BAYE'S THEOREM

Partition of a set

A set of events $E_1, E_2, \ldots, E_n$ is said to represent a partition of the sample space S if

(a) $E_i \cap E_j = \phi, i \neq j, i, j = 1,2,3, \ldots, n$

(b) $E_1 \cup E_2 \cup E_3 \ldots E_n = S$ and

(c) $P(E_i) > 0$ for all, $i = 1,2,3, \ldots, n$

In other words, the events $E_1, E_2, \ldots, E_n$ represent a partition of the sample space if they are pairwise disjoint, exhaustive and have non-zero probabilities.

Baye's Theorem

If an event A can take place alongside any of the n mutually exclusive and exhaustive events B_1, $B_2, \ldots, B_n$, and the probabilities $p\left(\frac{A}{B_1}\right), p\left(\frac{A}{B_2}\right), \ldots\ldots, p\left(\frac{A}{B_n}\right)$ are ascertainable, then

$$P\left(\frac{B_i}{A}\right) = \frac{P(B_i).P\left(\frac{A}{B_i}\right)}{\sum_{j=1}^{n} P(B_j).P\left(\frac{A}{B_j}\right)}$$

Proof

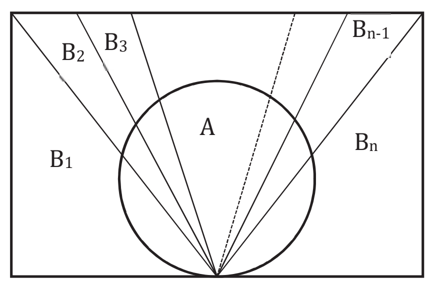
Event A transpires in conjunction with one of the n mutually exclusive and exhaustive events B_1, $B_2, B_3, \ldots, B_n$.

$$A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup \ldots \cdots (A \cap B_1)$$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots\ldots + P(A \cap B_1) = \sum_{i=1}^{n} P(A \cap B_i)$$

Now, $$P(A \cap B) = P(A) \cdot p\left(\frac{B}{A}\right) = P(B) \cdot p\left(\frac{A}{B}\right)$$

$$A = \frac{P(B_i) \cdot p\left(\frac{A}{B_i}\right)}{P(A)}$$



$$= \frac{P(B_i) \cdot p\left(\frac{A}{B_i}\right)}{\sum_{i=1}^{n} P(A \cap B_i)}$$

$$p\left(\frac{B}{A}\right) = \frac{P(B_i) \cdot p\left(\frac{A}{B_i}\right)}{\sum P(B_j) \cdot p\left(\frac{A}{B_j}\right)}$$

Ex. Provided with three identical boxes labeled I, II, and III, each holding two coins, where in box I both coins are gold, in box II both are silver, and in box III, one is gold and the other is silver. If a person randomly selects a box and withdraws a coin, and the coin is gold, what is the probability that the second coin in the box is also gold?

Sol. Denote by E_1, E_2, and E_3 the events corresponding to the selection of boxes I, II, and III, respectively.

Then $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Moreover, define event A as 'drawing a gold coin.'

Then $P(A|E_1) = P(\text{a gold coin from box I}) = \frac{2}{2} = 1$

$P(A|E_2) = P(\text{a gold coin from box II}) = 0$

$P(A|E_3) = P(\text{a gold coin from box III}) = \frac{1}{2}$

Now, the probability that the other coin in the box is gold is equal to the probability of drawing a gold coin from box I.

$= P(E_1|A)$

According to Bayes' theorem, it is known that

$$P(E_1 | A) = \frac{P(E_1)P(A|E_1)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + P(E_3)P(A|E_3)}$$

$$= \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1 + \frac{1}{3}\times 0 + \frac{1}{3}\times\frac{1}{2}} = \frac{2}{3}$$

Ex. Pal's gardener is unreliable, with a $\frac{2}{3}$ probability of forgetting to water the rose bush. The rose bush is already in questionable condition. If watered, there is a $\frac{1}{2}$ probability of it withering; if not watered, the probability of withering is $\frac{3}{4}$. Pal, who went out of town, returns to find the rose bush withered. Given this outcome, what is the probability that the gardener did not water the bush? [Bayes' theorem is to be applied since the result, i.e., the withering of the rose bush, is known.]

Sol. Let A = the event that the rose bush has withered

Let A_1 = the event that the gardener did not water.

A_2 = the event that the gardener watered.

According to Bayes' theorem, the probability needed is

$$P\left(\frac{A_1}{A}\right) = \frac{P(A_1).P\left(\frac{A}{A_1}\right)}{P(A_1).P\left(\frac{A}{A_1}\right) + P(A_2).P\left(\frac{A}{A_2}\right)}$$

Given, $p(A_1) = \frac{2}{3}, p\left(A_2 = \frac{1}{3}\right)$

$$P\left(\frac{A}{A_1}\right) = \frac{3}{4}, P\left(\frac{A}{A_2}\right) = \frac{1}{2}$$

From (i) $$P\left(\frac{A_1}{A}\right) = \frac{\frac{2}{3}\cdot\frac{3}{4}}{\frac{2}{3}\cdot\frac{3}{4} + \frac{1}{3}\cdot\frac{1}{2}} = \frac{6}{6+2} = \frac{3}{4}$$

Ex. In bag A, there are 2 white and 3 red balls, while bag B contains 4 white and 5 red balls. If a ball is randomly drawn from one of the bags and turns out to be red, determine the probability that it was drawn from bag B.

Sol. Let E_1 = The event of ball being drawn from bag A

E_2 = The event of ball being drawn from bag B.

E = The event of ball being red.

Given that both bags have an equal likelihood of being chosen,

Therefore $$P(E_1) = P(E_2) = \frac{1}{2}$$

$$P(E|E_1) = \frac{3}{5}$$

$$P(E|E_2) = \frac{5}{9}$$

Required probability $$P(E_2|E) = \frac{P(E_2)P(E|E_2)}{P(E_1)\times P(E|E_1) + P(E_2)P(E|E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{5}{9}}{\frac{1}{2}\times\frac{3}{5}+\frac{1}{2}\times\frac{5}{9}} = \frac{25}{52}$$

Ex. In class XI, there are 5 brilliant students, and in class XII, there are 8 brilliant students, with each class having a total of 50 students. The odds in favor of choosing class XI are 2:3. If class XI is not chosen, then class XII is selected. If a student is chosen and is found to be brilliant, determine the probability that the selected student is from class XI.

Sol. Define events E and F as 'Selection of Class XI' and 'Selection of Class XII,' respectively.

Then, $$P(E) = \frac{2}{5}, P(F) = \frac{3}{5}$$

Let A be the event 'Student chosen is brilliant'.

$$P\left(\frac{A}{E}\right) = \frac{5}{50} \text{ and } P\left(\frac{A}{F}\right) = \frac{8}{50}.$$

$$P(A) = P(E)\times p\left(\frac{A}{E}\right) + P(F)\times p\left(\frac{A}{F}\right)$$

$$= \frac{2}{5}\times\frac{5}{50}+\frac{3}{5}\times\frac{8}{50} = \frac{34}{250}$$

$$P\left(\frac{E}{A}\right) = \frac{P(E)\times P\left(\frac{A}{E}\right)}{P(E)\times P\left(\frac{A}{E}\right) + P(F)\times P\left(\frac{A}{F}\right)} = \frac{5}{17}$$