Chapter 2

Relation And Function

- Introduction
- Relation
- Types of relations
- Equivalence relations
- Function
  - ➢ Definition of Function
  - ➢ Function as a Special Type of Relation
  - ➢ Graph of a function
  - ➢ Domain, Co-Domain, and Range
- Types of Function
  - ➢ One to One function
  - ➢ Many to One Function
  - ➢ Onto Function
  - ➢ One – One and Onto Function
- Different Types of Functions
  - ➢ Other Types of Functions
  - ➢ Identity Function
  - ➢ Constant Function
  - ➢ Polynomial Function
  - ➢ Rational Function
  - ➢ Signum Function
  - ➢ Greatest Integer Function
- Different Types of Functions
  - ➢ Algebra of Real Functions
  - ➢ Addition of Two Real Functions
  - ➢ Subtraction of One Real Function from the Other
  - ➢ Multiplication by a Scalar
  - ➢ Multiplication of Two Real Functions
  - ➢ Division of Two Real Functions

INTRODUCTION

A "relation" represents a connection between sets of information. In this chapter, we will explore relations and functions, which constitute a special type of relation.

RELATIONS

In the expression A R b, it signifies that 'a is R-related to b,' implying that a is related to b under the relation R. If (a, b) belongs to R, the ordered pair (a, b) is denoted in such a way that the interchangeability of a and b is not permissible, given that a belongs to set A and b belongs to set B.

Ordered Pair:

It refers to a pair of objects arranged in a specific order, with two members separated by a comma and enclosed in parentheses. In the ordered pair (a, b), a is designated as the first component, the first element, or the first coordinate, while b is recognized as the second.

It's essential to note that ordered pairs (a, b) and (b, a) are distinct from each other.

$$(a, b) = (c, d) \text{ iff } a = c \text{ and } b = d$$

$$(1,3) = (1,3); (1,3) \neq (1,2) \neq (2,3) \neq (3,1)$$

CARTESIAN PRODUCT

Cartesian product of two sets A × B :

For any two non-empty sets A and B

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

This represents a set of all ordered pairs where the first element is from set A and the second element is from set B. The notation A×B is pronounced as 'A cross B' or the 'Product set of A and B.'

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}$$

$$(a, b) \in A \times B \Leftrightarrow a \in A \text{ and } b \in B.$$

$$B \times A = \{(b, a) : b \in B \wedge a \in A\}$$

$$A \times B \neq B \times A$$

$$n(A \times B) = n(A)n(B) \text{ and } n(P(A \times B)) = 2^{n(A)n(B)}$$

$$A = \phi \text{ and } B = \phi \Leftrightarrow A \times B = \phi$$

Cartesian product of n non-empty sets $A_1, A_2 \ldots \ldots A_n$ Sets is a set of all n tuples $(a_1, a_2 \ldots \ldots \ldots a_n)$ such that each

$$a_i \in A_i, i = 1,2..n.$$

$$A_1 \times A_2 \times \ldots\ldots.\times A_n = \prod_{i=1}^{n} A_i$$

$A \times A = A^2 : R \times R = R^2$ Is a set encompassing all points situated in the plane.

$R \times R \times R = R^3$ Represents the set of all points in three-dimensional space. If at least one of the sets A and B is infinite, then $A \times B$ is also an infinite set, given that the other set is non-empty.

Ex. Let $A = \{a, b\}, B = \{c, d\}, C = \{e, f\}$

Sol. Then $n(A \times B \times C) = n(A) \cdot n(B) \cdot n(C) = 8$

$$A \times B \times C = \{(a, c, e), (a, c, f), (a, d, e), (a, d, f), (b, c, e), (b, c, f), (b, d, e), (b, d, f)\}$$

Note: Each element of $A \times B$ is an ordered pair or 2–tuple

Each element of $A \times B \times C$ is an ordered triplet or 3–tuple

For $A_1 \times A_2 \times \ldots . . A_n$ each element is an n–tuple

Ex. $A_1 \times A_2 \times A_3 \times A_4 = \{(1,1,1,1), (2,4,8,16), (3,9,27,81) \ldots\ldots\ldots.\}$ Find A_1, A_2, A_3 and A_4.

