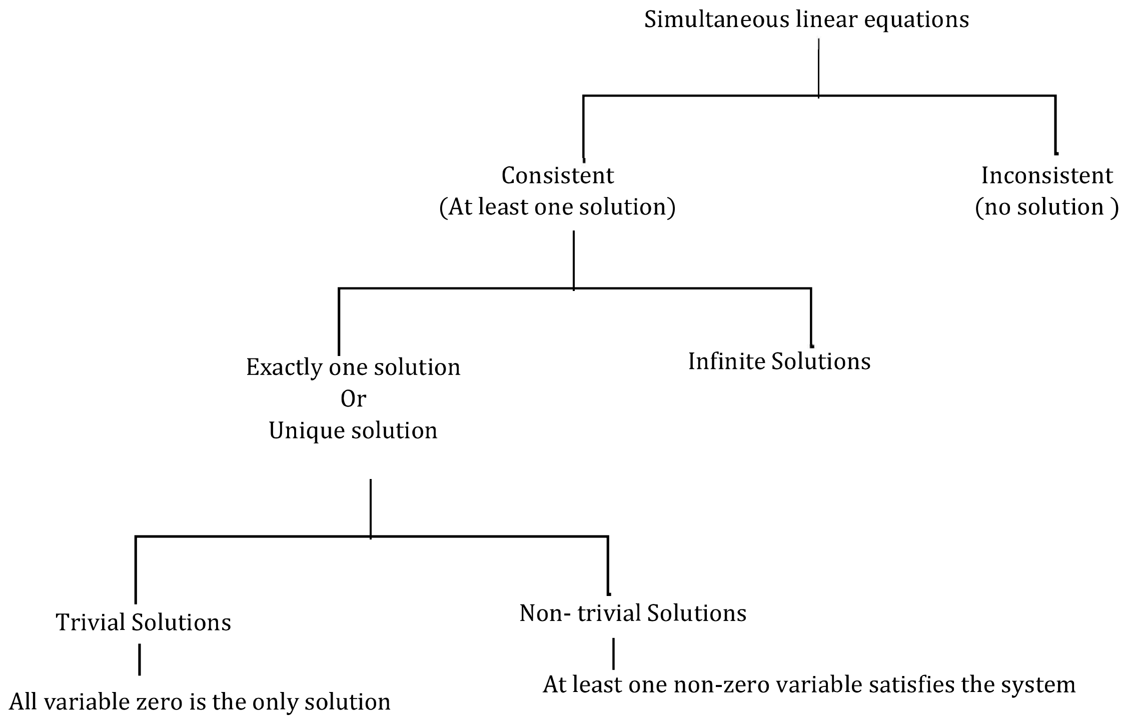
APPLICATIONS OF DETERMINANTS AND MATRICES
Cramer's Rule for Solution of System of Non-Homogeneous Linear Equations



Equations involving two variables

Consistent Equations : Definite & unique solution (Intersecting lines)

Inconsistent Equations : No solution (Parallel lines)

Dependent Equations : Infinite solutions (Identical lines)

Let, $a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$

then:

(1) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ $\Rightarrow$ Given, equations are consistent with unique solution

(2) $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ $\Rightarrow$ Given, equations are inconsistent

(3) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ $\Rightarrow$ Given, equations are consistent with infinite solutions

Solution of a system of equations
Equations Involving Three variables

Let $a_1x + b_1y + c_1z = d_1$ ......(i)

$a_2x + b_2y + c_2z = d_2$ ....(ii)

$a_3x + b_3y + c_3z = d_3$ ....(iii)

Then, $x = \frac{D_1}{D}, y = \frac{D_2}{D}, z = \frac{D_3}{D}.$

Where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Remember

(i) If $D \neq 0$ and at least one of D_1 , D_2 , $D_3 \neq 0$, then the provided system of equations is both consistent and possesses a unique non-trivial solution.

(ii) In this case where $D \neq 0$ & $D_1 = D_2 = D_3 = 0$, the given system of equations remains consistent but now only has a trivial solution.

(iii) When $D = D_1 = D_2 = D_3 = 0$, the provided system of equations is consistent and exhibits an infinite number of solutions.

Note: In case $a_1x + b_1y + c_1z = d_1$ If at least two of d_1, d_2, and d_3 are not equal, leading to $D = D_1$

$a_1x + b_1y + c_1z = d_2$

$a_1x + b_1y + c_1z = d_3$

$= D_2 = D_3 = 0$,it is important to note that these three equations depict three parallel planes. Consequently, the system becomes inconsistent.

(iv) If, $D = 0$ but at least one of D_1, D_2, D_3 is non-zero, the equations are inconsistent and do not possess a solution.

