SYMMETRIC & SKEW-SYMMETRIC MATRIX

A square matrix A is characterized as symmetric if its transpose A′ = A

i.e. Let $A = [a_{ij}]_n$. A is symmetric iff $a_{ij} = a_{ji}$ " i & j.

A square matrix A is said to be skew-symmetric if A′ = – A

i.e. Let $A = [a_{ij}]n$. A is skew-symmetric if $a_{ij} = - a_{ji}$ " i & j.

For example. $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix.

$B = \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Remember

(i) In a skew-symmetric matrix, all the diagonal elements are zero, as denoted by the condition

$(\because aii = - aii \quad \Rightarrow \quad aii = 0)$

(ii) For any square matrix A, A + A’ is symmetric and A – A’ is skew-symmetric.

(iii) Every square matrix can be distinctly represented as a sum of two square matrices, one being symmetric and the other is skew-symmetric.

A = B + C,

Where, $B = \frac{1}{2}(A + A')$ & $C = \frac{1}{2}(A - A')$.

Ex. If A is both symmetric and skew symmetric matrix, then A can be expressed as.

Sol. Let $A = [a_{ij}]$ Since A is skew symmetric $a_{ij} = -a_{ji}$

For $i = j, a_{ii} = -a_{ii}$

$\Rightarrow$ $a_{ii} = 0$

For $i \neq j, aij = -aji$ $[\because$ A is skew symmetric$]$

& $aij = aji$ $[\because$ A is symmetric$]$

$aij = 0$ for all $i \neq j$

So, aij = 0 for all 'i' and 'j' i.e. A is null matrix.

Ex. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a matrix given by, determine the values of θ satisfy the equation $A^T + A = I_2$.

Sol. We have, $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Now, $A^T + A = I_2$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2\cos\theta & 0 \\ 0 & 2\cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$2\cos\theta = 1$$

$$2\cos\theta = \frac{1}{2}$$

$$\cos\theta = \cos\frac{\pi}{3}$$

$$\theta = 2n\pi \pm \frac{\pi}{3}, n \in Z$$

Ex. Represent the matrix $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ as the combination of a symmetric matrix and a skew symmetric matrix.

Sol. $B' = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$

$$P = \frac{1}{2}(B + B') = \frac{1}{2}\begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & 3 & 1 \\ -\frac{3}{2} & 1 & -3 \end{bmatrix}$$

$$P' = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} = P$$

Thus, $P = \frac{1}{2}(B + B')$ is a symmetric matrix

Also, Let $Q = \frac{1}{2}(B - B') = \frac{1}{2}\begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$$

$$Q' = \begin{bmatrix} 0 & \frac{1}{2} & \frac{5}{3} \\ -\frac{1}{2} & 0 & -3 \\ -\frac{5}{2} & 3 & 0 \end{bmatrix} = -Q$$

Thus, $Q = \frac{1}{2}(B - B')$is a skew symmetric matrix

Now, $P + Q = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = B$$

Thus, B is represented as the sum of a symmetric and a skew symmetric matrix.

Orthogonal Matrix

A square matrix is considered an orthogonal matrix if $A A^T = I$

Note

(i) The determinant value of orthogonal matrix is either 1 or –1.

(ii) Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

$$A^T = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} a_1^2 + a_2^2 + a_3^2 & a_1b_1 + a_2b_2 + a_3b_3 & a_1c_1 + a_2c_2 + a_3c_3 \\ b_1a_1 + b_2a_2 + b_3a_3 & b_1^2 + b_2^2 + b_3^2 & b_1c_1 + b_2c_2 + b_3c_3 \\ c_1a_1 + c_2a_2 + c_3a_3 & c_1b_1 + c_2b_2 + c_3b_3 & c_1^2 + c_2^2 + c_3^2 \end{bmatrix}$$

If $AA^T = I,$

Then $\sum_{i=1}^{3} a_i^2 = \sum_{i=1}^{3} b_i^2 = \sum_{i=1}^{3} c_i^2 = 1$

And $\sum_{i=1}^{3} a_ib_i = \sum_{i=1}^{3} b_ic_i = \sum_{i=1}^{3} c_ia_i = 0$

Ex. Find the values of a, b, g, for $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ which is considered orthogonal.

Sol.
$$A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$$

But given A is orthogonal

$$AA^T = I$$

$$\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}\begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4\beta^2+\gamma^2 & 2\beta^2-\gamma^2 & -2\beta^2+\gamma^2 \\ 2\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 \\ -2\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Setting the corresponding elements equal, we get:

$4b^2 + g^2 = 1$ ....(i)

$2b^2 - g^2 = 0$ ....(ii)

$a^2 + b^2 + g^2 = 1$ ....(iii)

From (i) and (ii) $6\beta^2 = 1,\ \beta^2 = \frac{1}{6}$

And $\gamma^2 = \frac{1}{3}$

From (iii) $\alpha^2 = 1 - \beta^2 - \gamma^2$

$= 1 - \frac{1}{6} - \frac{1}{3} = \frac{1}{2}$

Hence $\alpha = \pm\frac{1}{\sqrt{2}}, \beta = \pm\frac{1}{\sqrt{6}}$ and $\gamma = \pm\frac{1}{\sqrt{3}}$

Properties of Symmetric and Skew Symmetric Matrices

(i) Any square matrix has a special representation that is the sum of two unique matrices: a symmetric matrix and a skew-symmetric matrix. To illustrate, consider matrix A, which can be expressed as follows:

$$A = \frac{A+A'}{2} + \frac{A-A'}{2}$$

$$\text{Let } P = \frac{A+A'}{2} \text{ and } Q = \frac{A-A'}{2}$$

$$\text{Here } P' = \left(\frac{A+A'}{2}\right)' = \frac{(A')'+A'}{2} = \frac{A+A'}{2} = \frac{A'+A}{2} = P$$

Hence P is symmetric Similarly

$$Q = \frac{A-A'}{2} \Rightarrow Q' = \frac{A'-(A')'}{2} = \frac{A'-A}{2} = -Q$$

(ii) If A is symmetric then A^n, $n \in N$ will be symmetric.

(iii) A is skew symmetric then A^n, $n \in N$ will be symmetric if n is even and An will be skew symmetric n is odd.

(iv) Null matrix is both symmetric and skew symmetric.

(v) A and B both are symmetric then AB + BA is symmetric and AB – BA is skew symmetric.