METHODS OF INTERGRATION

Integration by Substitution

If we perform the substitution $\phi(x) = t$ in an integral, then

(i) Every occurrence of x will be substituted with the new variable t.

(ii) The differential dx is also expressed in terms of dt.

Ex. Evaluate: $\int \frac{\sec^2 x}{3+\tan x} dx$

Sol. $I = \int \frac{\sec^2 x}{3+\tan x} dx$

$$3+\tan x = t$$
$$\sec^2 x dx = dt$$
$$\int \frac{dt}{t} = \text{In} t + C$$
$$\text{In} |(3+\tan x)| + C$$

Ex. Determine: $\int \frac{1}{1+e^{-x}} dx$

Sol. $I = \int \frac{1}{1+e^{-x}} dx$

$$\int \frac{e^x}{e^x+1}$$
$$\int \frac{e^x}{e^x+1}$$
$$\int \frac{\frac{d}{dx}\left(e^x+1\right)}{(e^x+1)}$$
$$\log_e \left|e^x+1\right| + C$$

Ex. Find: $\int \tan^4 x dx$

Sol. $\int \tan^4 x dx = \int \tan^2 x \cdot \tan^2 x dx$

$$\int \tan^2 x\left(\sec^2 x - 1\right) dx$$
$$\int \tan^2 x \sec^2 x dx - \int \tan^2 x dx$$
$$\int \tan^2 x \sec^2 x dx - \int \left(\sec^2 x - 1\right) dx$$
$$\frac{\tan^3 x}{3} - \tan x + x + C$$

Ex. Evaluate: $\int \frac{x}{x^4+x^2+1} dx$

Sol. $I = \int \frac{x}{x^4+x^2+1} dx$

$$\int \frac{x}{\left(x^2\right)^2 + x^2 + 1} dx$$
$$x^2 = t$$

$$x \cdot dx = \frac{dt}{2}$$

$$I = \frac{1}{2}\int \frac{1}{t^2 + t + 1} dt$$

$$\frac{1}{2}\int \frac{1}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dt$$

$$\frac{1}{2} \cdot \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2t + 1}{\sqrt{3}}\right) + C$$

$$\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + C$$

Integration Using Trigonometric Identities

Consider the following integral

$$t = \int \cos^2 x dx$$

Now, recall the identity, $\cos 2x = 2\cos^2 x - 1$

$\Rightarrow$ $\cos^2 x = \frac{1 + \cos 2x}{2}$

$\therefore$ $I = \int \cos^2 x dx = \int \frac{1 + \cos 2x}{2} dx$

$\therefore$ $I = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x dx$

$\therefore$ $I = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$

As it is clear from the above example when the integral involves some trigonometric functions, we use some known identities to convert the given integral into a convenient form.

Integrals of Some Particular Function

Integration of type

$$\int \frac{dx}{ax^2 + bx + c}, \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \int \sqrt{ax^2 + bx + c} dx$$

Express $ax^2 + bx + c$ as a perfect square and then utilize the standard results.

Ex. Evaluate: $\int \sqrt{x^2 + 2x + 5} dx$

Sol. We have, $\int \sqrt{x^2 + 2x + 5}$

$$\int \sqrt{x^2 + 2x + 1 + 4} dx$$

$$\int \sqrt{(x+1)^2 + 2^2}$$

$$\frac{1}{2}(x+1)\sqrt{(x+1)^2 + 2^2} + \frac{1}{2} \cdot (2)^2 \ln\left|(x+1) + \sqrt{(x+1)^2 + 2^2}\right| + C$$

$$\frac{1}{2}(x+1)\sqrt{x^2 + 2x + 5} + 2\ln\left|(x+1) + \sqrt{x^2 + 2x + 5}\right| + C$$

