PROPERTIES OF DETERMINANTS

(a) The value of a determinant remains unchanged when the rows and columns are interchanged.

e.g. $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(b) If any two rows (or columns) of a determinant are swapped, the determinant's value changes only in sign.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$D_1 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then $D_1 = -D$.

(c) When all the elements of a row (or column) are zero, the determinant's value is zero.

(d) If all the elements of a particular row (or column) are multiplied by the same number, the determinant is then multiplied by that number.

e.g. $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_3 \end{vmatrix}$

and $D_1 = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then $D_1 = KD$

(e) If every element in a row (or column) is proportional (or identical) to the element in another row, the determinant becomes zero, meaning its value is zero.

e.g. $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$D = 0;$

If $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$D_1 = 0$

(f) If each element in any row (or column) is represented as a sum of two (or more) terms, the determinant can be expressed as the sum of two (or more) determinants.

e.g. $\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Note: If $D_r = \begin{vmatrix} f(r) & g(r) & h(r) \\ a & b & c \\ a & b & c \end{vmatrix}$

Where, $r \in N$ and a, b, c, a_1, b_1, c_1 are constants, Then, $$\sum_{r=1}^{n} D_r = \begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

(g) Row - column operation:

The determinant's value remains unchanged when subjected to a column (C_i) operation of the form $C_i \rightarrow C_i + aC_j + bC_k$ (where j, k ≠ i) or a row (R_i) operation of the form $R_i \rightarrow R_i + aR_j + bR_k$ (where j, k ≠ i). In simpler terms, adding the elements of any row (or column) to the same multiples of corresponding elements in another row (or column) does not alter the determinant's value. For example, let

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$D = \begin{vmatrix} a_1 + \alpha a_2 & b_1 + \alpha b_2 & c_1 + \alpha c_2 \\ a_2 & b_2 & c_2 \\ a_3 + \beta a_1 & b_3 + \beta b_1 & c_3 + \beta c_1 \end{vmatrix} \qquad (R_1 \rightarrow R_1 + \alpha R_2; R_3 \rightarrow R_3 + \beta R_2)$$

Remember

(i) When employing the operation $R_i \rightarrow xR_i + yR_j + zR_k$ (where j, k ≠ i), the determinant's value becomes x times the original one.

(ii) To apply this property, at least one row (or column) must remain unchanged.

(h) Factor theorem:

If the elements of a determinant D are rational integral functions of x, and two rows (or columns) become identical when x = a, then (x – a) is a factor of D. Note that if r rows become identical when a is substituted for x, then

$(x - a)^{r-1}$ is a factor of D.

Ex. Simplify $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

Sol. Given determinant is equal to

$$= \frac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$$

Apply $C_1 \rightarrow C_1 - C_2$, $C_2 \rightarrow C_2 - C_3$

$$= \begin{vmatrix} a^2 - b^2 & b^2 - c^2 & c^2 \\ a^3 - b^3 & b^3 - c^3 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} a+b & b+c & c^2 \\ a^2 + ab + b^2 & b^2 + bc + c^2 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$= (a-b)(b-c)[ab^2+abc+ac^2+b^3+b^2C+bc^2-a^2b-a^2c-ab^2-abc-b^3-b^2c]$
$= (a - b)(b - c)[c(ab + bc + ca) - a(ab + bc + ca)]$
$= (a - b)(b - c)(c - a)(ab + bc + ca)$

Ex. Determine the value of the determinant $\begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix}$

Sol.
$$D = \begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & b & c^2 \end{vmatrix}$$
$$= a\begin{vmatrix} a & b & c \\ ab & b^2 & bc \\ ac & b & c^2 \end{vmatrix}$$
$$= abc\begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix} = 0$$

As all the rows are identical, the determinant's value is zero.

Ex. Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

Sol. Let $\Delta = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

Applying $R_1 \to R_1 - R_2 - R_3$ to D,

we get $\Delta = \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

Expanding along R_1,

we obtain $\Delta = 0\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - (-2c)\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + (-2\,b)\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$

$= 2\,c\,(a\,b + b^2 - bc) - 2\,b\,(b\,c - c^2 - ac)$
$= 2\,a\,b\,c + 2\,cb^2 - 2\,bc^2 - 2\,b^2c + 2\,bc^2 + 2\,abc$
$= 4\,abc$

Multiplication of Two Determinants

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1l_1 + b_1l_2 & a_1\,m_1 + b_1\,m_2 \\ a_2l_1 + b_2l_2 & a_2\,m_1 + b_2\,m_2 \end{vmatrix}$$

Similarly two determinants of order three are multiplied.

(a) In this case, we have performed multiplication row by column. Additionally, multiplication can also be done row by row, column by row, and column by column.

(b) If D_1 is the determinant formed by replacing the elements of determinant D of order n with their corresponding cofactors, then $D_1 = D^{n-1}$.

Ex. Let α & β be the roots of equation $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$ for $n \geq 1$.

Evaluate the value of the determinant $\begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$

Sol.

$$\Delta = \begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$$

$$= \begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= [(1-\alpha)(1-\beta)(\alpha-\beta)]^2$$

$$\Delta = (a - b)^2 (a + b - ab - 1)^2$$

Note: a & b are roots of the equation $ax^2 + bx + c = 0$

$$\alpha + \beta = \frac{-b}{a} \ \& \ \alpha\beta = \frac{c}{a}$$

$$|\alpha - \beta| = \frac{\sqrt{b^2 - 4ac}}{|a|}$$

$$\Delta = \frac{(b^2 - 4ac)}{a^2}\left(\frac{a+b+c}{a}\right)^2 \sqrt{b^2 - 4ac}$$

$$= \frac{(b^2 - 4ac)(a+b+c)^2}{a^4}$$

Important Determinants

(i) $\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

(v) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)\left(a^2+b^2+c^2-ab-bc-ca\right)$