USE OF EXPANSION IN LIMITS

1. $a^x = 1 + \frac{x\ln a}{1!} + \frac{x^2\ln^2 a}{2!} + \frac{x^3\ln^3 a}{3!} + \cdots \ldots \ldots, a > 0$
2. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \ldots$
3. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \ldots \ldots, \text{for } -1 < x \le 1$
4. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
5. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
6. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \ldots$
7. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$
8. $\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2 \cdot 3^2}{5!}x^5 + \frac{1^2 \cdot 3^2 \cdot 5^2}{7!}x^7 + \cdots$
9. $\sec^{-1} x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61\ \ ^6}{6!} + \cdots \ldots$
10. For $|x| < 1, n \in R; (1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \cdots \ldots \ldots .. \infty$
11. $(1+x)^{\frac{1}{x}} = e(1 - \frac{x}{2} + \frac{11}{24}x^2 - \cdots \ldots \ldots ..)$

Ex. Evaluate $\lim\limits_{x\to 0} \frac{\tan x - \sin x}{x^3}$

Sol. $\lim\limits_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim\limits_{x\to 0} \frac{(x+\frac{x^3}{3}+\cdots\ldots) - (x - \frac{x^3}{3!}+\cdots\ldots)}{x^3} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}.$

Ex. Find the values of a, b and c so that $\lim\limits_{x\to 0} \frac{ae^x - b \quad x + ce^{-x}}{x\sin x} = 2$

Sol. $\lim\limits_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$ ... (1)

at $x \to 0$ numerator must be equal to zero

$a - b + c = 0 \Rightarrow b = a + c$ ... (2)

From (1) and (2) $\lim\limits_{x\to 0} \frac{ae^x - (a+c)\cos x + ce^{-x}}{x\sin x} = 2$

$\lim\limits_{x\to 0} \frac{a(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\ldots) - (a+c)(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\ldots) + c(1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\ldots)}{x(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots.)} = 2$

$\lim\limits_{x\to 0} \frac{\frac{(a-c)}{x}+(a+c)+\frac{x}{3!}(a-c)+\cdots\ldots}{(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\ldots)} = 2$

Since R.H.S is finite, $a - c = 0$

$a = c$, then $\frac{0+2a+0+\cdots}{1} = 2$

$a = 1$ then $c = 1$

From (2), $b = a + c = 1 + 1 = 2$

Ex. $\lim\limits_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$

Sol. $\lim\limits_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$

$$\lim_{x\to 0} \frac{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\ldots-(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots.)-2x}{x-(x-\frac{x^3}{3!}+\frac{x^5}{5!}\ldots.)}$$

$$\lim_{x\to 0} \frac{2\cdot\frac{x^3}{6}+2\cdot\frac{x^5}{5!}+\cdots\ldots}{\frac{x^3}{6}+\frac{x^5}{5!}\ldots.}$$

$$\lim_{x\to 0} \frac{x^3(\frac{1}{3}+\frac{1}{60}x^2+\cdots.)}{x^3(\frac{1}{6}+\frac{1}{120}x^2+\cdots.)} = \frac{1/3}{1/6} = 2$$