Ex. Evaluate: $\int \frac{1}{x^2-2x+3}dx$

Sol. $I=\int \frac{1}{x^2-2x+3}dx$

$$\int \frac{1}{(x-1)^2+2}dx$$

$$\int \frac{1}{(x-1)^2+(\sqrt{2})^2}dx$$

$$\int \frac{1}{(x-1)^2+(\sqrt{2})^2}dx$$

$$\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+C$$

Ex. Evaluate: $\int \frac{1}{\sqrt{33+8x-x^2}}\ dx$

Sol. $\int \frac{1}{\sqrt{33+8x-x^2}}dx$

$$\int \frac{1}{\sqrt{-\left\{x^2-8x-33\right\}}}dx$$

$$\int \frac{1}{\sqrt{-\left\{x^2-8x+16-49\right\}}}dx$$

$$\int \frac{1}{\sqrt{-\left\{(x-4)^2-7^2\right\}}}dx$$

$$\int \frac{1}{\sqrt{7^2-(x-4)^2}}dx$$

$$\sin^{-1}\left(\frac{x-4}{7}\right)+C$$

Integration of type

$$\int \frac{px+q}{ax^2+bx+c}dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx, \int (px+q)\sqrt{ax^2+bx+c}\,dx$$

Represent px + q as A times the derivative of the denominator plus B.

Ex. Evaluate: $\int \frac{2x+3}{\sqrt{x^2+4x+1}}\ dx$

Sol. $\int \frac{2x+3}{\sqrt{x^2+4x+1}}dx$

$$\int \frac{(2x+4)-1}{\sqrt{x^2+4x+1}}dx \triangleleft$$

$$\int \frac{2x+4}{\sqrt{x^2+4x+1}}dx-\int \frac{1}{\sqrt{x^2+4x+1}}dx$$

$$\int \frac{dt}{\sqrt{t}}-\int \frac{1}{\sqrt{(x+2)^2-(\sqrt{3})^2}}dx$$

$t=\left(x^2+4x+1\right)$ (For I^{ST} integral)

$$2\sqrt{t} - \ln|(x+2)+|+\sqrt{x^2+4x+1}C$$

$$2\sqrt{x^2+4x+1} - \ln\left|x+2+\sqrt{x^2+4x+1}\right| + C$$

Ex. Evaluate: $\int (x-5)\sqrt{x^2+x}\, dx$

Sol. $(x-5) = \ln \cdot \frac{d}{dx}\left(x^2+x\right)+\mu.$

$$x-5 = \lambda(2x+1)+\mu$$

By comparing coefficients of similar powers of x, we obtain

$$1 = 2\lambda \text{ and } \lambda+\mu = -5$$

$$\lambda = \frac{1}{2} \text{ and } \mu = -\frac{11}{2}$$

$$\int (x-5)\sqrt{x^2+x}\,dx$$

$$\int\left(\frac{1}{2}(2x+1)-\frac{11}{2}\right)\sqrt{x^2+x}\,dx$$

$$\int \frac{1}{2}(2x+1)\sqrt{x^2+x}\,dx - \frac{11}{2}\int\sqrt{x^2+x}\,dx$$

$$\frac{1}{2}\int\sqrt{t}\,dt - \frac{11}{2}\int\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\,dx$$

(Where $t = x^2 + x$ for first integral)

$$\frac{1}{2}\cdot\frac{t^{3/2}}{3/2} - \frac{11}{2}\left[\left\{\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right]\right.$$

$$\left.-\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 \ln\left[\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right\}\right] + C$$

$$= \frac{1}{3}t^{3/2} - \frac{11}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x} - \frac{1}{8}\ln\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|\right] + C$$

$$= \frac{1}{3}\left(x^2+x\right)^{3/2} - \frac{11}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x} - \frac{1}{8}\ell n\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|\right] + C$$

Integration By partial Fraction

Integration of rational algebraic functions by using partial fractions:

(i) Partial Fractions:

If f(x) and g(x) represent two polynomials, the expression $\frac{f(x)}{g(x)}$ defines a rational algebraic function of x. Assume the degree of f(x) is less than the degree of g(x); if not, perform division of f(x) by g(x) until the degree of the numerator is lower than that of the denominator. Then, apply the concept of partial fractions as outlined below:

CASE I :

When the denominator can be expressed as the product of non-repeating linear factors.

Let's assume... $g(x)=(x-a_1)(x-a_2)\ldots(x-a_n)$.

Then, we assume that $\frac{f(x)}{g(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+......+\frac{A_n}{x-a_n}$

Where A_1, A_2, An are constants that can be determined by equating the numerator on the right-hand side to the numerator on the left-hand side and subsequently substituting them.

$$x = a_1, a_2, \ldots\ldots\ldots, a_n.$$

CASE II:

When the denominator g(x) can be represented as the product of linear factors, with some of them being repeated.

