ELEMENTARY OPERATION (TRANSFORMATION) OF A MATRIX

Elementary Row Operations

Elementary row operations have diverse applications, ranging from simplifying the solution of systems of equations to determining matrix rank. Additionally, these elementary row transformations serve as a method for finding the inverse of a matrix A without relying on explicit formulas $A^{-1} = \frac{(\text{adj}A)}{(\text{det}A)}$.

Let's explore the application of inverse row operations for achieving various tasks in a straightforward manner.

What are Elementary Row Operations?

When employing elementary row operations, we commonly denote the first row as R_1, the second row as R_2, and so forth. There are essentially three types of elementary row operations:

- Interchanging two rows.
  For instance, exchanging the first and second rows is indicated as $R_1 \leftrightarrow R_2$.
- Multiplying/dividing a row by a scalar.
  For instance, when the first row (comprising all elements of the first row) is multiplied by a scalar, such as 3, it is represented as $R_1 \to 3R_1$.
- Multiplying/dividing a row by a scalar and adding/subtracting the result to the corresponding elements of another row.
  For example, if the first row is multiplied by 3, and added to the second row, it can be written as either $R_1 \to 3R_1 + R_2$ (or) $R_2 \to R_2 + 3R_1$.

It is customary to express the identical row on the left side of the arrow and at the initial instance of the right side of the arrow.

Elementary Row Operations to Solve a System of Equations

To solve a system of equations presented in matrix form $AX = B$, one can construct the augmented matrix [A B] and perform elementary row operations to transform it into echelon form, preferably the upper triangular form. The application of the three aforementioned row operations does not modify the augmented matrix because:

- Interchanging two rows is equivalent to swapping two equations in the system, and this doesn't impact the solution.
- Multiplying a row by a scalar does not change the augmented matrix, as one can always multiply both sides of an equation by a scalar without altering the equation.
- Multiplying one row by a scalar and adding it to another row is akin to multiplying an equation by a scalar and adding it to another equation, a common practice in solving a system of equations.

This method of employing row operations to solve a system is referred to as Gauss elimination. An example illustrating the application of row transformations to solve a system of equations can be found in the "Elementary Row Operations Examples" section below.

Elementary Row Operations to Find Inverse of a Matrix

To determine the inverse of a square matrix A, the conventional approach involves employing the formula:

$$A^{-1} = \frac{(\text{adj}A)}{(\text{det}A)}$$

However, this method is cumbersome, involving multiple steps such as computing the cofactor matrix, adjoint matrix, determinant, and more. To simplify this process, elementary row operations can be utilized. The following steps outline this approach:

- Begin with the augmented matrix [A | I], where I is the identity matrix matching the order of A.
- Apply row operations to transform the left side matrix A into I.
- The resulting matrix on the right side (replacing the original matrix I) is equivalent to A-1.

Elementary Row operations to find Determinant

Typically, the determinant of a matrix is computed by summing the products of the elements in a row or column and their corresponding cofactors. However, this process becomes challenging when the matrix contains expression-based terms. The application of elementary row operations provides a more straightforward method for determining the determinant. Nevertheless, certain row operations impact the determinant as follows:

- Interchanging two rows of a determinant leads to a change in sign.
- Multiplying a row by a scalar result in the determinant being multiplied by the same scalar.
- Multiplying a row by a scalar and adding the result to another row does not modify the determinant.

Elementary Row Operations to Find Rank of a Matrix

The matrix rank is defined as the count of linearly independent rows (or columns) within it. Determining the rank of a matrix involves applying elementary row operations in two distinct approaches:

- Transform the matrix into Echelon form and count the non-zero rows to ascertain its rank.
- Convert the matrix to normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where is the identity matrix of order r. In this case, the rank of the matrix is equal to r.

Ex. Execute the given elementary row operations on the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 0 \\ -4 & 0 & 2 \end{bmatrix}$

(a) $R_1 \leftrightarrow R_2$ (b) $R_2 \to R_2\text{-}5\,R_1$

Sol. (a) Swapping (or interchanging) the first two rows is denoted by $R_1 \leftrightarrow R_2$

The outcome is: $\begin{bmatrix} 3 & 2 & 0 \\ 1 & 2 & -1 \\ -4 & 0 & 2 \end{bmatrix}$

(b) We are given R_1 (the first row) = [1 2 -1].

