INTEGRATION OF IRRATIONAL FUNCTIONS

a. **Integral of the type $\int f\left(x, \left(\frac{px+q}{p_1x+q_1}\right)^{\frac{m}{n}}\right)dx$ where $p, q_1 p_1, q_1 \in R$ and $m, n \in I - \{0\}$.**

We shall substitute $\frac{px+q}{p_1x+q_1} = t^n$ which will reduce the integrand into standard form.

b. **Integral of the type $\int f\left(x, (px+q)^{\frac{m}{n}}, (px+q)^{\frac{m_1}{m_1}}, \ldots\right)dx$, where $p, q \in R, m, n, m_1, n_1 \in Z$.**

We shall substitute $(px+q)= t^{\alpha}$, where α is L.C.M. of n_1, n_2....., which will reduce the integrand into standard form.

c. **Integral of the type $\int \frac{1}{L\sqrt{M}}dx$**

We shall use the following substitution for the given integral form in different cases

	Substitution
L and M are both linear expressions	$M = t^2$
L is linear and M is quadratic expression	$\frac{1}{L} = t$
L is quadratic and M is linear	$M = t^2$
L and M are both quadratic expressions	$\frac{M}{L} = t^2$ or $x = \frac{1}{t}$

d. **Integral of the type $\int f\{x^m(ax^n+b)^r\}dx$**

i. If r = a positive integer then expand

ii. Let $r = \frac{p}{g}$, where p, q are integers, $q \neq 0$, if $\frac{m+1}{n} =$ integer put $ax^n + b = t^q$

iii. If $\frac{m+1}{n} \neq$ integer but $\frac{m+1}{n} + \frac{p}{q} =$ integer take x^m out of bracket (i.e. $(x^m)^{\rho/q}$) and put $a + bx^{-n} = t^q$

Successive Integration by parts

Let $u'(x)$ and $v'(x)$ be two given functions of x. Let $u'(x), u''(x), u'''(x) \ldots\ldots u''(x)$ denote the successive differential coefficients of u w.r.t. x and $v_1 = \int v dx, v_2 = \int v_1 dx = \int(\int v dx)dx, v_3 = \int v_2 dx \ldots\ldots.. v_n = \int v_{n-1}dx$

Then $\int uv dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots\ldots\ldots + (-1)^{n-1}u^{n-1}v_n + (-1)^n \int u^n v_n dx$

Reduction Formulae

1. $I_n = \int \sin^n x dx$

$= \int \sin_1^{n-1} x \cdot \sin x dx$

$= \sin^{n-1} x \cdot (-\cos x) - \int (n-1)\sin^{n-2} x \cdot \cos x \cdot (-\cos x)dx$

$= -\sin^{n-1} x \cdot \cos x + (n-1)\int \sin^{n-2} x \cdot (1 - \sin^2 x)dx$

$= -\sin^{n-1} x \cdot \cos x + (n-1)l_{n-2} - (n-1)l_n$

$= I_n + (n-1)I_n = -\sin^{n-1} x \cdot \cos x + (n-1)V_{n-2}$

$$= I_n = \frac{-\sin^{n-1} x \cdot \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

Similarly $\int \cos^n x dx$ can be reduced as

$$l_n = \frac{\cos^{n-1} \cdot \sin x}{n} + \frac{n-1}{n} I_{n-2}$$

2. $I_n = \int \tan^n x dx$

$= \int \tan^{n-2} x \cdot \tan^2 xdx$
$= \int \tan^{n-2} x - (\sec^2 x - 1)dx$
$= \int \tan^{n-2} x \cdot \sec^2 xdx - I_{n-2}$
$= \int \tan^{n-2} x - d(\tan x) - I_{n-2}$
$\therefore I_n + I_{n-2} = \dfrac{\tan^{n-1} x}{n-1}$

3. $I_n = \int \sec^n xdx$

$= \int \sec_1^{n-2} x \cdot \sec^2 xdx$
$= \sec^{n-2} x \cdot \tan x - \int (n-2)\sec^{n-2} x \cdot \tan x \cdot \tan xdx$
$= \sec^{n-2} x \cdot \tan x - (n-2)\int \sec^{n-2} x \cdot (\sec^2 x - 1)dx$
$= \sec^{n-2} x \cdot \tan x - (n-2)I_n + (n-2)I_{n-2}$
$\Rightarrow I_n = \dfrac{\sec^{n-2} x \cdot \tan x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$

4. $I_n = \int x^n e^{ax} dx$

$= x^n \dfrac{e^{ax}}{a} - \int \dfrac{e^{ax}}{a} nx^{n-1} dx$ (Applying by parts)
$= \dfrac{e^{ax}}{a} x^n - \dfrac{n}{a} \int e^{ax} x^{n-1} dx$
$= \dfrac{e^{ax}}{a} x^n - \dfrac{n}{a} I_{n-1}$