Example:

$$\frac{1}{g(x)}=\frac{1}{(x-a)^k(x-a_1)(x-a_2)......(x-a_r)}$$

This can be expressed as

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+.....+\frac{A_k}{(x-a)^k}+\frac{B_1}{(x-a_1)}+\frac{B_2}{(x-a_2)}+....+\frac{B_r}{(x-a_r)}$$

To find the constants, we equate the numerators on both sides. Some of these constants are determined through substitution, as in Case I, while the remaining constants are obtained by equating the coefficients of the same power of x. The procedure is illustrated in the following example.

CASE III:

When certain factors of the denominator g(x) are quadratic but non-repeating, for each quadratic factor $ax^2 + bx + c$, we posit a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$. Here, A and B are constants determined by comparing coefficients of corresponding powers of x in the numerators on both sides. In practice, it is recommended to assume partial fractions of this form $\frac{A(2ax+b)}{ax^2+bx+c}+\frac{B}{ax^2+bx+c}$. The following example illustrates the procedure.

CASE IV :

When certain factors of the denominator g(x) are quadratic and recurring, fractions of the form

$$\left\{\frac{A_0(2ax+b)}{ax^2+bx+c}+\frac{A_1}{ax^2+bx+c}\right\}+\left\{\frac{A_1(2ax+b)}{(ax^2+bx+c)^2}+\frac{A_2}{(ax^2+bx+c)^2}\right\}+.....\left\{\frac{A_{2k-1}(2ax+b)}{(ax^2+bx+c)^k}+\frac{A_{2k}}{(ax^2+bx+c)^k}\right\}$$

Ex. Evaluate $\int \frac{(2x-1)}{(x-1)(x+2)(x-3)}\,dx$

Sol. Let $\frac{(2x-1)}{(x-1)(x+2)(x-3)} = \frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{x-3}$

$\Rightarrow \quad \frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{A(x+2)(x-3)+B(x-1)(x-3)+C(x-1)(x+2)}{(x-1)(x+2)(x-3)}$

Putting $x = 1, \quad -6A = 1$

$\Rightarrow \quad A = -\frac{1}{6}$

Putting $x = 3, \quad 10C = 5$

$\Rightarrow \quad C = \frac{1}{2}$

Putting $x = -2, 15B = 5$

$\Rightarrow \quad B = -\frac{1}{3}$

So $-\frac{1}{6}\int\frac{1}{x-1}dx-\frac{1}{3}\int\frac{1}{x+2}dx+\frac{1}{2}\int\frac{1}{x-3}dx$

$-\frac{1}{6}\log|x-1| -\frac{1}{3}\log_e|x+2| + \frac{1}{2}\log_e|x-3| + C$

Ex. Solve $\frac{x^3-6x^2+10x-2}{x^2-5x+6}$ into partial fractions.

Sol. In this case, the provided function is an improper rational function, meaning that the degree of the numerator is greater than the degree of the denominator.
On dividing,

We get, $\frac{x^3-6x^2+10x-2}{x^2-5x+6} = x-1+\frac{(-x+4)}{(x^2-5x+6)}$ .... (i)

We have, $\frac{-x+4}{x^2-5x+6} = \frac{-x+4}{(x-2)(x-3)}$

So, let $\frac{-x+4}{(x-2)(x-3)} = \frac{A}{x-2}+\frac{B}{x-3}$

Then $-x+4 = A(x-3)+B(x-2)$ .... (ii)

Putting $x - 3 = 0$ or $x = 3$ in (ii),

We get $1 = B\,(1)$

$\Rightarrow \quad B = 1.$

Putting $x - 2 = 0$ or $x = 2$ in (ii),

We get $2 = A\,(2-3)$

$\Rightarrow \quad A = -2$

$\therefore \quad \frac{-x+4}{(x-2)(x-3)} = \frac{-2}{x-2}+\frac{1}{x-3}$

Hence $\frac{x^3-6x^2+10x-2}{x^2-5x+6} = x-1-\frac{2}{x-2}+\frac{1}{x-3}$

Ex. Resolve $\frac{3x-2}{(x-1)^2(x+1)(x+2)}$ into partial fractions, and evaluate $\int\frac{(3x-2)dx}{(x-1)^2(x+1)(x+2)}$.

