RELATING $F^{-1}(X)$ WITH $F^{-1}(-X)$ AND $F^{-1}\left(\frac{1}{x}\right)$

Property 1: "–x"

The graphs of $\sin^{-1}x$, $\tan^{-1}x$, $\text{cosec}^{-1}x$ are symmetric about origin.

Therefore, we obtain the following relations:

$$\sin^{-1}(-x) = -\sin^{-1}x$$
$$\tan^{-1}(-x) = -\tan^{-1}x$$
$$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x.$$

Similarly, the graphs of $\cos^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$ are symmetric about the point $(0, \frac{\pi}{2})$.

From this, symmetry, we derive the relationships:

$$\text{Cos}^{-1}(-x) = \pi - \cos^{-1}x$$
$$\text{Sec}^{-1}(-x) = \pi - \sec^{-1}x$$
$$\text{Cot}^{-1}(-x) = \pi - \cot^{-1}x.$$

Property 2: "$\frac{1}{x}$"

(i) $\text{cosec}^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$, $|x| \geq 1$

(ii) $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$, $|x| \geq 1$

(iii) $\cos^{-1}x = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right) & x > 0 \\ \pi + \tan^{-1}\left(\frac{1}{x}\right) & x < 0 \end{cases}$

Remark :

The function $\sin(\sin^{-1}x)$, $\cos(\cos^{-1}x)$, ... $\cot(\cot^{-1}x)$ are aperiodic (non periodic) where as $\sin^{-1}(\sin x)$, ..., $\cot^{-1}(\cot x)$ are periodic functions.

Ex.: Evalute the value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$.

Sol:

$$\tan^{-1}(\tan x) = x$$

If $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$$
$$= \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{4}\right)\right)$$
$$= \tan^{-1}\left(-\tan\frac{\pi}{4}\right)$$
$$\tan^{-1}\left(-\tan\frac{\pi}{4}\right)$$
$$= -\tan^{-1}\left(\tan\frac{\pi}{4}\right)$$
$$\therefore \quad -\tan^{-1}\left(\tan\frac{\pi}{4}\right) = -\frac{\pi}{4}$$

Ex.: Solve the value of $\sin^{-1}(\sin 7)$ and $\sin^{-1}(\sin(-5))$.

Sol: Let $y = \sin^{-1}(\sin 7)$

$$\sin^{-1}(\sin 7) \neq 7$$

as $7 \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\sin^{-1}(\sin 7) = \sin^{-1}\sin(7 - 2\pi)$$
$$\sin^{-1}\sin(7 - 2\pi) = 7 - 2\pi$$
$$(\because 7 - 2\pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right])$$

Similarly, if we have to find $\sin^{-1}(\sin(-5))$

Then, Let $y = \sin^{-1}(\sin - 5)$

$\sin^{-1}(\sin -5) \neq -5$

as $-5 \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin^{-1}(\sin - 5) = -\sin^{-1}\sin 5$

$-\sin^{-1}\sin 5 = -\sin^{-1}\sin(5 - 2\pi)$

$-\sin^{-1}\sin(5 - 2\pi) = -(5 - 2\pi)$ $\left(\because 5 - 2\pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

Ex.: Evaluate the value of $\tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$

Sol: Let $y = \tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$ .... (i)

$\because$ $\cot^{-1}(-x) = \pi - \cot^{-1}x, x \in R$

(i) Can be written as $y = \tan\left\{\pi - \cot^{-1}\left(\frac{2}{3}\right)\right\}$

$y = -\tan\left(cot^{-1}\frac{2}{3}\right)$

$\because$ $\cot^{-1}x = \tan^{-1}\frac{1}{x}$

If $x > 0$

$\therefore$ $y = -\tan\left(\tan^{-1}\frac{3}{2}\right)$

$\Rightarrow$ $y = -\frac{3}{2}$

Ex.: Determine the value of $\tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$

Sol: Let $y = \tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$ .... (i)

Let $\cos^{-1}\frac{\sqrt{5}}{3} = \theta$

$\Rightarrow$ $\theta \in \left(0, \frac{\pi}{2}\right)$

And $\cos\theta = \frac{\sqrt{5}}{3}$

$\therefore$ (i) becomes $y = \tan\left(\frac{\theta}{2}\right)$ .... (ii)

$\because$ $\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{1-\frac{\sqrt{5}}{3}}{1+\frac{\sqrt{5}}{3}}$

$\frac{3-\sqrt{5}}{3+\sqrt{5}} = \frac{(3-\sqrt{5})^2}{4}$

$\tan\frac{\theta}{2} = \pm\left(\frac{3-\sqrt{5}}{2}\right)$ ....(iii)

$\frac{\theta}{2} \in \left(0, \frac{\pi}{4}\right)$

$\Rightarrow$ $\tan\frac{\theta}{2} > 0$

$\therefore$ From (iii),

we get $y = \tan\frac{\theta}{2} = \left(\frac{3-\sqrt{5}}{2}\right)$