5. $I_n = \int (\log x)^n dx$

$= x(\log x)^n - \int xn(\log x)^{n-1} \frac{1}{x} dx$ (Applying by parts)
$= x(\log x)^n - n\int (\log x)^{n-1} dx$
$= x(\log x)^n - nI_{n-1}$

6. $I_{m,n} = \int \sin^m x\cos^n xdx$

$= \int \sin^m x\cos^{n-1} x\cos xdx$
$= \sin^m x\cos^{n-1} x\sin x - \int [\sin^m x(n-1)\cos^{n-2} x(-\sin x) + \cos^{n-1} x m\sin^{m-1} x\cos$
$= \sin^{m+1} x\cos^{n-1} x + (n-1)\int \sin^m x\cos^{n-2} x\sin^2 xdx - m\int \sin^m x\cos^n xdx$
$= \sin^{m+1} x\cos^{n-1} x + (n-1)\int \sin^m x\cos^{n-2} x(1 - \cos^2 x)dx - mI_{m,n}$
$= \sin^{m+1} x\cos^{n-1} x + (n-1)I_{m,n-2} - (n-1)I_{m,n} - mI_{m,n}$
$= \dfrac{\sin^{m+1} x\cos^{n-1} x}{m+n} + \dfrac{n-1}{m+n} I_{m,n-2}$

Integrals of the type $\int \sin^m x\cos^n xdx$

In order to integrate such type of integrals, the suggested substitution are as follows

(i) If m is an odd positive integer, put $\cos x = U$
(ii) If n is an odd positive integer, put $\sin x = u$
(iii) If m and n are both positive odd integers, put $\sin x = u$ or $\cos x = u$
(iv) If neither of m, n is an odd positive integer, then we look for their sum $m + n$

a. If $m + n =$ negative and even integer then convert the given integrand in terms of $\tan x$ and $\sec^2 x$ and then put $\tan x = u$ and $dx = \frac{1}{1+u^2} du$
b. If m and n are small even integers, then convert them in terms of multiple angles by using the formulae
$\cos^2 x = \dfrac{1 + \cos 2x}{2}, \sin^2 x = \dfrac{1 - \cos 2x}{2}$

$\sin x \cos x = \frac{\sin 2x}{2}$ and $\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$

c. If m and n are large even positive integers then change in multiple angles with the help of Euler's formulae in complex numbers

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \sin x = \frac{e^{ix} - e^{-i}}{2i}$$
$$\text{or } \cos x = \frac{1}{2}(z + \frac{1}{z}), \sin x = \frac{1}{2i}(z - \frac{1}{z}), \text{ where } z = e^{jx}$$

Integral of Type $\int \tan^n x \sec^n x dx$ or $[\cot^m x \operatorname{cosec}^n x dx$

The suggested substitutions are as follows :

i. If n is an even positive integer, we put $\tan x = t$ or $\cot x = t$

ii. If n is not an even positive integer, then we look for the value of m

a. If m is an odd positive integer we put $\sec x = t$ or $\operatorname{cosec} x = t$

b. If $m = 0$, then we integrate the integrand by means of integration by parts

c. If m is an even positive integer, we write $\sec^2 x - 1$ in place of $\tan^2 x$ and $\operatorname{cosec}^2 x \ -1$ for $\cot^2 x$ and then use integration by parts.

Integral of the type $\int \left(\frac{p\cos x + q\sin x}{a\cos x + b\sin x}\right) dx$ and $\int \frac{p\cos x + q\sin x + r}{a\cos x + b\sin x + c} dx$

For the first type of integral, express the numerator as

Numerator = k(denominator) + m (denominator) and find k, m by comparing the coefficients of

sin x and dx cos x and then split the given integral into the sum of two integrals as

Integral of the type $\int f(\sin x, \cos x) dx$

where fis a rational function of sin x and cos x

The suggested substitutions for the given type of integral are as follows

i. If f(sin x, cos x) is of the form g(sin x)cos x, substitute sin x = t

ii. If f(sin x, cos x) is of the form g(cos x)sin x, substitute cos x = t

iii. If f(sin x, cos x) is dependent on tan x, substitute tan x = f

iv. If f(- sin x, - cos x) = f(sin x, cos x) substitute tan x = t

v. If f(- sin x, cos x) = - f * (sin x, cos x) substitute cos x = t

vi. If f(sin x, - cos x) = - f * (sin x, cos x) substitute sin x = t

Integral of the type $\int f(x \pm \sqrt{x^2 + a^2}) dx$

We substitute $x \pm \sqrt{x^2 + a^2} = t$

so that $\frac{dx}{\sqrt{x^2+a^2}} = \frac{dt}{t}$ and $\sqrt{x^2 + a^2} = \frac{1}{2}(t + \frac{a^2}{t}) = \frac{t^2+a^2}{2t} \Rightarrow dx = \pm \frac{t^2+a^2}{2t^2} dt$

Hence, the given integral will reduce to $\pm \int f(t) \cdot \frac{t^2+a^2}{2t^2} dt$

which can be integrated using the standard results.