Sol. Let $\frac{3x-2}{(x-1)^2(x+1)(x+2)}=\frac{A_1}{x-1}+\frac{A_2}{(x-1)^2}+\frac{A_3}{x+1}+\frac{A_4}{x+2}$

$\Rightarrow$ $3x - 2$

$A_1(x-1)(x+1)(x+2)+A_2(x+1)(x+2)+A_3(x-1)^2(x+2)+A_4(x-1)^2(x+1)$ ......(i)

Putting x – 1 = 0 or, x = 1 in (i)

we get $1 = A_2(1 + 1)(1 + 2)$

$\Rightarrow$ $A_2 = -\frac{5}{4}$

Putting x + 1 = 0 or, x = –1 in (i)

we get $-5 = A_3(-2)^2(-1 + 2)$

$\Rightarrow$ $A_3 = -\frac{5}{4}$

Putting x + 2 = 0 or, x = –2 in (i)

we get $-8 = A_4(-3)^2(-1)$

$\Rightarrow$ $A_4 = \frac{8}{9}$

Now equating coefficient of x^3 on both sides,

we get $0 = A_1 + A_3 + A_4$

$\Rightarrow$ $A_1 = -A_3 - A_4$

$\frac{5}{4}-\frac{8}{9}=\frac{13}{36}$

$$\frac{3x-2}{(x-1)^2(x+1)(x+2)}=\frac{13}{36(x-1)}+\frac{1}{6(x-1)^2}-\frac{5}{4(x+1)}+\frac{8}{9(x+2)}$$

Hence $\int\frac{(3x-2)dx}{(x-1)^2(x+1)(x+2)}$

$\frac{13}{36}\text{In}|x-1|-\frac{1}{6(x-1)}-\frac{5}{4}\text{In}|x+1|+\frac{8}{9}\text{In}|x+2|+C$

Ex. Evaluate $\int\frac{x^2}{(x^2+4)(x^2+1)}dx$

Sol. $\int\frac{x^2}{(x^2+4)(x^2+1)}dx$

$$=\frac{1}{3}\int\left[\frac{4}{x^2+4}-\frac{1}{x^2+1}\right]dx$$

$$=\frac{4}{3}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)-\frac{1}{3}\tan^{-1}x+C$$

$$=\frac{2}{3}\tan^{-1}\left(\frac{x}{2}\right)-\frac{1}{3}\tan^{-1}x+C$$

Ex. Resolve $\frac{2x-3}{(x-1)(x^2+1)^2}$ into partial fractions.

Sol. Let $\frac{2x-3}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$. Then,

$$2x-3 = A(x^2+1)^2 + (Bx+C)(x-1)(x^2+1) + (Dx+E)(x-1) \dots\dots (i)$$

Putting $x = 1$ in eq (i), we get $-1 = A(1+1)^2 \Rightarrow A = -\frac{1}{4}$

Comparing coefficients of like powers of x on both side of (i), we have

$A + B = 0, C - B = 0, 2A + B - C + D = 0, C + E - B - D = 2$ and

$$A - C - E = -3.$$

Putting $A = -\frac{1}{4}$ and solving these equations, we get

$$B = \frac{1}{4} = C, D = \frac{1}{4} \text{ and}$$

$E = \frac{5}{2} \therefore \frac{2x-3}{(x-1)(x^2+1)^2} = \frac{-1}{4(x-1)} + \frac{x+1}{4(x^2+1)} + \frac{9x+5}{2(x^2+1)^2}$

Ex. Resolve $\frac{2x}{x^3-1}$ into partial fractions.

Sol. We have, $\frac{2x}{x^3-1} = \frac{2x}{(x-1)(x^2+x+1)}$

So, let $\frac{2x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

Then, $2x = A(x^2 + x + 1) + (Bx + C)(x - 1)$ ....(i)

Putting $x - 1 = 0$ or, $x = 1$ in (i),

We get $2 = 3A$

$\Rightarrow$ $A = \frac{2}{3}$

Putting $x = 0$ in (i),

We get $A - C = 0$

$\Rightarrow$ $C = A = \frac{2}{3}$

Putting $x = -1$ in (i),

We get $-2 = A + 2B - 2C$.

