

EVALUATION OF DEFINITE INTEGRALS BY SUBSTITUTION**Some Properties Of Definite Integrals**

Property 1 $\int_a^b f(x)dx = \int_a^b f(z)dz$

Property 2 $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Property 3 $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \dots + \int_{c_{n-1}}^{c_n} f(x)dx + \int_{c_n}^b f(x)dx$

Where $a < c_1 < c_2 < \dots < c_{n-1} < c_n < b$

Ex. $f(x) = \begin{cases} x^2 & 0 \leq x \leq 1 \\ \sqrt{x} & 1 \leq x \leq 2 \end{cases}$ find $\int_0^2 f(x)dx$.

Sol. We have
$$\begin{aligned} \int_0^2 f(x)dx &= \int_0^1 f(x)dx + \int_1^2 f(x)dx \\ &= \int_0^1 x^2 dx + \int_1^2 \sqrt{x} dx \\ &= \left[\frac{x^3}{3} \right]_0^1 + \left[\frac{x^{3/2}}{3/2} \right]_1^2 \\ &= \frac{1}{3} + \frac{2^{3/2}}{3/2} - \frac{1}{3/2} - \frac{2}{3} (2\sqrt{2}) - \frac{1}{3} - \frac{1}{3} [4\sqrt{2} - 1] \end{aligned}$$

Property 4 $\int_0^a f(x)dx = \int_0^d f(a-x)dx$

Ex. Evaluate $\int_0^{\pi/4} \log(1 + \tan x) dx$

Sol.
$$\begin{aligned} I &= \int_0^{\pi/4} \log(1 + \tan x) dx \\ I &= \int_0^{\pi/4} \log \left(1 + \tan \left(\frac{\pi}{4} - x \right) \right) dx \\ I &= \int_0^{\pi/4} \log \left(1 + \frac{1 - \tan x}{1 + \tan x} \right) dx \\ I &= \int_0^{\pi/4} \log \left(\frac{2}{1 + \tan x} \right) dx \\ I &= \int_0^{\pi/4} \log \{ \log 2 - \log(1 + \tan x) \} dx \\ I &= \log 2 \int_0^{\pi/4} dx - I \\ 2I &= \frac{\pi}{4} \log 2 \\ I &= \frac{\pi}{8} \log 2. \end{aligned}$$

Property 5 $\int_{-a}^a f(x)dx = \begin{cases} 0 & \text{if } f(-x) = -f(x) \\ 2\int_0^a f(x)dx & \text{if } f(-x) = f(x) \end{cases}$

Ex. Evaluate

(i) $\int_{-1/2}^{1/2} \cos x \log\left(\frac{1+x}{1-x}\right) dx$

(ii) $\int_{-\pi}^{\pi} \sin mx \cos nx dx$

Sol. In (i) $\log \frac{1+x}{1-x}$ is an odd function, then integrand is an odd function and in (ii) $\sin(mx)$ is an odd function so integration is an odd function.

(i) $I = 0$

(ii) $I = 0$

Property 6

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

Proof:

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx$$

In second integral of RHS substitute $x = 2a - y, dx = -dy$

Then,

$$\begin{aligned} & \int_0^{2a} f(x)dx \\ &= \int_0^a f(x)dx - \int_a^0 f(2a-y)dy \\ &= \int_0^a f(x)dx + \int_0^a f(2a-x)dx \end{aligned}$$

Property 7 In particular, $\int_0^{2a} f(x)dx = \begin{cases} 0 & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)dx & \text{if } f(2a-x) = f(x) \end{cases}$

Property 8 In Particular $\int_a^{a+n} f(x)dx = n \int_0^1 f(x)dx; n \in \mathbb{I}$

$$\int_{mT}^{nT} f(x)dx = (n-m) \int_0^T f(x)dx; m, n \in \mathbb{I}$$