Geometrical Applications of Determinant

(a) The lines: $a_1x + b_1y + c_1 = 0$ ...... (i)

$a_2x + b_2y + c_2 = 0$ ...... (ii)

$a_3x + b_3y + c_3 = 0$ ...... (iii)

are concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

This represents the criterion for the consistency of a system of three simultaneous linear equations in two variables.

(b) The Equation $ax^2 + 2\,hxy + by^2 + 2\,gx + 2\,fy + c = 0$ represents a pair of straight lines if :

$abc + 2\,fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

(c) The area of a triangle with vertices (x_r, y_r) ; $r = 1, 2, 3$ is given by

$$D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

If $D = 0$ then the three points are collinear.

(d) Equation of a straight line passing through points (x_1, y_1) & (x_2, y_2) is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Ex. Determine the area of the triangle with vertices at (3, 8), (– 4, 2) and (5, 1).

Sol. The area of triangle is given by $\Delta = \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}$

$$= \frac{1}{2}[3(2-1) - 8(-4-5) + 1(-4-10)]$$

$$= \frac{1}{2}(3 + 72 - 14) = \frac{61}{2}$$

Singular & non-singular matrix

A square matrix A is considered singular if its determinant |A| is zero and non-singular if |A| is non-zero.

Minors & Cofactors

Consider D as a determinant. The minor of the element a_{ij}, represented by M_{ij}, is defined as the determinant of the submatrix obtained by removing the i^{th} row and j^{th} column of D. The cofactor of the element a_{ij}, denoted as C_{ij}, is defined as

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

e.g. 1 $D = M_{11} = d = C_{11}$

$M_{12} = c$, $C_{12} = -c$

$M_{21} = b$, $C_{21} = -b$

$M_{22} = a = C_{22}$

e.g.2 $\Delta = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

$$M_{11} = \begin{vmatrix} q & r \\ y & z \end{vmatrix} = qz - yr = C_{11}.$$

$$M_{23} = \begin{vmatrix} a & b \\ x & y \end{vmatrix} = ay - bx, C_{23} = -(ay - bx) = bx - ay \text{ etc.}$$

Ex. Determine the minors and cofactors of elements of the matrix

$$A = [a_{ij}] = \begin{bmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{bmatrix}.$$

Sol. Let M_{ij} and C_{ij} represent the minor and cofactor, respectively, of the element a_{ij} in matrix A.

Then, $M_{11} = \begin{vmatrix} -5 & 6 \\ 5 & 2 \end{vmatrix} = -10 - 30 = -40$

$\Rightarrow$ $C_{11} = M_{11} = -40$

$M_{12} = \begin{vmatrix} 4 & 6 \\ 3 & 2 \end{vmatrix} = 8 - 18 = -10$

$\Rightarrow$ $C_{12} = -M_{12} = 10$

$M_{13} = \ = \begin{vmatrix} 4 & -5 \\ 3 & 5 \end{vmatrix} = 20 + 15 = 35$

$\Rightarrow$ $C_{13} = M_{13} = 35$

$M_{21} = \begin{vmatrix} 3 & -2 \\ 5 & 2 \end{vmatrix} = 6 + 10 = 16$

$\Rightarrow$ $C_{21} = -M_{21} = -16$

$M_{22} = \ = \begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} = 2 + 6 = 8$

$\Rightarrow$ $C_{22} = M_{22} = 8$

$M_{23} = \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} z = 5 - 9 = -4$

$\Rightarrow$ $C_{23} = -M_{23} = 4$

Ex. Determine the minors and cofactors of elements '−3', '5', '−1' & '7' in the determinant

$$\begin{vmatrix} 2 & -3 & 1 \\ 4 & 0 & 5 \\ -1 & 6 & 7 \end{vmatrix}$$

Sol. Minor of $-3 = \begin{vmatrix} 4 & 5 \\ -1 & 7 \end{vmatrix} = 33$ Cofactor of $-3 = -33$

Minor of $5 = \begin{vmatrix} 2 & -3 \\ -1 & 6 \end{vmatrix} = 9$ Cofactor of $5 = -9$

Minor of –1 = $\begin{vmatrix} -3 & 1 \\ 0 & 5 \end{vmatrix}$ =-15   Cofactor of –1 = –15

Minor of 7 = $\begin{vmatrix} 2 & -3 \\ 4 & 0 \end{vmatrix}$ =12   Cofactor of 7 = 12

Transpose of a Determinant

The transpose of a determinant equal the determinant of transpose of the corresponding matrix.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$D^T = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$