L'HOPITAL'S RULE

If f (x) and g (x) are functions of x such that f (a) = 0 and g (a) = 0, then.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

In this context, $f'(x)$ and $g'(x)$ represent the derivatives of f(x) and g(x) with respect to x.

1. When employing this rule, it is necessary to differentiate f (x) and g (x) separately.
2. L. Hospital's Rule is true even if f(x) = g(x) = 0
   when $x \to \infty$
   $$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$
3. Forms $0 \times \infty$ and $\infty - \infty$, reduced either to form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ for using L. Hospital's rule
4. $0^0, 1^\infty, \infty^0$ such types of form can be reduced to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by taking log of the given expression.

DERIVATIVES

Derivative of a Function at a Point

The instantaneous rate of change of f(x) at x, expressed mathematically as f'(x) or $f'(x)$ or $\frac{df}{dx}$is referred to as the derivative of f, at x. geometrically, it signifies the slope of the tangent at the point (x, y) on the curve y = f(x).

$f'(x) = \lim\limits_{h\to0}\frac{f(x+h)-f(x)}{h}$ (derivative by first principle)$f'(a)$

$\lim\limits_{h\to0}\frac{f(a+h)-f(a)}{h}$ provided the derivatives exist.

Ex. Apply the first principle to determine the derivatives of the following functions.

1. x^n
2. $a^x, a > 0$
3. $\tan x, x \neq (2n+1)\frac{\pi}{2}, n \in Z$
4. $\sec x, x \neq (2n+1)\frac{\pi}{2}, n \in Z$
5. $\sin^{-1} x, -1 \le x \le 1$

Sol. 1 We have

$f'(x) = \lim\limits_{h\to0}\frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to0}\frac{(x+h)^n - x^n}{h}$

$\lim\limits_{h\to0}\frac{x^n[(1+\frac{h}{x})^n-1]}{h}$

$\lim\limits_{h\to0}\frac{x^n[(1+\frac{h}{x})^n-1]}{x(1+\frac{h}{x}-1)} = x^{n-1}\lim\limits_{t\to1}\frac{t^n-1}{t-1}; t = 1+\frac{h}{x}$ say

$x^{n-1}\cdot n = nx^{n-1}$

2. We have

$f'(x) = \lim\limits_{h\to0}\frac{f(x+h)-f(x)}{h}$

$\lim\limits_{h\to0}\frac{a^{x+h}-a^x}{h}$

$\lim\limits_{h\to0}a^x(\frac{a^h-1}{h})$

$a^x\lim\limits_{h\to0}(\frac{a^h-1}{h})$

$a^x\ln a$

3. We have

$f'(x) = \lim\limits_{h\to0}\frac{f(x+h)-f(x)}{h}$

$\lim\limits_{h\to0}\frac{\tan(x+h)-\tan x}{h}$

$\lim\limits_{h\to0}\frac{\tan(x+h)-\tan x}{1+\tan(x+h)\cdot\tan x}\cdot\frac{1+\tan(x+h)\tan x}{h}$

$\lim\limits_{h\to0}\frac{\tan(x+h-x)}{h}[1+\tan(x+h)\tan x]$

$\lim\limits_{h\to0}(\frac{\tan h}{h})[1+\tan(x+h)\tan x]$

$1\times[1+\tan^2 x] = \sec^2 x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

4. We have

$f'(x) = \lim\limits_{h\to0}\frac{f(x+h)-f(x)}{h}$

$\lim\limits_{h\to0}\frac{\sec(x+h)-\sec x}{h}$

$\lim\limits_{h\to0}\frac{\cos x-\cos(x+h)}{h\cos x\cos(x+h)}$

$\lim\limits_{h\to0}2\sin(x+\frac{h}{2})\frac{\sin\frac{h}{2}}{2\cdot\frac{h}{2}}\cdot\frac{1}{\cos x\cos(x+h)}$

