ADJOINT MATRIX & COFACTOR MATRIX

Consider $A = [a_{ij}]_n$ as a square matrix. The matrix obtained by replacing each element of A with its corresponding cofactor is termed the cofactor matrix of A, denoted as cofactor A. The transpose of the cofactor matrix of A is referred to as the adjoint of A, denoted as adj A.

i.e. if $A = [a_{ij}]_n$

then, cofactor $A = [c_{ij}]_n$ when c_{ij} is the cofactor of a_{ij} " for each pair (i, j).

$$\text{Adj } A = [d_{ij}]_n$$

Where, $d_{ij} = c_{ji}$ " i & j.

Properties of cofactor A and adj A

(a) $A \,.\text{adj } A = |A|\, I_n = (\text{adj } A)\, A$ where $A = [a_{ij}]_n$.

(b) $|\text{adj } A| = |A|^{n-1}$, where n is order of A.

In particular, for 3×3 matrix, $|\text{adj } A| = |A|^2$

(c) If A is a symmetric matrix, then adj A are also symmetric matrices.

(d) If A is singular, then adj A is also singular.

Ex Find $B \times (\text{adj } A)$, If $A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & 2 \\ 0 & -3 & 1 \end{bmatrix}$ and $B = [-1 \;\; 2 \;\; -1]$ is given.

Sol. The cofactors are

$$c_{11} = +(4+6) = 10$$
$$c_{12} = -(-1 - 0) = 1$$
$$c_{13} = +(3-0) = 3$$
$$c_{21} = -(-1+9) = -8$$
$$c_{22} = +(2-0) = 2$$
$$c_{23} = -(-6+0) = 6$$
$$c_{31} = +(-2-12) = -14$$
$$c_{32} = -(4+3) = -7$$
$$c_{33} = +(8-1) = 7$$

$$\text{adj}A = \begin{bmatrix} 10 & 1 & 3 \\ -8 & 2 & 6 \\ -14 & -7 & 7 \end{bmatrix}^T = \begin{bmatrix} 10 & -8 & -14 \\ 1 & 2 & -7 \\ 3 & 6 & 7 \end{bmatrix}$$

Then $B \cdot (\text{adj}A) = [-1 \quad 2 - 1] \begin{bmatrix} 10 & -8 & -14 \\ 1 & 2 & -7 \\ 3 & 6 & 7 \end{bmatrix}$

$$= [-10 + 2 - 3 \quad 8 + 4 - 6 \quad 14 - 14 - 7]$$
$$= [-11 \quad 6 \quad -7]$$

Inverse of a Matrix (Reciprocal Matrix)

Consider A as a non-singular matrix. In this case, the matrix $\frac{1}{|A|}$ adj A serves as the Multiplicative inverse of A, commonly referred to as the inverse of A, and is Symbolized as A^{-1}.

We have $A\,(\text{adj } A) = |A|\, I_n = (\text{adj } A)\, A$

$\Rightarrow \quad A\left(\frac{1}{|A|}\text{adj}A\right) = I_n = \left(\frac{1}{|A|}\text{adj}A\right)A$ for A is non-singular

$\Rightarrow \quad A^{-1} = \frac{1}{|A|}$ adj A

REMEMBER

(i) The essential condition for the existence of the inverse of A is that A is non-singular.

(ii) A^{-1} is always non-singular.

(iii) If $A = \text{dia}(a_{11}, a_{22}, \ldots, a_{nn})$

Where $a_{ii} \neq 0$ " i,

Then $A^{-1} = \text{diag}(a_{11}^{-1}, a_{22}^{-1}, \ldots, a_{nn}^{-1})$.

(iv) $(A^{-1})' = (A')^{-1}$ for any non-singular matrix A. Also adj (A′) = (adj A)′.

(v) $(A^{-1})^{-1} = A$ if A is non-singular.

(vi) Let k be a non-zero scalar & A be a non-singular matrix.

Then $(kA)^{-1} = \frac{1}{K}A^{-1}$.

(vii) $|A^{-1}| = \frac{1}{|A|}$ for $|A| \neq 0$

(viii) Let A be a non-singular matrix.

Then AB = AC

$\Rightarrow$ B = C

& BA = CA

$\Rightarrow$ B= C.

(ix) A is non-singular and symmetric

$\Rightarrow$ A^{-1} is symmetric.

(x) $(AB)^{-1} = B^{-1}A^{-1}$ if A and B are non- singular.

(xi) In general AB = 0 does not imply A = 0 or B = 0. But if A is non-singular and AB = 0, then B = 0.

Similarly B is non-singular and AB = 0

$\Rightarrow$ A = 0.

Therefore, AB = 0

$\Rightarrow$ either both are singular or one of them is 0.

Ex. If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ then verify the equation A adj A = | A | I. additionally, determine the find A^{-1}

Sol. We have $|A| = 1(16 - 9) - 3(4 - 3) + 3(3 - 4) = 1 \neq 0$

Now $A_{11} = 7$,

$A_{12} = -1$,

$A_{13} = -1$,

$A_{21} = -3$,

$A_{22} = 1$

$A_{23} = 0$

$A_{31} = -3$,

$A_{32} = 0$,

$A_{33} = 1$

Therefore $\text{adj } A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Now $A(\text{adj } A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|.I$$

Also $A^{-1} \frac{1}{|A|} \text{adj } A = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Ex. Demonstrate that the matrix A = [(2,3), (1,2)] satisfies the equation $A^2 - 4A + I = O$, where I is the 2 × 2 identity matrix, and O is the 2 × 2 zero matrix. Utilizing this equation, determine the inverse of A, denoted as A^{-1}.

Sol. We have $A^2 = A \cdot A$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$$

Hence $A^2 - 4A + I$

$$= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Now $A^2 - 4A + I = 0$

Therefore $AA - 4A = -I$

or $AA(A^{-1}) - 4AA^{-1} = -IA^{-1}$

(Post multiplying by A^{-1} because $|A| \neq 0$)

or $A(AA^{-1}) - 4I = -A^{-1}$

or $AI - 4I = -A^{-1}$

or $A^{-1} = 4I - A$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

Hence $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Ex. calculate the inverse of the matrix, $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

Sol. We have, $|A| = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix} = (14 - 12) = 12 \neq 0.$

So, A^{-1} exists.

The cofactors of the elements of the determinant |A| are expressed as:

$A_{11} = 7,$

$A_{12} = (-4) = 4.$

$A_{21} = -(-3) = 3,$

$A_{22} = 2.$

$$(\text{adj}A) = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

Hence, $$A^{-1} = \frac{1}{|A|} \cdot (\text{adj}A)$$

$$= \frac{1}{2} \cdot \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & \frac{3}{2} \\ 2 & 1 \end{bmatrix}.$$