## AXIOMATIC DEFINITION OF PROBABILITY

In a set of possibilities, for each event A, there's a number called P (A), which is the likelihood of A happening. This setup follows certain rules known as the axioms of probability.

For every A in S, $0 \le P(A) \le 1$

The probability of the entire sample space, denoted as P(S), is equal to 1.

For mutually exclusive events A and $B(A \cap B = \phi), P(A \cup B) = P(A) + P(B)$.

## Basic theories of probability

1 For an event A and its complement Ac in the sample space S

$$P(A^C) = 1 - P(A)$$
$$A \cap A^C = \phi \text{ and } A \cup A^C = S$$
$$P(A \cup A^C) = P(A) + P(A^C)$$
$$P(S) = P(A) + P(A^C)$$
$$1 = P(A) + P(A^C)$$

**Ex.** In a cricket club with 15 members, where only 5 can bowl, what is the probability of creating a team of 11 members with at least 3 bowlers?

**Sol.** Total number of ways of forming the team = ${}^{15}C_{11} = {}^{15}C_4$

Of these, number of ways of formation of the team

1. with one bowler $= {}^5C_1 \cdot {}^{10}C_{10} = 5$
2. with two bowlers $= {}^5C_2 \cdot {}^{10}C_9 = 100$

Probability that at least 3 bowlers are in the team $= 1 - \frac{105}{{}^{15}C_4} = \frac{12}{13}$

**Ex.** When tossing five coins simultaneously, determine the probability of the event where at least one head appears. (Assume fair coins.)

**Sol.** Consider event A as 'at least one head turns up.' Since each coin can show either a head or a tail, the sample space includes $2^5 = 32$ outcomes, each with a probability of occurrence of $\frac{1}{32}$. Then, Ac represents the event 'No head turns up.' Consequently, Ac consists of only one outcome.

$$P(A^C) = \frac{1}{32}$$
$$P(A) = 1 - \frac{1}{32} = \frac{31}{32}$$

## Addition Rule Of Probability

For events A and B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

In general for n events $A_1, A_2 \ldots A_n$ of sample space S

$$P(A_1 \cup A_2 \cup \ldots \ldots A_n)$$
$$= \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j < n} P(A_i \cap A_j) + \sum_{1 \le i < j < k} P(A_i \cap A_j \cap A_k) \ldots \ldots + (-1)^{n-1} P(A_1 \cap \ldots . \cap A_n)$$

**Ex.** When rolling a fair die, what is the probability of obtaining either an odd number or a number less than 4?

**Sol.**
$$S = \{1,2,3,4,5,6\}$$

Let A be the event that odd number occurs then $P(A) = \frac{3}{6} = \frac{1}{2}$

$$A = \{1,3,5\}$$

Let B be the event that a number less than 4 occurs then $B = \{1, 2, 3\}$

$$P(B) = \frac{3}{6} = \frac{1}{2}$$
$$A \cap B = \{1,3\} \quad \text{(Odd number less than 4)}$$
$$P(A \cap B) = \frac{2}{6} = \frac{1}{3}$$
$$P(A \cup B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{2}{3}$$

For exhaustive events $A_1, A_2 \ldots A_n$ in a sample spaces.

$$P(A_1 \cup A_2 \cup \ldots \ldots . \cup A_n) = 1$$

For mutually exclusive events $A_1$ $A_2$…….. $A_n$ in a sample space S

$$P(A_1 \cup A_2 \cup A_3 \ldots..\cup A_n) = P(A_1) + P(A_2) + \cdots \ldots..+P(A_n)$$

**Ex.** If the likelihood that a garage will have 10-20, 21-30, 31-40, or over 40 cars to service on any workday is 0.20, 0.35, 0.25, and 0.12, respectively, what is the probability that on a specific workday the garage has at least 21 cars to service?

**Sol.** As these events are mutually exclusive, the probability needed is

$$0.35 + 0.25 + 0.12 = 0.72$$

For mutually exclusive and exhaustive events $A_1$, $A_2$… $A_n$ in a sample spaces.

$$P(A_1 \cup A_2 \cup A_3 \ldots \ldots \cup A_n) = P(A_1) + P(A_2) + \cdots..+P(A_n) = 1$$

For an event A in sample space S, 'the odds in favor of A' are $\frac{P(A)}{P(A^c)}$ where $A^c$ is the complement of the event A is S. Also 'the odds against A' are $\frac{P(A^c)}{P(A)}$.