Solution of a homogeneous system of a linear equation

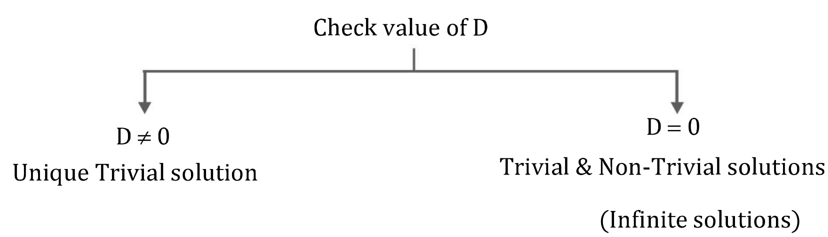
Let $a_1x + b_1y + c_1z = 0$ .... (i)

$a_2x + b_2y + c_2z = 0$ .... (ii)

$a_3x + b_3y + c_3z = 0$ .... (iii)

$\Rightarrow$ $D_1 = D_2 = D_3 = 0$

The system consistently has at least one solution with $x = 0, y = 0, z = 0$, termed the trivial solution. Therefore, this system is invariably consistent.



Note: If a provided system of linear equations exclusively yields zero solutions for all its variables, it is characterized as having a trivial solution. Additionally, it is important to observe that if the system of equations

$$a_1x + b_1y + c_1 = 0;$$

$$a_2x + b_2y + c_2 = 0;$$

$$a_3x + b_3y + c_3 = 0$$

is always consistent then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ but converse is not true.

Ex. Determine the type of solution for the provided system of equations:

$$x + 2y + 3z = 1;$$
$$2x + 3y + 4z = 3;$$
$$3x + 4y + 5z = 0$$

Sol. $D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$

Now $D_1 = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 4 \\ 0 & 4 & 5 \end{vmatrix} = 5$

$D = 0$ but $D_1 \neq 0$

Hence no solution.

Ex. Determine the value of λ that ensures the consistency of the following equations:

$$x + y - 3 = 0;$$
$$(1 + \lambda)x + (2 + \lambda)y - 8 = 0;$$
$$x - (1 + \lambda)y + (2 + \lambda) = 0$$

Sol. If the provided equations in two unknowns are consistent, then $D = 0$

i.e. $\begin{vmatrix} 1 & 1 & -3 \\ 1+\lambda & 2+\lambda & -8 \\ 1 & -(1+\lambda) & 2+\lambda \end{vmatrix} = 0$

Applying $C_2 \to C_2 - C_1$

and $C_3 \to C_3 + 3C_1$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1+\lambda & 1 & 3\lambda - 5 \\ 1 & -2-\lambda & 5+\lambda \end{vmatrix} = 0$$

$$(5 + \lambda) - (3\lambda - 5)(-2 - \lambda) = 0$$

$$\lambda = 1, -\frac{5}{3}$$

Ex. If x, y, z are not all simultaneously zero, and they satisfy the system of equations

$$\text{Sin}\,(3\theta)\,x - y + z = 0;$$
$$\text{Cos}\,(2\theta)x + 4y + 3z = 0;$$
$$2x + 7y + 7z = 0,$$

then find the values of θ (where $0 \le \theta \le 2\pi$).

Sol. The provided system of equations is a set of homogeneous linear equations with a non-zero solution set; consequently, $D = 0$

$$D = \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix}$$

$$D = \begin{vmatrix} \sin 3\theta & -1 & 0 \\ \cos 2\theta & 4 & 7 \\ 2 & 7 & 14 \end{vmatrix} \qquad (C_3 \to C_3 + C_2)$$

$$D = \begin{vmatrix} \sin 3\theta & -1 & 0 \\ \cos 2\theta - 1 & 0.5 & 0 \\ 2 & 7 & 14 \end{vmatrix} \qquad (R_2 \to R_2 - \tfrac{R_3}{2})$$

$D = 14\left(\frac{\sin 3\theta}{2} + \cos 2\theta - 1\right)$

$D = 0$

$\sin 3\theta + 2\cos 2\theta - 2 = 0$

$\Rightarrow$ $3\sin\theta - 4\sin^3\theta = 4\sin^2\theta$

$\Rightarrow$ $(\sin\theta)(4\sin^2\theta + 4\sin\theta - 3) = 0$

$\Rightarrow$ $(\sin\theta)(2\sin\theta - 1)(2\sin\theta + 3) = 0$

$\sin\theta = 0 \sin\theta = \frac{1}{2} \sin\theta = -\frac{5}{2}$

$\sin\theta = 0$

$\theta = 0, \pi, 2\pi$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\sin\theta = -\frac{3}{2}$

No solution

$\theta = 0, \frac{\pi}{6}, \frac{5\pi}{4}, \pi, 2\pi$