FUNDAMENTAL THEOREM OF CALCULUS

Definition of the Definite Integral

In this segment, we will provide a formal definition of the definite integral and discuss various properties associated with it. To commence, let's delve into the definition of a definite integral.

Definite Integral

For a function f(x) that exhibits continuity over the interval [a, b], we partition the interval into n subintervals of uniform width, Δx, and select a point from each interval, the definite integral of f(x) from a to b is then defined as:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

The definite integral is precisely defined as the limit and summation method explored in the previous section to determine the net area between a function and the x-axis. It's worth noting that the notation for the definite integral closely resembles that of an indefinite integral, with the reason for this similarity becoming clear over time.

In addition, there is some terminology to clarify here. The value "a" positioned at the bottom of the integral sign is termed the lower limit of the integral, while the value "b" at the top is referred to as the upper limit of the integral. Despite being presented as an interval, it's important to note that the lower limit does not necessarily have to be smaller than the upper limit. Together, a and b are commonly referred to as the interval of integration.

Let's work a quick example

This illustration will leverage various properties and information covered in the concise review of summation notation in the Extras chapter.

Ex. Apply the definition of the definite integral to calculate the following.

$$\int_0^2 x^2 + 1dx$$

Sol. Initially, we cannot employ the definition unless we ascertain the points within each interval that we will use for. To simplify matters, we will utilize the right endpoints of each interval.

From the previous section we know that for a general n the width of each subinterval is,

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

The subintervals are then,

$$\left[0,\frac{2}{n}\right],\left[\frac{2}{n},\frac{4}{n}\right],\left[\frac{4}{n},\frac{6}{n}\right],\ldots,\left[\frac{2(i-1)}{n},\frac{2i}{n}\right],\ldots,\left[\frac{2(n-1)}{n},2\right]$$

Observing that the right endpoint of the ith subinterval is evident,

$$x_i^* = \frac{2i}{n}$$

The sum in the definition of the definite integral is subsequently,

$$\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left(\left(\frac{2i}{n}\right)^2 + 1\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left(\frac{8i^2}{n^3} + \frac{2}{n}\right)$$

Now, we need to take a limit of this. This implies that we have to "evaluate" this summation, meaning we must utilize the formulas provided in the summation notation review to eliminate the actual summation and derive a formula for a general n.

To accomplish this, it's essential to recognize that n is treated as a constant in terms of the summation notation. As we cycle through the integers from 1 to n in the summation, only i changes, and anything that isn't an i will be a constant that can be factored out of the summation. Notably, any n within the summation can be factored out if necessary.

Here is the "evaluation" of the summation.

$$\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = \sum_{i=1}^{n}\frac{8i^2}{n^3} + \sum_{i=1}^{n}\frac{2}{n}$$

$$= \frac{8}{n^3}\sum_{i=1}^{n} i^2 + \frac{1}{n}\sum_{i=1}^{n} 2$$

$$= \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{n}(2n)$$

$$= \frac{4(n+1)(2n+1)}{3n^2} + 2$$

$$= \frac{14n^2 + 12n + 4}{3n^2}$$

We can now compute the definite integral.

$$\int_0^2 x^2 + 1dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

$$= \lim_{n\to\infty}\frac{14n^2 + 12n + 4}{3n^2} = \frac{14}{3}$$

We've presented several methods to handle the limit in this problem, and we'll leave it to you to verify the results. While the process may seem involved for a relatively simple function, there's a more straightforward method for evaluation, which we'll explore later. The main goal of this section is to establish the fundamental properties and facts about the definite integral. Practical computation methods will be discussed in the next section. Now, let's start by exploring some properties of the definite integral.

Properties

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx.$

   The limits on any definite integral can be interchanged by simply attaching a minus sign to the integral during the exchange.

2. $\int_a^a f(x)dx = 0.$

   When the upper and lower limits are the same, no computation is required, and the integral equals zero.

3. $\int_a^b cf(x)dx = c\int_a^b f(x)dx,$

   Where c is any constant. Much like in limits, derivatives, and indefinite integrals, we can factor out a constant.

4. $\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$

   Decomposition of definite integrals is possible across a sum or difference..

5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)$

dx, where c is any constant. This property possesses greater significance than we might initially realize. Its primary application lies in assisting us in integrating a function over adjacent intervals, specifically [a, c] and [c, b]. It's important to note that c doesn't necessarily have to be situated between a and b.

6. $\int_a^b f(x)dx = \int_a^b f(t)dt$

The core idea behind this property is to understand that, as long as the function and limits remain consistent, the choice of the variable of integration in the definite integral will not affect the result. For the proof of properties 1–4, please consult the "Proof of Various Integral Properties" section in the Extras chapter. Property 5 is challenging to prove and is not included there. Property 6, strictly speaking, isn't a complete property. Its inclusion is simply to acknowledge that, as long as the function and limits remain constant, the variable letter used in the integration doesn't affect the result—the outcome remains the same.