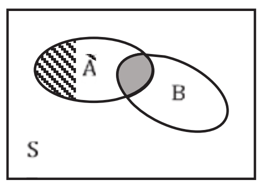
**For example,** in throw of a die the odds in favor of "a multiple of 3 occurs" is 2: 4 i.e. 1: 2.

## Conditional probability

Frequently, there is a need to determine the probability of an event B given that event A has occurred. This probability is referred to as the conditional probability of B given A and is represented as P(B | A). In this scenario, A functions as a new (reduced) sample space, and the probability is the fraction of that portion of set A which corresponds to the intersection of A and B.

$$p(\frac{B}{A}) = \frac{p(A \cap B)}{p(A)}$$

$$P(A) \neq 0$$




The shaded area indicates the favorable region, while the lined portion represents the reduced sample space.

Similarly, the conditional probability of A given B is

$$p(\frac{A}{B}) = \frac{P(A \cap B)}{P(B)}$$

From the two expressions mentioned above, we can express the probability of the intersection of two events A and B.

$$P(A) \neq 0 \text{ and } P(B) \neq 0$$

$$p(A \cap B) = p(A) \cdot p(\frac{A}{B}) \text{orp } p(B)p(\frac{A}{B}).$$

(Multiplication theorem)

**Ex.** In the production of screws, consider A as "screw too slim" and B as "screw too short."

Let P A = 0.1, and let the conditional probability that a slim screw is also too short be PB/A = 0.2. What is the probability that a randomly selected screw from the produced lot will be both too slim and too short?

**Sol.** We need the probability of the simultaneous occurrence of both events, which can be expressed as

$$P(A \cap B) = P(A)P(B/A)$$

$$0.1 \times 0.2 = 0.02 = 2\%$$

## Independent events

If events A and B are such that

$$P(A \cap B) = P(A)P(B),$$

They are called independent events. Assuming, $P(A) \neq 0, P(B) \neq 0$, in this case

$$p(\frac{A}{B}) = p(A), p(\frac{B}{A}) = p(B)$$

This implies that the probability of A is not influenced by the occurrence or non-occurrence of B, and vice versa. This justifies the term "independent."

Similarly, m events $A_1 \ldots, A_m$ are called independent if

$$P(A_1 \cap \ldots \cap A_m) = P(A_1) \ldots \ldots . P(A_m)$$

As well as for every k different events $A_{j_1}, A_{j_2}, \ldots \ldots, A_{j_k}$

$$P(A_{j_1} \cap A_{j_2} \cap \ldots \cap A_{j_k}) = P(A_{j_1})P(A_{j_2}) \ldots P(A_{j_k})$$

Where $k = 2,3, \ldots \ldots m-1$

Therefore, three events A, B, C are considered independent if

$$P(A \cap B) = P(A)P(B),$$
$$P(B \cap C) = P(B)P(C),$$
$$P(C \cap A) = P(C)P(A)$$
$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

**Ex.** A, and B are two independent witnesses in a case. The probability that A will speak the truth is $\frac{3}{4}$, and that of B is $\frac{4}{5}$. What percentage of cases are they likely to contradict each other when stating the same fact?

**Sol.** If E is the event of their contradicting each other then $E = (A \cap \overline{B}) \cup (\overline{A} \cap B)$ also $(A \cap \overline{B})$ and $(\overline{A} \cap B)$ are two mutually exclusive events.

$$P(E) = P(A \cap \overline{B}) + P(\overline{A} \cap B)$$
$$= \frac{3}{4} \cdot \frac{1}{5} + \frac{4}{5} \cdot \frac{1}{4} = \frac{7}{20}$$

In 35% of the cases they are likely to contradict each other.

**Ex.** The likelihood of a sportsman shooting an animal at a distance $r(> a)$ is $\frac{a^2}{r^2}$ He shoots when $r = 2a$, and if he misses, he reloads and shoots when $r = 3a, 4a, 5a$, and so on. If he misses at a distance na, the animal escapes. What are the odds against the sportsman?