$\sin x\cdot 1\cdot\frac{1}{\cos x\cos x}$

$\tan x\sec x$

$\frac{d}{dx}(\sec x) = \sec x\tan x$

5. We have,

$y = f(x) = \sin^{-1} x$

$$x = \sin y \Rightarrow y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$
$$\frac{dx}{dy} = \lim_{h \to 0} \frac{\sin(y+h) - \sin y}{h}$$
$$\lim_{h \to 0} \frac{2\cos(y+\frac{h}{2})\sin\frac{h}{2}}{2\frac{h}{2}}$$
$$\lim_{h \to 0} \cos(y + \frac{h}{2}) \cdot (\frac{\sin\frac{h}{2}}{\frac{h}{2}})$$
$$\cos y = \sqrt{1 - \sin^2 y}$$
$$\sqrt{1 - x^2}$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} = \frac{d}{dx}(\sin^{-1} x)$$

Derivatives Of Standard Function

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$
$$\frac{d}{dx}(\text{ constant }) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}; n \in R$$
$$\frac{d}{dx}(\frac{1}{x^n}) = \frac{-n}{x^{n+1}}; x \neq 0$$
$$\frac{d}{dx}(e^x) = e^x$$
$$\frac{d}{dx}(a^x) = a^x \ln a; a > 0, a \neq 1$$
$$\frac{d}{dx}(\ln x) = \frac{1}{x}; x > 0 (\frac{d}{dx}(\ln |x|) = \frac{1}{x}, x \neq 0)$$
$$\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e; x > 0, a > 0, a \neq 1$$
$$\frac{d}{dx}(|x|) = \frac{x}{|x|} \text{ or } \frac{|x|}{x}; x \neq 0$$

While the idea of the inverse of a function is covered in the twelfth grade curriculum, the derivatives presented below aid in addressing limit problems using L Hospital's Rule.

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}; -1 < x < 1$$
$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}; -1 < x < 1$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}; x \in R$$

$$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}; x \in R$$
$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}; |x| > 1$$
$$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}; |x| > 1$$

Basic Rules of Differentiation

The subsequent rules facilitate the determination of a function's derivative without relying on the first principle.

$$\frac{d}{dx}\{ku(x)\} = k\frac{du}{dx}, \text{k being a constant.}$$
$$\frac{d}{dx}(u(x) \pm v(x)) = \frac{du}{dx} \pm \frac{dv}{dx}$$

Generalization

$$\frac{d}{dx}(u(x) \pm v(x) \pm w(x) \pm \cdots..) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm \cdots..$$
$$\frac{d}{dx}(u(x) \cdot v(x)) = \frac{du}{dx} \cdot v + u\frac{dv}{dx}$$

Generalization

$$\frac{d}{dx}(u(x)v(x)w(x)) = \frac{du}{dx}vw + u\frac{dv}{dx}w + uv\frac{dw}{dx}$$
$$\frac{d}{dx}(u(x)v(x)w(x)t(x) \ldots.) = \frac{du}{dx}(vwt \ldots..) + u\frac{dv}{dx}(wt \ldots.) + uv\frac{dw}{dx}(t \ldots.) + uvw\frac{dt}{dx}(\ldots.) + \cdots..$$

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Ex. Find $\frac{dy}{dx}$ when

1. $y = \frac{x}{\log_a x}$
2. $y = \frac{1}{x\log_a x}$
3. $y = \cos x\cos 2x\cos 4x\cos 8x$
4. $y = (1+x)(1+x^2)(1+x^4)(1+x^8)\ \ldots..(1+x^{2^n})$
5. $y = (\cos x - i\sin x)(\cos 3x - i\sin 3x)(\cos 5x - i\sin 5x)\ \ldots.(\cos 2007x - i\sin 2007x)$
6. $y = \sin x\cos x\tan x\operatorname{cosec} x\sec x\cot x$, if all the functions are well defined
7. If, $y = f(x)$, where $f(x + \frac{1}{x}) = x^4 + x^{-4}$

Sol. We have

1. $y = \frac{x}{\log_a x} = \frac{x}{\log_e x \times \log_a e} = \frac{1}{\log_a e}\left(\frac{x}{\log_e x}\right)$

$$\frac{dy}{dx} = \frac{1}{\log_a e}\frac{d}{dx}\left(\frac{x}{\log_e x}\right)$$

$$\frac{1}{\log_a e}\cdot\left[\frac{\frac{dx}{dx}\times\log_e x - x\frac{d}{dx}(\log_e x)}{(\log_e x)^2}\right]$$