Now, let's go through a couple of examples that involve these properties.

Ex. Use the results from the first example to evaluate each of the following.

1. $\int_2^0 x^2 + 1dx$ 2. $\int_0^2 10x^2 + 10dx$ 3. $\int_0^2 t^2 + 1dt$

Sol. (a) In this scenario, the sole distinction lies in the interchange of the limits.

So, using the first property gives,

$$\int_2^0 x^2 + 1dx = -\int_0^2 x^2 + 1dx = -\frac{14}{3}$$

(b) In this part, note that we can factor out a factor of 10 from both terms and then from the integral using the third property.

$$\int_0^2 10x^2 + 10dx = \int_0^2 10\left(x^2 + 1\right)dx$$

$$= 10\int_0^2 x^2 + 1dx$$

$$= 10\left(\frac{14}{3}\right)$$

$$= \frac{140}{3}$$

(c) In this case, the only difference is the selection of the variable, making it possible to address this using property 6.

$$\int_0^2 t^2 + 1dt = \int_0^2 x^2 + 1dx = \frac{14}{3}$$

Ex. Compute the value of the given definite integral. $\int_{130}^{130} \frac{x^3 - x\sin(x) + \cos(x)}{x^2 + 1}dx$

Sol. Once we realize that the limits are the same, there isn't much to handle with this integral.

Applying the second property, it transforms into:

$$\int_6^{-10} f(x)dx = 23 \text{ and} \int_{-10}^6 g(x)dx = -9$$

Ex. Given that $\int_6^{-10} f(x)dx = 23$ and $\int_{-10}^6 g(x)dx = -9$ determine the value of

$$\int_{-10}^6 2f(x) - 10g(x)dx$$

Sol. At the outset, we'll use the fourth property to break down the integral and the third property to extract the constants.

$$\int_{-10}^{6} 2f(x)-10g(x)dx \ = \int_{-10}^{6} 2f(x)dx - \int_{-10}^{6} 10g(x)dx$$

$$= 2\int_{-10}^{6} f(x)dx - 10\int_{-10}^{6} g(x)dx$$

Now, take note that the limits on the first integral are exchanged with those on the provided integral. Adjust them using the first property mentioned above (and introduce a minus sign, if required). After this adjustment, substitute the known values of the integrals.

$$\int_{-10}^{6} 2f(x)-10g(x)dx \ = -2\int_{6}^{-10} f(x)dx - 10\int_{-10}^{6} g(x)dx$$

$$= -2(23)-10(-9)$$

$$= 44$$

Ex. Given that $\int_{12}^{-10} f(x)dx = 6, \int_{100}^{-10} f(x)dx = -2,$ and $\int_{100}^{-5} f(x)dx = 4$ determine the value of

$$\int_{-5}^{12} f(x)dx.$$

Sol. This example predominantly illustrates the application of property 5, with a few instances of employing property 1 in the solution.

To effectively decompose the integral using property 5 and leverage the given information, we must identify the appropriate approach. Initially, observe an integral with a "-5" in one of the limits. While it's not the lower limit, property 1 can be utilized to rectify that eventually. The other limit is 100, which becomes the value of c used in property 5.

$$\int_{-5}^{12} f(x)dx = \int_{-5}^{100} f(x)dx + \int_{100}^{12} f(x)dx$$

We can calculate the value of the first integral, but the second one is still not one of the known integrals. Nevertheless, we do have a second integral with a limit of 100. The other limit for this second integral is -10, and this will be the value of c in the application of property 5.

$$\int_{-5}^{12} f(x)dx = \int_{-5}^{100} f(x)dx + \int_{100}^{-10} f(x)dx + \int_{-10}^{12} f(x)dx$$

At this point, all that needs to be done is to apply property 1 to the first and third integrals to match the limits with the known integrals. After this adjustment, we can substitute the values for the known integrals.

$$\int_{-5}^{12} f(x)dx \ = -\int_{100}^{-5} f(x)dx + \int_{100}^{-10} f(x)dx - \int_{12}^{-10} f(x)dx$$

$$= -4-2-6$$

$$= -12$$

Moreover, there are several helpful properties that we can utilize to compare the overall magnitude of definite integrals. Here they are.

More Properties

7. $\int_{a}^{b} c dx = c(b-a)$, c is any number

8. If $f(x) \geq 0$ for $a \leq x \leq b$ then $\int_{a}^{b} f(x)dx \geq 0$

9. If $f(x) \ge g(x)$ for $a \le x \le b$ then $\int_a^b f(x)dx \ge \int_a^b g(x)dx$

10. If $m \le f(x) \le M$ for $a \le x \le b$ then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

11. $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx$

Interpretations of Definite Integral

Here, we can provide a couple of brief interpretations of the definite integral.

