LIMIT OF THE FORM $\lim\limits_{x\to a}(f(x))^{g(x)}$: Form $0^\circ, \infty\infty^\circ$

Let $L = \lim\limits_{x\to a}(f(x))^{g(x)}$. Then,

$\log_e L = \log_e[\lim\limits_{x\to a}\{f(x)\}^{g(x)}]$

$\lim\limits_{x\to a}[\log_e\{f(x)\}^{g(x)}] \Rightarrow \lim\limits_{x\to a} g(x)\log_e[f(x)]$

$L = e^{\lim\limits_{x\to a} g(x)\log_e f(x)}$

Ex. Evaluate $\lim\limits_{x\to\infty} x^{1/x}$.

Sol. $\lim\limits_{x\to\infty} x^{1/x} = e^{\log \lim\limits_{x\to\infty} x^{\frac{1}{x}}}$

$e^{\lim\limits_{x\to\infty} \log x^{1/x}}$

$e^{\lim\limits_{x\to\infty} \frac{\log x}{x}}$

($\because$ x increasing faster than $\log_e x$ when $x \to \infty$)

e^0

1

Form - 1^∞

$(1)^\infty$ type of problems can be solved by the following method.

1. $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$
2. $\lim\limits_{x\to a}[f(x)]^{g(x)}$; where $f(x) \to 1$; $g(x) \to \infty$ as $x \to a$

$\lim\limits_{x\to a}[1 + f(x) - 1]^{\frac{1}{f(x)-1}\{(x)-1\}\cdot g(x)} = \lim\limits_{x\to a}([1 + (f(x) - 1)]^{\frac{1}{f(x)-1}})^{(f(x)-1)g(x)} = e^{\lim\limits_{x\to a}[f(x)-1]g(x)}$

Ex. Evaluate $\lim\limits_{x\to 0}(1+x)^{\text{cose}}$.

Sol. $\lim\limits_{x\to 0}(1+x)^{\text{cosec}\,x} = \lim\limits_{x\to 0}[(1+x)^{\frac{1}{x}}]^{\frac{x}{\sin x}}$

$= [\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}]^{\lim\limits_{x\to 0}\frac{x}{\sin}} = e^1$

Ex. Evaluate $\lim\limits_{x\to 0}(\frac{a^x+b^x+c^x}{3})^{2/x}$; $(a, b, c > 0)$

Sol. We have, $\lim\limits_{x\to 0}(\frac{a^x+b^x+c^x}{3})^{2/x} = e^{\lim\limits_{x\to 0}(\frac{a^x+b^x+c^x}{3}-1)\cdot\frac{2}{x}}$

$= e^{\frac{2}{3}\lim\limits_{x\to 0}\left(\frac{a^x+b^x+c^x-3}{x}\right)} \Rightarrow e^{\frac{2}{3}\lim\limits_{x\to 0}\left(\frac{a^x-1}{x}+\frac{b^x-1}{x}+\frac{c^x-1}{x}\right)}$

$= e^{\frac{2}{3}\left\{\lim\limits_{x\to 0}\frac{a^x-1}{x}+\lim\limits_{x\to 0}\frac{b^x-1}{x}+\lim\limits_{x\to 0}\frac{c^x-1}{x}\right\}}$

$= e^{(\frac{2}{3})\{\ln a+\ln b+l\ \ c\}=} e^{(2/3)\ln(abc)} = e^{\ln(abc)^{\frac{2}{3}}} = (abc)^{\frac{2}{3}}$