INVERSE TRIGONOMETRIC FUNCTIONS

We've observed that trigonometric functions such as sin and cos are periodic, meaning they repeat the same numerical values at an infinite number of points. Consequently, equations like sin x = $\frac{1}{2}$ have numerous solutions, such as

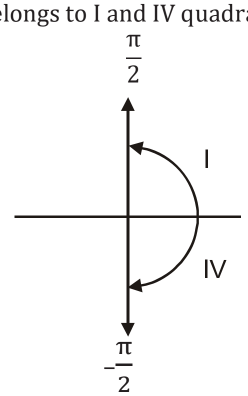
$x = \frac{\pi}{6}, \pi - \frac{\pi}{6}$, and so on. When faced with the question, "What is the angle whose sine is $\frac{1}{2}$?", there isn't a unique answer due to the non-one-to-one nature of the function f: R → R, where f(x) = sin x. This lack of uniqueness prompts us to restrict the domain and codomain of sin x, making the resulting function invertible.

Thus, g: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1,1]$ we define a new function g(x) = sin x, which is one-to-one, onto, and has an inverse (denoted by h = $\sin^{-1}$, read as sine inverse or arc sin) h: $[-1,1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ given by h(y) = x if y = sin x. The function $\sin^{-1}$ serves as the inverse of the sine function under these restricted conditions.

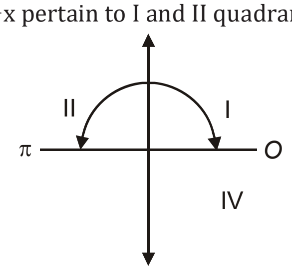
Similarly, we extend this concept to define other inverse trigonometric functions.

Intervals for Inverse Functions

Here, $\sin^{-1}x$, $\operatorname{cosec}^{-1}x$, $\tan^{-1}x$ belongs to I and IV quadrant.



Similarly, $\cos^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$ pertain to I and II quadrant.



1. The I quadrant is a shared region among all the inverse functions.
2. The III quadrant is not utilized in the context of inverse function.
3. The IV quadrant is employed in the clockwise direction i.e., $-\frac{\pi}{2} \le y \le 0$.

Domain, Range and Graphs of Inverse Functions

1. If sin y = x, then $y = \sin^{-1}x$, under certain condition.

	$-1 \le \sin y \le 1$;
But	sin y = x.
∴	$-1 \le x \le 1$
Again,	sin y = –1
⇒	$y = -\frac{\pi}{2}$
And	sin y = 1
⇒	$y = \frac{\pi}{2}$

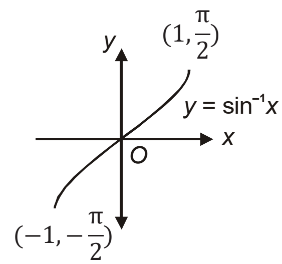
Considering the smallest numerical angles or real numbers, we have

$\therefore \quad -\frac{\pi}{2} \le y \le -\frac{\pi}{2}$

These restrictions on the values of x and y establish the domain and range for the function $y = \sin^{-1}x$.

i.e., Domain: $x \in [-1, 1]$

Range: $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



2. Let cos y = x then $y = \cos^{-1}x$, under certain condition $-1 \le \cos y \le 1$.

$\Rightarrow \quad -1 \le x \le 1$ as cos y = –1

$\Rightarrow \quad y = \pi$, cos y = 1

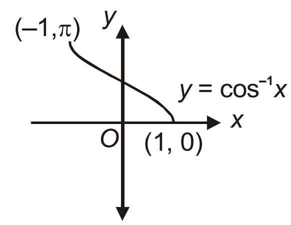
$\Rightarrow \quad y = 0$

$\therefore \quad 0 \le y \le \pi$ {as cos x is a decreasing function in $[0, \pi]$; hence $\cos\pi \le \cos y \le \cos 0$}

These restrictions on the values of x and y provide us the domain and range for the function $y = \cos^{-1}x$.

i.e., Domain: $x \in [-1, 1]$

Range: $y \in [0, \pi]$



3. If $\tan y = x$

Then $y = \tan^{-1}x$, (under certain conditions.)