Initially, as hinted in the preceding section, one potential interpretation of the definite integral is to provide the net area between the graph of f(x) and the x-axis over the interval [a, b]. For instance, the net area between the graph of 2 + 1f (x) =x^2 +1 and the x-axis on [0, 2] is given by:

$$\int_0^2 x^2 + 1dx = \frac{14}{3}$$

If you refer to the previous section, you'll notice that this was precisely the area provided for the initial set of problems we examined in this context.

Another interpretation is occasionally referred to as the Net Change Theorem. This interpretation posits that if f(x) represents a quantity (with $f'(x)$ denoting the rate of change of f(x)), then:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

This represents the net change in f(x) over the interval [a, b]. In simpler terms, when you compute the definite integral of a rate of change, you obtain the net change in the quantity. It's evident that the value of the definite integral, f (b) − f(a), indeed provides the net change in f(x). Therefore, there isn't anything to prove with this statement; it essentially acknowledges what the definite integral of a rate of change reveals.

As a brief illustration, suppose V(t) represents the volume of water in a tank, then:

$$\int_{t_1}^{t_2} V'(t)dt = V\left(t_2\right) - V\left(t_1\right)$$

This denotes the net change in volume as we transition from time 1t_1 to time 2t_2.

Similarly, if s(t) is the function describing the position of an object at time t, we understand that the velocity of the object at any time t is given by

v(t)=s (t). Consequently, the displacement of the object from time 1t_1 to time 2t_2 is:

$$\int_{t_1}^{t_2} v(t)dt = s\left(t_2\right) - s\left(t_1\right)$$

In this scenario, it's important to highlight that if v(t) is both positive and negative (indicating the object moves both to the right and left within the given time frame), this will NOT provide the total distance traveled. Instead, it will only give the displacement—essentially, the difference between the object's starting and ending positions. To calculate the total distance traveled by an object, we would need to compute:

$$\int_{t_1}^{t_2} |\,v(t)\,|\,dt$$

It's crucial to emphasize that the Net Change Theorem is meaningful when we're integrating the derivative of a function.

Fundamental Theorem of Calculus, Part I

As indicated by the title above, this constitutes just the initial segment of the Fundamental Theorem of Calculus. We will present the second part in the following section, as it forms the basis for effortlessly calculating definite integrals—an aspect that will be explored further in the upcoming section.

The first part of the Fundamental Theorem of Calculus instructs us on differentiating specific categories of definite integrals, shedding light on the intimate connection between integrals and derivatives.

Fundamental Theorem of Calculus, Part I

If f(x) exhibits continuity over the interval [a, b], then...

$$g(x)=\int_a^x f(t)dt$$

Is continuous over [a, b] and differentiable on the open interval (a, b), and...

$$g'(x)=f(x)$$

An alternative notation for the derivative component of this is,

$$\frac{d}{dx}\int_a^x f(t)dt=f(x)$$

Let's check out a couple of quick examples using this.

Ex. Differentiate each of the following.

(a) $g(x)=\int_{-4}^{x}\mathbf{e}^{2t}\cos^2(1-5t)dt$ (b) $\int_{x^2}^{1}\frac{t^4+1}{t^2+1}dt$

Sol. (a) An alternative notation for the derivative component of this is,

$$g'(x)=\mathbf{e}^{2x}\cos^2(1-5x)$$

(b) This one requires some adjustments before we can apply the Fundamental Theorem of Calculus. The initial observation is that the Fundamental Theorem of Calculus mandates the lower limit to be a constant and the upper limit to be the variable. Therefore, by employing a property of definite integrals, we can interchange the limits of the integral, remembering to include a minus sign afterward. This transformation yields:

$$\frac{d}{dx}\int_{x^2}^{1}\frac{t^4+1}{t^2+1}dt$$

$$\frac{d}{dx}\left(-\int_{1}^{x^2}\frac{t^4+1}{t^2+1}dt\right)$$

$$-\frac{d}{dx}\int_{1}^{x^2}\frac{t^4+1}{t^2+1}dt$$

The subsequent observation is that the Fundamental Theorem of Calculus also mandates an x in the upper limit of integration, and we currently have x^2. To perform this derivative, we'll require the following version of the chain rule:

$$\frac{d}{dx}(g(u))=\frac{d}{du}(g(u))\frac{du}{dx} \quad \text{where } u=f(x)$$

So, if we let $u=x^2$ we use the chain rule to get,

$$\frac{d}{dx}\int_{x^2}^{1}\frac{t^4+1}{t^2+1}dt$$

$$=-\frac{d}{dx}\int_{1}^{x^2}\frac{t^4+1}{t^2+1}dt$$

$$= -\frac{d}{du}\int_1^u \frac{t^4+1}{t^2+1}dt\,\frac{du}{dx} \quad \text{where } u = x^2$$

$$= -\frac{u^4+1}{u^2+1}(2x)$$

$$= -2x\frac{u^4+1}{u^2+1}$$

The final step is to get everything back in terms of x.

$$\frac{d}{dx}\int_{x^2}^1 \frac{t^4+1}{t^2+1}dt = -2x\frac{\left(x^2\right)^4+1}{\left(x^2\right)^2+1}$$

$$= -2x\frac{x^8+1}{x^4+1}$$

Using the chain rule as we did in the last part of this example we can derive some general formulas for some more complicated problems.

