ALGEBRA OF MATRICES

i. Equality of Matrices

Two matrices are deemed comparable when they share the same number of rows and columns. Matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are considered equal if they meet two conditions:
(i) They are comparable.
(ii) Each corresponding element aij in matrix A is equal to the corresponding element b_{ij} in matrix B for every pair of row index i and column index j.

According to this definition, the concept of matrix equality establishes an equivalence relation within the set of matrices.

ii. Addition of matrices : $A + B = [a_{ij} + b_{ij}]$ where A & B are of the same order.

(A) Addition of matrices is commutative
i.e. $A + B = B + A$

(B) Matrix addition is associative
$(A + B) + C = A + (B + C)$

(C) Additive inverse
If $A + B = O = B + A$, then B is referred to as the additive inverse of A.

(D) Existence of additive identity
Consider $A = [a_{ij}]$ as an $m \times n$ matrix and O as an $m \times n$ zero matrix,
In this case $A + O = O + A = A$. Simply put, O serves as the additive identity for matrix addition.

(E) Cancellation laws in Matrix Addition
The cancellation laws are applicable in the context of matrix addition. If A,B,C are matrices of the same order,

Then $A + B = A + C$
$\Rightarrow$ $B = C$ (left cancellation law)
and $B + A = C + A$
$\Rightarrow$ $B = C$ (right-cancellation law)

Note: The zero matrix functions similarly in matrix addition as the number zero does in numerical addition.

Ex. Given $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix}$ such that $A + B - D = 0$ which is zero matrix. Find the matrixD.

Sol. Let $D = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

$$A + B - D = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

$$\begin{bmatrix} 1-1-a & 3-2-b \\ 3+0-c & 2+5-d \\ 2+3-e & 5+1-f \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$-a = 0$
$\Rightarrow$ $a = 0,$
$1 - b = 0$
$\Rightarrow$ $b = 1,$
$3 - c = 0$
$\Rightarrow$ $c = 3,$
$7 - d = 0$
$\Rightarrow$ $d = 7,$
$5 - e = 0$
$\Rightarrow$ $e = 5,$
$6 - f = 0$
$\Rightarrow$ $f = 6$

$$D = \begin{bmatrix} 0 & 1 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$$

iii. Multiplication of matrix by scalar

Let l be a scalar (either a real or complex number) and A = [aij]m × n be a matrix.
The product lA is defined as l_A = [bij]m × n where bij = laij for each i and j.

For example: $A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & -1 & -2 \end{bmatrix}$

and $-3A \equiv (-3)A = \begin{bmatrix} -6 & 3 & -9 & -15 \\ 0 & -6 & -3 & 9 \\ 0 & 0 & 3 & 6 \end{bmatrix}$

Note: If A is a scalar matrix, then A = lI, where l is a diagonal entry of A

Multiplication Of Matrices

Consider matrices A and B be such that the number of columns of A is same as the number of rows of B.

i.e., $A = [a_{ij}]_{m \times p}$ & $B = [b_{ij}]_{p \times n}$.

In this case, $AB = [c_{ij}]_{m \times n}$ where $c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$, is the dot product of i^{th} row vector of A and j^{th} column vector of B.

For example $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 3 & 4 & 9 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix}$

i. The product AB is valid when the number of columns in matrix A equal the number of rows in matrix B. In this context, A is referred to as premultiplier and B is the post multiplier. Both AB and BA are defined.
ii. In general AB ≠ BA, even when both the products are defined.
iii. A (BC) = (AB) C, whenever it is defined.

Properties Of Matrix Multiplication

Examine the collection of all matrices with order 'n'. Represent the set of all square matrices of order 'n' as M_n (F) where F represents the field, which can be Q, R or C.

(A) A, B ∈ Mn (F) ⇒ AB ∈ Mn (F)
(B) In general AB ≠ BA
(C) (AB) C = A(BC)
(D) In, the identity matrix of order n, is the multiplicative identity.
AIn = A = In A " A ∈ Mn (F)
(E) For every non-singular matrix A (i.e., |A| ≠ 0) of Mn (F) there exist a unique (particular) matrix B belonging to Mn (F) such that AB = In = BA. In this case, we describe A and B as multiplicative inverse of each other. Using notations, we express this relationship B = A–1 or A = B–1.
(F) If l is a scalar (lA) B = l(AB) = A(lB).
(G) A(B + C) = AB + AC " A, B, C ∈ Mn (F)
(H) (A + B) C = AC + BC " A, B, C ∈ Mn (F).
(i) Consider a matrix A = [aij]m × n. Then AIn = A & Im A = A, where In & Im are identity matrices of order n & m respectively.
(ii) For a square matrix A, A_2 represents the product AA, A_3 represents AAA etc.

Ex. If $[1 \quad x \quad 1]\begin{bmatrix}1 & 3 & 2\\ 2 & 5 & 1\\ 15 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\ 2\\ x\end{bmatrix} = 0$ Determine the value of x.

