## CONTINUITY OF SPECIAL TYPES OF FUNCTIONS

### Continuity of Functions in which Greatest Integer Function is involved

$f(x) = [x]$ is discontinuous when x is an integer.

Similarly, $f(x) = [g(x)]$ is discontinuous at all integers when g(x) is an integer, but this is true only when g(x) is monotonic [g(x) is strictly increasing or strictly decreasing].

For example, $f(x) = [\sqrt{x}]$ is discontinuous at all integers when $[\sqrt{x}]$ is integer, as $[\sqrt{x}]$ is strictly increasing (monotonic functions.)

$f(x) = [x^2]$, $x \geq 0$, is discontinuous at all integers when $x^2$ is an integer, as $x^2$ is strictly increasing x $\geq 0$.

Now, consider $f(x) = [\sin x]$, $x \in [0, 2\pi]$. $g(x) = \sin x$ is not monotonic in $[0, 2\pi]$. For this type of function, points of discontinuity can be determined easily by graphical methods. We can note that $x = \frac{3\pi}{2}$, sin x takes integral value −1, but at $x = \frac{3\pi}{2}$, $f(x) = [\sin x]$ is continuous.

**Ex.** Discuss the continuity of the following functions ([·] represent the GIF)

(A) $f(x) = [\log_e x]$

(B) $f(x) = [\sin^{-1} x]$

(C) $f(x) = [\frac{2}{1+x^2}], x \geq 0$

**Sol.** (A) $\log_e x$ is a monotonically increasing function.

Hence, $f(x) = [\log_e x]$ is discontinuous, where $\log_e x = k$ or $x = e^k, k \in Z$

Thus, f(x) is discontinuous at $x = e^{-2}, e^{-1}, e^0, e^1, e^2$

(B) $\sin^{-1} x$ is a monotonically increasing function

Hence, $f(x) = [\sin^{-1} x]$ is discontinuous where $\sin^{-1} x$ is an integer.

Therefore, $\sin^{-1} x = -1,0,1$ or $x = -\sin 1, 0, \sin 1$.

(D) $\frac{2}{1+x^2}, x \geq 0$, is a monotonically decreasing function.

Hence, $f(x) = \left[\frac{2}{1+x^2}\right], x \geq 0$, is discontinuous, when $\frac{2}{1+x^2}$ is an integer.

Therefore, $\frac{2}{1+x^2} = 1,2$ or $x = 1,0$

### Continuity of functions in which Signum function is involved

We know that $f(x) = \text{sgn}(x)$ is discontinuous at $x = 0$.

In general, $f(x) = \text{sgn}(g(x))$ is discontinuous $x = a$ if $g(a) = 0$.

**Ex.** Discuss the continuity of

(A) $f(x) = \text{sgn}(x^3 - x)$

(B) $f(x) = \text{sgn}(2\cos x - 1)$

(C) $f(x) = \text{sgn}(x^2 - 2x + 3)$

**Sol.** (A) $f(x) = \text{sgn}(x^3 - x)$

Here, $\mathbf{x}^3 - \mathbf{x} = 0 \Rightarrow x = 0, -1, 1$.

Hence, f(x) is discontinuous at $x = 0, -1, 1$

(B) $f(x) = \text{sgn}(2\cos x - 1)$

Here, $2\cos x - 1 = 0 \Rightarrow \cos x = \frac{1}{2}$

$\Rightarrow x = 2n\pi + \left(\frac{\pi}{3}\right)$, $n \in Z$, where f(x) is discontinuous.

(C) $f(x) = \text{sgn}(x^2 - 2x + 3)$

Here, $x^2 - 2x + 3 > 0$ for all x.

Thus, $f(x) = 1$ for all x.

Hence, it is continuous for all x.

**Continuity of Functions Involving Limit $\lim\limits_{n\to\infty} a^n$**

We know that $\lim\limits_{n\to\infty} a^n = \begin{cases} 0, & 0 \le a < 1 \\ 1, & a = 1 \\ \infty, & a > 1 \end{cases}$

**Ex.** Discuss the continuity of $f(x) = \lim\limits_{n\to\infty} \frac{x^{2n}-1}{x^{2n}+1}$

**Sol.** $f(x) = \lim\limits_{n\to\infty} \frac{(x^2)^n-1}{(x^2)^n+1}$

$= \lim\limits_{n\to\infty} \frac{1-\frac{1}{(x^2)^n}}{1+\frac{1}{(x^2)^n}}$

$= \begin{cases} -1, & 0 \le x^2 < 1 \\ 0, & x^2 = 1 \\ 1, & x^2 > 1 \end{cases}$

$= \begin{cases} 1, & x < -1 \\ 0, & x = -1 \\ -1, & -1 < x < 1 \\ 0, & x = 1 \\ 1, & x > 1 \end{cases}$

Thus, f(x) is discontinuous at $x = \pm 1$.