Sol. Each ordered pair $\{x_1, x_2, x_2, x_1\}$ is of the form $\{x, x^2, x^3, x^4\}$

Hence $x_1 \in A_1 \Rightarrow A_1 = \{x : x \in N\} = \{1,2,3,4, \ldots . .\}$

$$x_2 \in A_2 \Rightarrow A_2 = \{x^2 : x \in N\} = \{1^2, 2^2, 3^2, 4^2, \ldots . .\}$$

$$x_3 \in A_3 \Rightarrow A_3 = \{x^3 : x \in N\} = \{1^3, 2^3, 3^3, 4^3, \ldots . .\}$$

$$x_4 \in A_4 \Rightarrow A_4 = \{x^4 : x \in N\} = \{1^4, 2^4, 3^4, 4^4, \ldots .\}$$

Key Results on Cartesian product

If A, B, C, D are four sets, then.

1. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
2. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
3. $A \times (B - C) = (A \times B) - (A \times C)$
4. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
5. $A \subseteq B$, then $(A \times C) \subseteq (B \times C)$
6. $A \subset B$, then $(A \times B) \cap (B \times A) = A^2$
7. $A \subset B$, then $A \times A \subset (A \times B) \cap (B \times A)$
8. If $A \subset B$ and $C \subset D$, then $(A \times C) \subset (B \times D)$
9. $A \times B = B \times A \Leftrightarrow A = B$
10. $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$
11. $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$

If A and B are two non-empty sets having n elements in common then $(A \times B)$ and $(B \times A)$ have n^2 elements in common.

Ex. If $n(A) = 7, n(B) = 8$ and $n(A \cap B) = 4$, then match the following columns.

1.	$n(A \cup B)$	(A)	56
2.	$n(A \times B)$	(B)	16
3.	$n((B \times A) \times A)$	(C)	392
4.	$n((A \times B) \cap (B \times A))$	(D)	96
5.	$n((A \times B) \cup (B \times A))$	(E)	11

Sol. 1. – (E) 2. – (A) 3. – (C) 4. – (B) 5. – (D)

1. $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 7 + 8 - 4 = 11$
2. $n(A \times B) = n(A)n(B) = 7 \times 8 = 56 = n(B \times A)$
3. $n((B \times A) \times A) = n(B \times A) \cdot n(A) = 56 \times 7 = 392$
4. $n((A \times B) \cap (B \times A)) = (n((A \cap B))^2 = 4^2 = 16$
5. $n\big((A \times B) \cup (B \times A)\big) = n(A \times B) + n(B \times A) - n(A \times B) \cap (B \times A)$

$$= 56 + 56 - 16 = 96$$

Ex. If $A = \{2, 4\}$ and $B = \{3, 4, 5\}$, then the expression $(A \cap B) \times (A \cup B)$is.

1. $\{(2, 2), (3, 4), (4, 2), (5, 4)\}$
2. $\{(2, 3), (4, 3), (4, 5)\}$
3. $\{(2, 4), (3, 4), (4, 4), (4, 5)\}$
4. $\{(4, 2), (4, 3), (4, 4), (4, 5)\}$

Sol. Answer 4.

$$A \cap B = \{4\} \text{ and } A \cup B = \{2,3,4,5\}$$

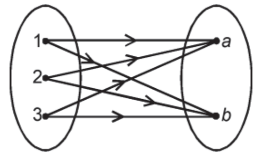
$$(A \cap B) \times (A \cup B) = \{(4,2), (4,3), (4,4), (4,5)\}$$

Pictorial Representation of Cartesian product of Two Sets:

Arrow diagram:

Let $A = \{1, 2, 3\}$ $B = (a, b)$

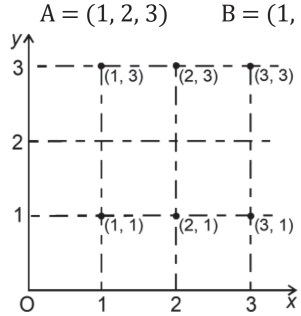
$A \times B$



Lattice-Diagram:

Axis OX represents elements of A and perpendicular axis OY represents set B. Each -dot represents an ordered pair of $A \times B$.