First, $$\frac{d}{dx}\int_a^{u(x)} f(t)dt = u'(x)f(u(x))$$

This is simply the chain rule for these kinds of problems.

Next, we can get a formula for integrals in which the upper limit is a constant and the lower limit is a function of x. All we need to do here is interchange the limits on the integral (adding in a minus sign of course) and then use the formula above to get,

$$\frac{d}{dx}\int_{v(x)}^b f(t)dt = -\frac{d}{dx}\int_b^{v(x)} f(t)dt = -v'(x)f(v(x))$$

Finally, we can also get a version for both limits being functions of x. In this case we'll need to use Property 5 above to break up the integral as follows,

$$\int_{v(x)}^{u(x)} f(t)dt = \int_{v(x)}^a f(t)dt + \int_a^{u(x)} f(t)dt$$

We can use pretty much any value of a when we break up the integral. The only thing that we need to do is to make sure that f(a) exists. So, assuming that f(a) exists after we break up the integral we can then differentiate and use the two formulas above to get,

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)dt = \frac{d}{dx}\left(\int_{v(x)}^a f(t)dt + \int_a^{u(x)} f(t)dt\right)$$

$$= -v'(x)f(v(x)) + u'(x)f(u(x))$$

Ex. Differentiate the following integral. $\int_{\sqrt{x}}^{3x} t^2 \sin\left(1+t^2\right)dt$

Sol. This will use the final formula that we derived above.

$$\frac{d}{dx}\int_{\sqrt{x}}^{3x} t^2 \sin\left(1+t^2\right)dt$$

$$= -\frac{1}{2}x^{-\frac{1}{2}}(\sqrt{x})^2\sin\left(1+(\sqrt{x})^2\right) + (3)(3x)^2\sin\left(1+(3x)^2\right)$$

INTEGRALS BY SUBSTITUTION

Evaluation of Definite Integral by change of variable

When the variable x in a definite integral is replaced with t, the substitution impacts three locations.

(1) The expression inside the integral is modified.

(2) dx is replaced with df(t).

(3) The upper and lower limits are modified.

Ex. Evaluate $\int_0^{\infty} \frac{x}{(1+x)\left(1+x^2\right)} dx$

Sol. Substitute $x = \tan\theta$, $dx = \sec^2\theta d\theta$

(limits of integration are changed to 0 and $\frac{\pi}{2}$)

$$I = \int_0^{\infty} \frac{x}{(1+x)\left(1+x^2\right)} dx$$

$$= \int_0^{\pi/2} \frac{\tan\theta}{(1+\tan\theta)\sec^2\theta} \sec^2\theta d\theta$$

$$= \int_0^{\pi/2} \frac{\tan\theta}{1+\tan\theta} d\theta$$

$$I = \int_0^{\pi/2} \frac{\sin\theta}{\sin\theta+\cos\theta} d\theta$$

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)+\cos\theta\left(\frac{\pi}{2}-\theta\right)} d\theta$$

$$I = \int_0^{\pi/2} \frac{\cos\theta}{\cos\theta+\sin\theta} d\theta$$

Adding (1) and (2),

We get, $2I = \int_0^{\pi/2} d\theta = \pi/2$

$$I = \frac{\pi}{4}$$

Some more properties of Definite Integral

Property 9 If $f(x) \geq 0 \forall x \in [a,b]$, then $\int_a^b f(x)dx \geq 0$

Property 10 If f(x) is an odd function of x, then $\int_0^{\hat{f}} f(t)dt$ is an even function of x.

Ex. Show that if f(t) is an odd function then $F(x) = \int_a^x f(t)dt$ is an even function.

Sol. We have $F(x) = \int_a^0 f(t)dt + \int_0^x f(t)dt$

$$F(-x) = \int_a^0 f(t)dt + \int_0^{-x} f(t)dt$$

Substituting t = -u in the 2^{nd} integral

$$F(-x) = \int_a^0 f(t)dt + \int_0^x f(-u)(-du)$$

$$= \int_a^0 f(t)dt + \int_0^x f(u)(du) \quad (\because f(-u) = -f(u))$$

$$= \int_a^x f(t)dt + F(x)$$ therefore F(x) is an even function