**Continuity of Functions in Which f(x) is Defined Differently for Rational and Irrational Values of x**

$F(x) = \begin{cases} g(x), ; x \in Q \\ h(x), x \notin Q \end{cases}$

$g(x) = h(x) \rightarrow$ solution of this equation

**Ex.** $F(x) = \begin{cases} x + 1; x \in Q \\ 6 - x; x \notin Q \end{cases}$

**Sol.** $x + 1 = 6 - x$

$2x = 6 - 1$

$2x = 5$

$x = 2.5$

f(x) is continuous at 2.5 and discontinuous for rest of the values.

**Continuity of Composite Functions**

If f is continuous at x = c and g is continuous at x = f(c),

Then the composite g[f(x)] is continuous at x = c.

f(x) = f(g(x)) is discontinuous also at those values of x where g(x) is discontinuous.

eg. $f(x) = \frac{x\sin x}{x^2 + 2}$ & $g(x) = |x|$ are continuous at $x = 0$

hence the composite function (gof) $(x) = |\frac{x\text{si}\ \ x}{x^2+2}|$ will also be continuous at x = 0.

For example, $f(x) = \frac{1}{1 - x}$ is discontinuous at $x = 1$.

Now, $f(f(x)) = \frac{1}{1-\frac{1}{1-x}} = \frac{x-1}{x}$ is not only discontinuous at $x = 0$ but also at x = 1.

Now, $f(f(f(x))) = \frac{\frac{x-1}{x}-1}{\frac{x-1}{x}} = x$ seems to be continuous, but it is discontinuous at x = 1 and x = 0, where f(x) and f(f(x)) are discontinuous, respectively

**Ex** If $f(x) =, \begin{cases} x-2, x \le 0 \\ 4-x^2, x > 0 \end{cases}$ then discuss the continuity of $y = f(f(x))$.

**Sol.** f(x) is discontinuous at x = 0.

Hence, f(f(x)) may be discontinuous at x = 0.

f(f(0+)) = f(4) = 4 – 16 = 12

and f(f(0–)) = f(–2) = –4

Hence, f(x) is discontinuous at x = 0.

f(f(x)) is also discontinuous when f(x) = 0. Therefore,

x – 2 = 0 when x ≤ 0 or $x^2$ – 4 = 0 when x > 0

So, it is discontinuous at x = 2.

Also, we can see that f(f(2)) = 0, f(f(2+)) = f(0–) = –2, and f(f(2–)) = f(0+) = 4.

Hence, f(f(x)) is discontinuous at x = 0 and x = 2.

## Properties of Continuous Function in [a, b]

i. If a function f is continuous on a closed interval [a, b] then it is bounded.

ii. A continuous function whose domain is some closed interval must have its range also in closed interval.

iii. If f is continuous and onto on [a, b] and is onto then f –1(from the range of f ) is also continuous.

iv. Some Discontinuous Functions

| Functions | Points of discontinuity |
|---|---|
| [x], {x} | Every Integer |
| tan x, sec x | $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots\dots$ |
| cot x, cosec x | $x = 0, \pm\pi, \pm2\pi, \dots\dots$ |
| $\sin\frac{1}{x}, \cos\frac{1}{x}, \frac{1}{x}, e^{1/x}$ | x = 0 |

v. Some continuous Functions

| Function f(x) | Interval in which f(x) is continuous |
|---|---|
| Constant function | $(-\infty, \infty)$ |
| $x^n$, n is an integer $\ge 0$ | $(-\infty, \infty)$ |
| $x^{-n}$, n is a positive integer | $(-\infty, \infty) - \{0\}$ |
| $\lvert x-a \rvert$ | $(-\infty, \infty)$ |
| $p(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots..+a_n$ | $(-\infty, \infty)$ |
| $\frac{p(x)}{q(x)}$, where p(x) and q(x) are polynomial in x | $(-\infty, \infty) - \{x: q(x) = 0\}$ |
| sin x, cos x, $e^x$ | $(-\infty, \infty)$ |
| tan x, sec x | $(-\infty, \infty) - \{(2n+1)\pi/2 : n \in I\}$ |
| cot x, cosec x | $(-\infty, \infty) - \{n\pi: n \in I\}$ |
| ln x | $(0, \infty)$ |