**Sol.**

$$P(r) = \frac{a^2}{r^2}$$
$$P(2a) = \frac{1}{4}, p(3a) = \frac{1}{9}, p(4a) = \frac{1}{16}. \text{ etc.}$$

The sportsman achieves success if

(a) He scores a hit on the initial attempt or

(b) Fails to hit on the initial attempt but succeeds on the second try or

(c) Fails to hit on the first and second attempts but succeeds on the third and so forth.

Probability of success

$$= \frac{1}{4} + \frac{3}{4} \times \frac{1}{9} + \frac{3}{4} \times \frac{8}{9} \times \frac{1}{16} + \frac{3}{4} \times \frac{8}{9} \times \frac{15}{16} \times \frac{1}{25} + \cdots \ldots . (n-1) \text{ Terms}$$
$$p = \frac{1}{4} + \frac{1}{12} + \frac{1}{24} + \frac{1}{40} + \cdots \text{ upto } (n-1) \text{ terms}$$
$$2p = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots \ldots \ldots . . \text{ up to } (n-1) \text{ terms}$$
$$= (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) + \cdots \ldots \ldots + (\frac{1}{n-1} - \frac{1}{n})$$
$$= 1 - \frac{1}{n} = \frac{n-1}{n}$$

odds against the sportsman $= \frac{q}{p} = \frac{n+1}{n-1}$

**Ex.** Among the three independent events, the probability of only the first occurring is a, of only the second occurring is b, and of only the third occurring is c. Demonstrate that the probabilities of occurrence of these three events are as follows: $\frac{a}{a+x}, \frac{b}{b+x}, \frac{c}{c+x}$ where x is a root of the equation $(a+x)(b+x)(c+x) = x^2$.

**Sol.** Let $E_1, E_2, E_3$ be three independent events and $E_1', E_2', E_3'$ be their complements. Then

$$P(E_1 \cap E_2' \cap E_3') = P(E_1) \cdot P(E_2') \cdot P(E_3') = a \quad \text{... (1)}$$

Since $E_1$, $E_2$, $E_3$ are independent

$$P(E_1' \cap E_2 \cap E_3') = P(E_1')P(E_2)P(E_3') = b \quad \text{... (2)}$$

$$P(E_1' \cap E_2' \cap E_3) = P(E_1')P(E_2')P(E_3) = c \quad \text{... (3)}$$

Denote $P(E_1')P(E_2')P(E_3')$ by x

$$\frac{P(E_1)}{P(E_1')} = \frac{a}{x},$$

This implies

$$\frac{P(E_1)}{1-P(E_1)} = \frac{a}{x}$$

$$P(E_1) = \frac{a}{a+x}$$

Similarly, we get $P(E_2) = \frac{b}{b+x}$ and $P(E_3) = \frac{c}{c+x}$ ... (4)

Multiplying (1), (2) and (3), we get

$$\frac{abc}{(a+x)(b+x)(c+x)}x^2 = abc \text{ or } (a+x)(b+x)(c+x) = x^2$$

**Ex.** The independent probabilities of A, B, and C solving a mathematical problem are $\frac{1}{3}, \frac{1}{3},$ and $\frac{1}{4}$, respectively. Determine the probability that only two of them solve the problem.

**Sol.** Given that $P(A) = \frac{1}{3}, P(B) = \frac{1}{3}, P(C) = \frac{1}{4}$ and A, B, C are independent events.

The problem gets solved by any two of them solving but the third one fails.