$\Rightarrow$ $-2 = \frac{2}{3} + 2B - \frac{4}{3}$

$\Rightarrow$ $B = -\frac{2}{3}$

$\therefore$ $\frac{2x}{x^3-1} = \frac{2}{3}.\frac{1}{x-1} + \frac{\left(-\frac{2}{3}\right)x + \frac{2}{3}}{x^2+x+1}$

Or $\frac{2x}{x^3-1} = \frac{2}{3}\frac{1}{x-1} + \frac{2}{3}\frac{1-x}{x^2+x+1}$

Integration by parts:

The integration of the product of two functions, f(x) and g(x), can be accomplished using the formula:

$$\int(f(x)g(x))dx = f(x)\int(g(x))dx - \int\left(\frac{d}{dx}(f(x))\int(g(x))dx\right)dx$$

(i) Upon determining the integral $\int g(x)\,dx$, it will be devoid of any arbitrary constants.

(ii) $\int g(x)\,dx$ Should be considered identical in both instances.

(iii) The selection of f(x) and g(x) can be guided by the ILATE rule. The function that appears later is treated as the integral function (g(x)).

I → Inverse function
L → Logarithmic function
A → Algebraic function
T → Trigonometric function
E → Exponential function

Ex. Evaluate: $\int x\log_e x dx$

Sol. Let

$$I = \int g(x)\,dx$$

$$- \log_e x\int x dx - \int\left\{\frac{d}{dx}(\log x)\int x dx\right\}dx$$

$$\log_e x\left(\frac{x^2}{2}\right) - \int\frac{1}{x}\times\frac{x^2}{2} - dx$$

$$= \frac{x^2}{2}\log_e x - \frac{x^2}{4} + C$$

Ex. Evaluate: $\int x\,In(1+x)\,dx$

Sol. Let

$$I = I = \int x\,In(1+x)\,dx$$

$$In(x+1)\frac{x^2}{2} - \int\frac{1}{x+1}.\frac{x^2}{2}dx$$

$$\frac{x^2}{2}In(x+1) - \frac{1}{2}\int\frac{x^2}{x+1}dx$$

$$\frac{x^2}{2}In(x+1) - \frac{1}{2}\int\frac{x^2-1+1}{x+1}dx$$

$$\frac{x^2}{2}In(x+1) - \frac{1}{2}\int\left(\frac{x^2-1}{x+1}+\frac{1}{x+1}\right)dx$$

$$\frac{x^2}{2}In(x+1) - \frac{1}{2}\int\left((x-1)+\frac{1}{x+1}\right)dx$$

$$\frac{x^2}{2}In(x+1) - \frac{1}{2}\int\left(\frac{x^2}{2} - x + In|x+1|\right) + C$$

Ex. Evaluate: $\int e^{2x}\sin 2x\,dx$

Sol. We know that $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$

$$a = 2$$

and $$b = 2 = \frac{e^{2x}}{8}\ (2\sin 2x - 2\cos 2x) + C$$

Note :

(i) $$\int e^x [f(x) + f'(x)]\, dx = e^x \cdot f(x) + C$$

(ii) $$\int [f(x) + xf'(x)]\, dx = x\, f(x) + C$$

Ex. Evaluate: $\int \left[\text{In}\ (\text{In}\, x) + \frac{1}{(\text{In}x)^2} \right] dx$

Sol. Let $$I = \int \left(\text{In}(\text{In}x) + \frac{1}{(\text{In}x)^2} \right) dx$$

put $$x = e^t$$

$$dx = e^t dt$$

$$I = \int e^t \left(\text{In}t + \frac{1}{t^2} \right) dt$$

$$\int e^t \left(\text{In}t - \frac{1}{t} + \frac{1}{t} + \frac{1}{t^2} \right) dt$$

$$e^t \left(\text{In}t - \frac{1}{t} \right) + C$$

$$x \left[\text{In}(\text{In}x) - \frac{1}{\text{In}x} \right] + C$$