Then $-5R_1$ = [-5 -10 5].

$R_2 - 5R_1$ = [3 2 0] + [-5 -10 5] = [-2 -8 5]

The operation $R_2 \to R_2 - 5R_1$ implies replacing R_2 with the row obtained by performing $R_2 - 5R_1$.

The resulting matrix is: $\begin{bmatrix} 1 & 2 & -1 \\ -2 & -8 & 5 \\ -4 & 0 & 2 \end{bmatrix}$

Ex. Apply elementary row transformations to solve the given system of equations:

$2x - y + 3z = 8$,

$-x + 2y + z = 4$,

$3x + y - 4z = 0$.

Sol. The system is represented by the matrix equation:

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

The matrix obtained by appending the augmented part is:

$$[AB] = \begin{bmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{bmatrix}$$

In this step, we aim to make the last two elements of the first column (-1 and 3) zero. This involves the use of R_1.

Apply $R_2 \to 2R_2 + R_1$ and $R_3 \to 2R_3 - 3R_1$,

we get: $\begin{bmatrix} 2 & -1 & 3 & 8 \\ 0 & 3 & 5 & 16 \\ 0 & 5 & -17 & -24 \end{bmatrix}$

We transform the last element of the second column (5) to zero, employing R_2 in this procedure.

Now, apply $R_3 \to 3R_3 - 5R_2$,

$$\begin{bmatrix} 2 & -1 & 3 & 8 \\ 0 & 3 & 5 & 16 \\ 0 & 0 & -76 & -152 \end{bmatrix}$$

Now, we express the matrix obtained as a set of equations:

$2x - y + 3z = 8$ ... (1)

$3y + 5z = 16$ ... (2)

$$-76z = -152 \quad \text{... (3)}$$

From (3), $z = \frac{-152}{-76} = 2.$

From (2), $3y + 5(2) = 16$

$\Rightarrow$ $3y = 6$

$\Rightarrow$ $y = 2.$

From (1), $2x - 2 + 3(2) = 8$

$\Rightarrow$ $2x = 4$

$\Rightarrow$ $x = 2.$

Answer: (x, y, z) = (2, 2, 2).

Ex. Determine the inverse of the matrix $A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ through the application of elementary row operations.

Sol. Begin by considering the augmented matrix composed of matrix A and the identity matrix I.

$$[A \mid I] = \begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 3 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \end{bmatrix}$$

Proceed to transform the right-side matrix into the identity matrix.

Apply $R_3 \rightarrow 2R_3 + R_1$,

$$= \begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 3 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 0 & 2 & 0 & 5 & 3 \end{bmatrix}$$

Now apply $R_1 \rightarrow R_1 + R_2$ and $R_3 \rightarrow R_3 + 5R_2$

$$\begin{bmatrix} 1 & 1 & 0 & -2 & 0 & 4 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 5 & 2 & 0 & 0 & 8 \end{bmatrix}$$

Apply $R_1 \rightarrow 2R_1 - R_3$ and $R_2 \rightarrow 8R_2 - R_3$,

$$\begin{bmatrix} 1 & -3 & -2 & -4 & 0 & 0 \\ -1 & 3 & -2 & 0 & -8 & 0 \\ 1 & 5 & 2 & 0 & 0 & 8 \end{bmatrix}$$

Now, divide R_1 by -4, R_2 by -8, and R_3 by 8:

$$\begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & \frac{2}{4} & 1 & 0 & 0 \\ \frac{1}{8} & -\frac{3}{8} & \frac{2}{8} & 0 & 1 & 0 \\ \frac{1}{8} & \frac{5}{8} & \frac{2}{8} & 0 & 1 & 0 \end{bmatrix}$$

Now, the right-side matrix has been transformed into the identity matrix (I). Consequently, the left-side matrix is A-1.

$$A^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & \frac{2}{4} \\ \frac{1}{8} & -\frac{3}{8} & \frac{2}{8} \\ \frac{1}{8} & \frac{5}{8} & \frac{2}{8} \end{bmatrix}$$