$

Q.88 The solution of the differential equation $ydx + (x + x^2y)dy = 0$ is

(A) $\log y = cx$ (B) $-\frac{1}{xy} + \log y = c$

(C) $\frac{1}{xy} + \log y = c$ (D) $-\frac{1}{xy} = c$

Q.89 Solution of $ydx - xdy = x^2ydx$ is

(A) $y^2e^{x^2} = cx^2$ (B) $ye^{-x^2} = cx^2$

(C) $y''' + y'' + y = 0$ (D) $y' = y$

Q.90 Solution of differential equation $\frac{dy}{dx}\tan y = \sin(x + y) + \sin(x - y)$, is

(A) $\sec y + 2\cos x = c$ (B) $\sec y - 2\cos x = c$

(C) $\cos y - 2\sin x = c$ (D) $\tan y - 2\sec x = c$

EXERCISE LEVEL –II

EL– II

Q.1 Find the order & degree of following differential equations.

(1) $\frac{dy}{dx}+y=\frac{1}{\frac{dy}{dx}}$

(2) $\sin^{-1}\left(\frac{dy}{dx}\right)=x+y$

(3) $e^{\left(\frac{dy}{dx}-\frac{d^2y}{dx^3}\right)}=\ln\left(\frac{d^5y}{dx^5}+1\right)$

(4) $\left[\left(\frac{dy}{dx}\right)^{\frac{1}{2}}+y\right]^2=\frac{d^2y}{dx^2}$

Q.2 Derive a differential equation for the family of straight lines that pass through the origin.

Q.3 Derive the differential equation for all circles that touch the x-axis at the origin and have their centers on the y-axis.

Q.4 Derive the differential equation for the family of curves described by y = a sin bx + c, where a and c are arbitrary constants.

Q.5 Demonstrate that the differential equation for the system of parabolas y^2 = 4a x - b is provided by:

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$$

Q.6 Derive a differential equation for the family of parabolas with the focus at the origin and the axis of symmetry along the x-axis.

Q.7 Find the solution for the given differential equation:

(1) $x^2y\frac{dy}{dx}=(x+1)(y+1)$ (2) $\frac{dy}{dx}=e^{x+y}+x^2e^y$

(3) $xy\frac{dy}{dx}=1+x+y+xy$ (4) $\frac{dy}{dx}=1+e^{x-y}$

(5) $\frac{dy}{dx}=\sin(x+y)+\cos(x+y)$ (6) $\frac{dy}{dx}=x\tan(y-x)+1$

Q.8 Determine the solution for the differential equation: $(1+x)ydx=(y-1)xdy$

Q.9 Find $e^{\frac{dy}{dx}}=x+1$, given that when x = 0, y = 3

Q.10 Evaluate $\frac{dy}{dx}=(4x+y+1)^2$

Q.11 Solve $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

Q.12 Find the solutions to the following differential equations:

(1) $\left(x\frac{dy}{dx} - y\right)\tan^{-1}\frac{y}{x} = x$ Given that y = 0 at x = 1

(2) $x\frac{dy}{dx} = y - x\tan\frac{y}{x}$

(3) $\frac{dy}{dx} = \frac{x+2y-3}{2x+y+3}$

(4) $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3}$

(5) $\frac{dy}{dx} = \frac{3x+2y-5}{3y-2x+5}$

Q.13 Find $x^2dy + y(x + y)\,dx = 0$

Q.14 Solve: $(x^2 - y^2)\,dx + 2xydy = 0$ given that y = 1 when x = 1

Q.15 Evaluate the differential equation $\frac{dy}{dx} = \frac{x+2y-5}{2x+y-4}$

Q.16 Solve $\frac{dy}{dx} = \frac{yf'(x) - y^2}{f(x)}$

Q.17 Evaluate $\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1}$

Q.18 Find the solutions for the following differential equations:

(1) $x\left(x^2+1\right)\frac{dy}{dx} = y\left(1-x^2\right) + x^2 \text{In} x$

(2) $\left(x+2y^3\right)\frac{dy}{dx} = y$

(3) $x\frac{dy}{dx} + y = y^2 \text{In} x$

(4) $xy^2\left(\frac{dy}{dx}\right) - 2y^3 = 2x^3$ given y = 1 at x = 1

Q.19 Find $\frac{dy}{dx} + \frac{3x^2}{1+x^3}y = \frac{\sin^2 x}{1+x^3}$

Q.20 Solve: $x\,\text{In}\,x\frac{dy}{dx} + y = 2\,\text{In}\,x$

Q.21 Evaluate: $y\sin x\frac{dy}{dx} = \cos x\left(\sin x - y^2\right)$

Q.22 Solve: $\frac{dy}{dx} - \frac{y}{x} = \frac{y^2}{x^2}$

ANSWER KEY – LEVEL – I

Q.	1	2	3	4	5	6	7	8	9	10
Ans.	C	A	D	C	A	A	B	C	B	B
Q.	11	12	13	14	15	16	17	18	19	20
Ans.	A	B	C	B	A	B	C	C	A	C
Q.	21	22	23	24	25	26	27	28	29	30
Ans.	B	D	A	C	A	C	C	A	A	C
Q.	31	32	33	34	35	36	37	38	39	40
Ans.	C	A	D	C	B	C	C	B	C	C
Q.	41	42	43	44	45	46	47	48	49	50
Ans.	C	D	C	B	A	C	C	B	C	A
Q.	51	52	53	54	55	56	57	58	59	60
Ans.	B	C	A	D	C	D	A	A	D	D
Q.	61	62	63	64	65	66	67	68	69	70
Ans.	A	C	A	A	C	C	A	B	D	D
Q.	71	72	73	74	75	76	77	78	79	80
Ans.	C	D	A	C	C	D	A	C	B	B
Q.	81	82	83	84	85	86	87	88	89	90
Ans.	C	A	C	D	D	A	B	B	A	A