Sol. We have $[1 \; x \; 1]_{1\times 3}\begin{bmatrix}1 & 3 & 2\\ 2 & 5 & 1\\ 15 & 3 & 2\end{bmatrix}_{3\times 3}\begin{bmatrix}1\\ 2\\ x\end{bmatrix}_{3\times 1} = 0$

$\Rightarrow \quad [1 + 2x + 15 \quad 3 + 5x + 3 \quad 2 + x + 2]\begin{bmatrix}1\\ 2\\ x\end{bmatrix} = 0$

$\Rightarrow \quad [16 + 2x \quad 6 + 5x \quad 4 + x]\begin{bmatrix}1\\ 2\\ x\end{bmatrix} = 0$

$\Rightarrow \quad [(16 + 2x) . 1 + (6 + 5x) . 2 + (4 + x) . x] = 0$

$\Rightarrow \quad (16 + 2x) + (12 + 10x) + (4x + x^2) = 0$

$\Rightarrow \quad x^2 + 16x + 28 = 0$

$\Rightarrow \quad (x + 14)(x + 2) = 0$

$\Rightarrow \quad x + 14 = 0$ or $x + 2 = 0$

$\Rightarrow \quad x = -14$ or $x = -2$

Hence, $x = -14$ or $x = -2$

iv. Negative of a matrix

The negative of a matrix is denoted by $-A$. We define $-A = (-1)A$

e.g., Let $A = \begin{bmatrix}1 & 3\\ -1 & 4\end{bmatrix}$,

then $-A$ is given by $(-1)\begin{bmatrix}1 & 3\\ -1 & 4\end{bmatrix} = \begin{bmatrix}-1 & -3\\ 1 & -4\end{bmatrix}$

v. Difference of matrices

If $A = [a_{ij}]$, $B = [b_{ij}]$ are two matrices of the same order. Say $m \times n$, then difference $A - B$ is defined as a matrix $C = [c_{ij}]$, where $c_{ij} = a_{ij} - b_{ij}$, for all value of i and j.

In other words, $C = A - B = A + (-1)B$ i.e., sum of matrix A and the matrix $-B$

vi. Product of Matrices

Special Square Matrices

(A) Idempotent Matrix:

A square matrix is considered idempotent if $A^2 = A$.

For idempotent matrix observe the following:

(i) $A^n = A \quad n \geq 2, n \in N$.

(ii) The determinant of an idempotent matrix is either 0 or 1

(B) Periodic Matrix :

A square matrix that fulfills the condition $A^{k+1} = A$, where k is a positive integer, is termed a periodic matrix. The period of the matrix is the smallest value of K for which this relationship is satisfied.

Note: that period of an idempotent matrix is 1.

(C) Nilpotent Matrix :

A square matrix of the order 'n' is defined as nilpotent matrix of order 'm', where $m \in N$,

if $A^m = 0$ & $A^{m-1} \neq 0$.

(D) Involutary Matrix :

If $A_2 = I$, the matrix is referred to as an involuntary matrix. In other words, a matrix whose square is equal to the identity matrix is considered involuntary.

Note: The determinant value of involuntary matrix is either 1 or –1.

Ex. Demonstrate that the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent matrix of order 3.

Sol. Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^5 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-27 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$A^3 = O$$

i.e., $Ak = O$

Here $k = 3$

Hence, A is nilpotent of order 3.

Ex. Demonstrate that a square matrix A is involuntary, if $(I - A)(I + A) = O$

Sol. Let A be involuntary

Then, $A^2 = I$

$(I - A)(I + A)$

$= I^2 + IA - AI - A^2$

$= I - A^2 = O$

Conversely, let $(I - A)(I + A) = O$

$\Rightarrow I^2 + IA - AI - A^2 = O$

$\Rightarrow I - A^2 = O$

$\Rightarrow$ A is involuntary

vii. Trace of a Matrix

Let A be a square matrix of order n. Then the sum of the elements of A lying along the principal diagonal is called the trace of A, written as tr(A).

$$trA = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}, \text{ where } A = [a_{ij}]_{n\times n}.$$

e.g., for matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ -2 & -4 & 0 \\ 0 & -6 & 5 \end{bmatrix}$, $trA = 3 - 4 + 5 = 4$

Let A and B be square matrices of order n, then following results hold.

(i) $tr(\lambda A) = \lambda trA$ (ii) $tr(A + B) = trA + trB$ (iii) $tr(AB) = tr(BA)$

viii. Transpose of a Matrix

Let $A = [a_{ij}]_{m \times n}$. Then the $n \times m$ matrix obtained from A by changing its rows into columns and its columns into rows is called the transpose of the matrix A, denoted by A^T or A^t or A'.

For example, the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & 2 \\ 3 & 1 & -1 & 5 \end{bmatrix}_{3\times4}$ has its transpose as $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 0 & -1 \\ 4 & 2 & 5 \end{bmatrix}_{4\times3}$

We also have the following results :

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$, A and B being of same order.
- $(kA)^T = kA^T$, k being any complex number
- $(AB)^T = B^T A^T$, A and B being conformable for multiplication.