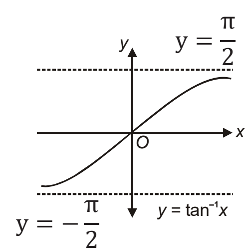
Here, $\tan y \in R$

$\Rightarrow \quad x \in R$; $-\infty < \tan y < \infty$

$\Rightarrow \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$

Thus, Domain $x \in R$

Range $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



4. If $\cot y = x$,

Then $y = \cot^{-1}x$ (under certain conditions)

$\cot y \in R$

$\Rightarrow \quad x \in R$;

$< \cot y < \infty$

$\Rightarrow \quad 0 < y < \pi$

These conditions on x and y make the function,

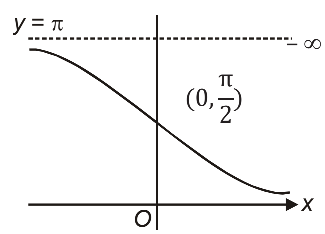
$\cot y = x$

One-one and onto so that the inverse function exists.

i.e., $y = \cot^{-1}x$ is meaningful.

i.e., Domain: $x \in R$

Range: $y \in (0, \pi)$



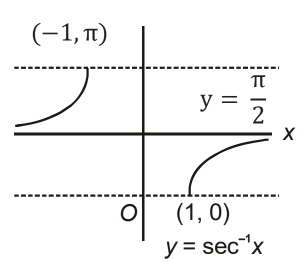
5. If $\sec y = x$,

Then $y = \sec^{-1}x$,

Where $| x | \ge 1$ and $0 \le y \le \pi, y \ne \frac{\pi}{2}$

Here, Domain: $x \in R - (-1, 1)$

Range: $y \in [0, \pi] - \frac{\pi}{2}\}$



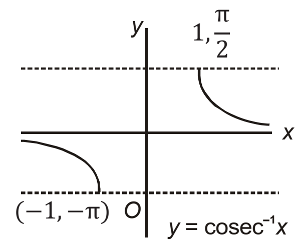
6. If $\text{cosec}\, y = x$ then $y = \text{cosec}^{-1}x$,

Where $|x| \geq 1$

And $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$

Here, domain $\in R - (-1, 1)$

Range $\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$



Function	Domain	Range
$\sin^{-1} x$	$[-1,1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1,1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right]$
$\text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, 0\right] \cup \left[0, \frac{\pi}{2}\right]$

Ex. Calculate domain of $\sin^{-1}(2x^2 - 1)$

Sol.: Let $y = \sin^{-1}(2x^2 - 1)$

For y to be defined $-1 \leq (2x^2 - 1) \leq 1$

$0 \leq 2x^2 \leq 2$

$0 \leq x^2 \leq 1$

$x \in [-1, 1]$.

Ex. Determine the value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$ is equal to-

Sol. $\tan^{-1}(1) + \cot^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

$\frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$

Ex. Determine the value of $\cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right)$ is equal to-

Sol. Let $\cos^{-1}\frac{1}{8} = \theta$,

Where $0 < \theta < \frac{\pi}{2}$.

Then $\frac{1}{2}\cos^{-1}\frac{1}{8} = \frac{1}{2}\theta$

$\Rightarrow$ $\cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right) = \cos\frac{1}{2}\theta$

Now $\cos^{-1}\frac{1}{8} = \theta$

$\Rightarrow$ $\cos\theta = \frac{1}{8}$

$\Rightarrow$ $2\cos^2\frac{\theta}{2} - 1 = \frac{1}{8}$

$\Rightarrow$ $\cos^2\frac{\theta}{2} = \frac{9}{16}$

$\Rightarrow$ $\cos\frac{\theta}{2} = \frac{3}{4}$ $\left[\because 0 < \frac{\theta}{2} < \frac{\pi}{4}, \text{so} \cos\frac{\theta}{2} \neq -\frac{3}{4}\right]$