ADJOINT OF A SQUARE MATRIX

Consider a square matrix A of order n, represented as $A = [a_{ij}]$. Let C_{ij} be the cofactor of a_{ij} in A. The adjoint of A, denoted as adj A, is defined as the transpose of the cofactor matrix.

Then, $\text{adj } A = [C_{ij}]T$

$$\Rightarrow \quad \text{adj}A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{23} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

Theorem A (adj. A) = (adj. A).A = |A| In.

Proof: $$A(\text{adj}A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$\begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A|\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A.(\text{Adj}.A) = |A|\, I$$

(whatever may be the value only |A| will come out as a common element)

If $|A| \neq 0$ then $\frac{A(\text{adj}.A)}{|A|} = I$ =unit matrix of the same order as that of A

Properties of adjoint matrix

If A be a square matrix of order n, then

(i) $|\text{adj } A| = |A|^{n-1}$

(ii) $\text{adj}(\text{adj } A) = |A|^{n-2} A$, where $|A| \neq 0$

(iii) $\text{adj}(\text{adj}A) = |A|^{(i-1)^2}$, where $|A| \neq 0$

(iv) $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$

(v) $\text{adj}(KA) = K^{n-1}(\text{adj } A)$, K is a scalar

(vi) $\text{adj } A^T = (\text{adj } A)^T$

Method to find Adjoint of a 2 ×2 Square Matrix, Directly

Consider A as a 2 × 2 square matrix. To obtain the adjoint, interchange the diagonal elements and reverse the sign of the off-diagonal elements (remaining elements). For example, if

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

$$\text{adj}A = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$$

Ex. If $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$, then adj(adjA) is equal to -

Sol. $$|A| = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{vmatrix} = 8$$

Now, $$\text{adj}(\text{adj}A) = |A|^{3-2}A$$

$$8\begin{vmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{vmatrix} = 16\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Inverse of a Matrix (Reciprocal Matrix)

A square matrix A is termed invertible (non-singular) if there exists a matrix B such

That $AB = I = BA$

B is referred to as the inverse (reciprocal) of A and is symbolized as A^{-1}

Thus $A^{-1} = B$

$AB = I = BA$.

We have, $A.(\text{adj } A) = \frac{1}{2} A \frac{1}{2} I_n$

$A^{-1} A (\text{adj } A) = A^{-1} I_n |A|$

$I_n (\text{adj } A) = A^{-1} \frac{1}{2} A \frac{1}{2} In$

$$A^{-1} = \frac{(\text{adj}A)}{|A|}$$

Note: The condition that is both necessary and sufficient for a square matrix A to be invertible is that $\frac{1}{2}A\frac{1}{2} \neq 0$.

Imp. Theorem :

If matrices A and B are invertible and have the same order,

Then $(AB)^{-1} = B^{-1} A^{-1}$.

This is reversal law for inverse.

Remember

(i) If A be an invertible matrix , then A^T is also invertible & $(A^T)^{-1} = (A^{-1})^T$

(ii) If A is invertible,

a) $(A^{-1})^{-1} = A$;

b) $(Ak)^{-1} = (A^{-1})k = A - k, k \in N$

(iii) If A is an Orthogonal Matrix. $AA^T = I = A_TA$

(iv) A square matrix is said to be orthogonal if , $A^{-1} = A^T$.

(v) $|A^{-1}| = \frac{1}{|A|}$

Ex. Demonstrate that if A is a non-singular matrix and it is symmetric, then the inverse of A, denoted as A^{-1}, is also symmetric.

Sol. $A^T = A$ [$\because$ A is a symmetric matrix]

$(A^T)^{-1} = A^{-1}$ [since A is non-singular matrix]

$(A^{-1})T = A^{-1}$ Hence proved

Ex. If $A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $M = AB$, then M^{-1} is equal to-

Sol.
$$M = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix}$$
$$|M| = 6, \text{adj}M = \begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$$
$$M^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

Ex. Demonstrate that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = O$, where I is 2×2 identity matrix and O is 2×2 zero matrix. Utilize this equation to determine the A^{-1}

Sol. We have $A^2 = A \cdot A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

Hence $A^2 - 4A + I$
$$= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Now $A^2 - 4A + I = 0$

Therefore $AA - 4A = -I$

Or $AA(A^{-1}) - 4AA^{-1} = -IA^{-1}$

(Post multiplying by A–1 because $|A| \neq 0$)

or $A(AA^{-1}) - 4I = -A^{-1}$

or $AI - 4I = -A^{-1}$

Or, $A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Hence, $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Matrix Polynomial

If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_nx_0$, then we define a matrix Polynomial.

$$f(a) = a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \dots + a_nI^n.$$

In the context of a given square matrix A, if the function $f(a)$ evaluates to the null matrix, then A is referred to as the zero or root of the polynomial $f(x)$.

System of Equation & Criterion For Consistency Gauss -Jordan Method

$$x + y + z = 6$$
$$x - y + z = 2$$
$$2x + y - z = 1$$
$$\begin{bmatrix} x + y + z \\ x - y + z \\ 2x + y - z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$
$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$X = A^{-1}B = \frac{(\text{adj}\cdot A)\cdot B}{|A|}.$$