Required probability

$$P(A \cap B \cap C^c) + P(A \cap B^c \cap C) + P(A^c \cap B \cap C)$$

$$= \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{3}{4} + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{4}$$

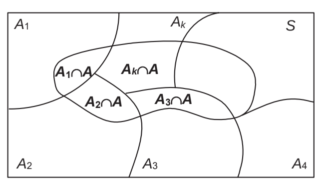
$$= \frac{3}{36} + \frac{2}{36} + \frac{2}{36} = \frac{7}{36}$$

## Total probability

Take a sample space S; let Ai, where i = 1 to n, be a collection of n mutually exclusive and exhaustive sets within the sample space S.

$$A_i \cap A_j = \phi \text{ for } 1 \le i < j \le n$$

$$\sum_{i=1} P(A_i) = 1 \text{ as } \cup_{i=1} A_i = S$$




Consider any event A within the sample space S. The total probability of event A is expressed as

$$P(A) = \sum_{i=1}^{n} P(A_i)P(A/A_i)$$

Here, $P\left(\frac{A}{Ai}\right)$ represents the contribution of Ai to the occurrence of A.

This result is obtained as

$$A = (A_1 \cap A) \cup (A_2 \cap A) \cup \dots \dots \cup (A_n \cap A)$$

$$P(A) = P(A_1 \cap A) + P(A_2 \cap A) + \cdots \dots + P(A_n \cap A)$$

$$P(A_1)P\left(\frac{A}{A_1}\right) + P(A_2)P\left(\frac{A}{A_2}\right) + \cdots \dots + P(A_n)P\left(\frac{A}{A_n}\right)$$

$$\sum_{i=1}^{n} P(A_i)P\left(\frac{A}{A_i}\right)$$

**Ex.** A fair coin is flipped. If it lands heads, two fair dice are rolled, and the sum of the numbers on the two faces is recorded. If it lands tails, a card is drawn from a well-shuffled deck of eleven cards numbered 2, 3 … 12, and the number on the card is recorded. What is the probability that the recorded number is either 7 or 8?

**Sol.** Let $E_1$ be the event "toss results in a head",
$E_2$ be the event "toss results in a tail"
A be the event "the noted number is 7 or 8"

$$P(E_1) = P(E_2) = \frac{1}{2}$$

$$p(\frac{A}{E_1}) = P(7) + P(8)$$

$$= \frac{6}{36} + \frac{5}{36} = \frac{11}{36}$$

$$7 = \{1 + 6, 2 + 5, 3 + 4, 4 + 3, 5 + 2, 6 + 1\}$$

$$8 = \{2 + 6, 3 + 5, 4 + 4, 5 + 3, 6 + 2\}$$

$$p(\frac{A}{E_2}) = \frac{2}{11}$$

We know that $P(A) = p(\frac{A}{E_1}) \times P(E_1) + p(\frac{A}{E_2}) \times P(E_2)$

$$= \frac{11}{36} \times \frac{1}{2} + \frac{2}{11} \times \frac{1}{2} = \frac{193}{792}$$

**Ex.** A batch consists of 20 articles. The probability of having exactly 2 defective articles is 0.4, and the probability of having exactly 3 defective articles is 0.6. Articles are randomly drawn from the batch one by one without replacement until all the defective ones are found. What is the probability that the testing procedure concludes at the twelfth test?

**Sol.** Suppose A is the event that the testing procedure ends at the twelfth testing.
$A_1$ = {the event that the lot contains 2 defective}
$A_2$ = {the event that the lot contains 3 defective}

$$P(A_1) = 0.4, P(A_2) = 0.6$$

$$P(A) = P(A_1)P(\frac{A}{A_1}) + P(A_2)P(\frac{A}{A_2})$$

$$= 0.4\{\frac{^{18}C_{10} \cdot {}^{2}C_1}{^{20}C_{11}} \cdot \frac{1}{9}\} + 0.6\{\frac{^{17}C_9 \cdot {}^{3}C_2}{^{20}C_{11}} \cdot \frac{1}{9}\}$$

$$= \frac{1}{9}\{\frac{4}{10} \cdot \frac{|18}{|10|18} \cdot \frac{|11|9}{|20} \cdot 2 + \frac{6}{10} \cdot \frac{|17}{|9|18} \cdot \frac{|11|9}{|20} \cdot 3\}$$

$$= \frac{1}{9}\{\frac{4}{10} \times \frac{11 \times 9}{19 \times 20} \cdot 2 + \frac{6}{10} \times \frac{9 \times 10 \times 11}{18 \cdot 19 \cdot 20} \cdot 3\}$$

$$= \frac{44}{1900} + \frac{55}{1900} = \frac{99}{1900}$$