Let $A = (1, 2, 3)$ $B = (1, 3)$



RELATIONS

In the case of any two non-empty sets A and B, each subset of A×B establishes a relation from A to B, and every relation from A to B is a subset of A×B.

$$aRb \subseteq A \times B \forall R$$

If $(a, b) \in R$, then a R b is read as 'a is related to b '

If $(a, b) \notin R$, then a $\not R$ is read as 'a is not related to b '

Ex. Consider $A = \{1, 3, 4, 5, 7\}$ and $B = \{1, 4, 6, 7\}$ with R being the relation 'is one less than' from A to B, then list the domain, range and co-domain sets of R.

Sol.
$$R = \{(3, 4), (5, 6)\}$$
$$\text{Dom}(R) = \{3, 5\}$$
$$\text{Range of } R = \{4, 6\}$$

Co-domain of $R = B = \{1, 4, 6, 7\}$

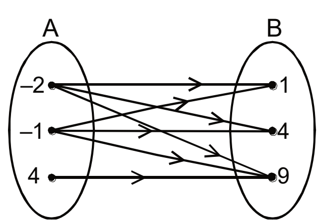
Clearly Range of R $\subseteq$ co-domain of R.

Representation of a Relation

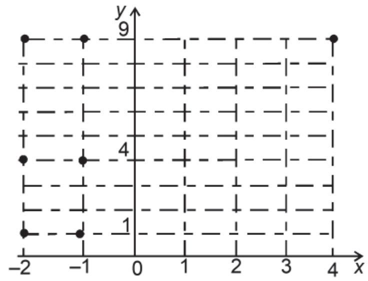
Consider $A = \{-2, -1, 4\}$ $B = \{1, 4, 9\}$

Define a relation from A to B, denoted as a R b, where a is less than b. This can be expressed in the following manners:

1. **Roster form:** $R = \{(-2,1), (-2,4), (-2,9), (-1,1), (-1,4), (-1,9), (4,9)\}$
2. **Set builder notation:** $R = \{(a, b): a \in A \text{ and } b \in B, a \text{ is less than } b\}$
3. **Arrow - diagram:**



4. **Lattice-diagram:**



5. **Tabular form:**

R	1	4	9
-2	1	1	1
-1	1	1	1
4	0	0	1

Note: If (a, b) ∈ R, we record '1' in the row corresponding to a and column corresponding to b and if (a, b) ∉ R, we record '0' in the respective row and column.

Ex. Let $A = \{1,2,3,4\}, B = \{1,2,3 \dots \dots 10\}$

$R_1 = \{(1,4), (2,5), (3,6), (4,7)\}$

$R_2 = \{(2,5), (3,6), (4,7), (5,8)\}$

$R_3 = \{(1,1), (2,4), (3,9)\}$

From, R_1, R_2, R_3, select those that represent a relation from A to B, and represent the relations in set-builder form.

Sol. $R_1 \subseteq A \times B; R_2 \not\subseteq\subseteq A \times B, R_2 \subseteq A \times B$

R_2 is not a relation as $(5,8) \notin A \times B$

Hence R_2 is not a relation as $(5, 8) \notin A \times B$

$R_1 = \{(a, b): a \in A, b \in B \text{ and } a + 3 = b\}$

$R_3 = \{(a, b): a \in A, b \in B \text{ and } a^2 = b\}$

Any subset of $A \times A$ is a relation on A.

If $n(A) = p$ and $n(B) = q$ then $n(A \times B) = pq$

Total number of subsets of $(A \times B) = 2^{pq}$

Hence 2^{pq} different relations are possible from A to B.