$$\frac{(\log_e a)\left[\log_e x - x\cdot\frac{1}{x}\right]}{(\log_e x)^2}$$

$$\frac{(\log_e a)\log_e\left(\frac{x}{e}\right)}{(\log_e x)^2}$$

2. $y = \frac{1}{x\log_a x}$

$$\frac{dy}{dx} = \frac{\frac{d(1)}{dx}\times x\log_a x - 1\cdot\frac{d}{dx}(x\log_e x\times\log_a e}{(x\log_a x)^2}$$

$$\frac{0\times x\log_a e - 1\times\log_a e[\frac{dx}{dx}\times\log_e x + x\frac{d}{dx}(\log_e x)]}{(x\log_a x)^2}$$

$$\frac{-\log_a e[1\times\log_e x + x\cdot\frac{1}{x}]}{(x\log_a x)^2}$$

$$= \frac{-(1+\log_e x)\log_a e}{(x\log_a x)^2}$$

3. $y = \cos x\cos 2x\cos 4x\cos 8x$

$$\frac{2\sin x}{2\sin x}\cos x\cos 2x\cos 4x\cos 8x$$

$$\frac{2\sin 2x\cos 2x\cos 4x\cos 8x}{2\times 2\sin x}$$

$$\frac{2\sin 4x\cos 4x\cos 8x}{2\times 4\sin x}$$

$$= \frac{2\sin 8x\cos 8x}{2\times 8\sin x}$$

$$= \frac{\sin 16x}{16\sin x}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin 16x}{16\,si}\right)$$

$$\frac{1}{16}\cdot\frac{\sin x\frac{d}{dx}(\sin 16x) - \sin 16x\frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$\frac{1}{16}\cdot\frac{\sin x\cdot 16\cos 16x - \sin 16x\cdot\cos x}{\sin^2 x}$$

$$\frac{1}{16}\left[\frac{16\sin x\cos 16x - \sin 16x\cos x}{\sin^2 x}\right]$$

5. $y=(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots.(1+x^{2^n})$

$\frac{(1-x)}{(1-x)}(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots..(1+x^{2^n})$

$\frac{(1-x^2)(1+x^2)(1+x^4)(1+x^8)\ldots.(1+x^{2^n})}{(1-x)}$

$\frac{(1-x^4)(1+x^4)(1+x^8)\ldots.(1+x^{2^n})}{(1-x)}$

$\frac{(1-x^8)(1+x^8)\ldots..(1+x^{2^n})}{(1-x)}$

$y=\frac{(1-x^{2^n})(1+x^{2^n})}{1-x}$

$\frac{1-x^{2^{n+1}}}{1-x}$

$\frac{dy}{dx}=\frac{(1-x)\frac{d}{dx}(1-x^{2^{n+1}})-(1-x^{2^{n+1}})\frac{d}{dx}(1-x)}{(1-x)^2}$

$\frac{(1-x)(0-2^{n+1}\cdot x^{2^{n+1}-1})-(1-x^{2^{n+1}})(0-1)}{(1-x)^2}$

$\frac{(1-x^{2^{n+1}})-2^{n+1}(1-x)(x^{2^{n+1}-1})}{(1-x)^2}$

6. $y=(\cos x-i\sin x)(\cos 3x-i\sin 3x)(\cos 5x-i\sin 5x)\ldots.(\cos 2007x-i\sin 2007x)$

$e^{-ix}\cdot e^{-3ix}\cdot e^{-5ix}\ldots\ldots e^{-i2007x}$, $[e^{-i\theta}=\cos\theta-i\sin\theta]$

$e^{-i(1+3+5+7+\cdots..+2007)x}$

$e^{-i(1004)^2x}=\cos(1004)^2x-i\sin(1004)^2x$

$\frac{dy}{dx}=-\sin(1004)^2x\times(1004)^2-i\times(1004)^2\cos(1004)^2x$

$-(1004)^2[\sin(1004)^2x+i\cos(1004)^2x]$

7. $y=\sin x\cos x\tan x\operatorname{cosec} x\sec x\cot x$

$(\sin x\operatorname{cosec} x)(\cos x\sec x)(\tan x\cot x)$

$1\times1\times1=1$

$\frac{dy}{dx}=\frac{d}{dx}(1)=0$

8. We have

$f(x+\frac{1}{x})=x^4+\frac{1}{x^4}=(x^2+\frac{1}{x^2})^2-2=[(x+\frac{1}{x})^2-2]^2-2$

$f(x)=(x^2-2)^2-2$

x^4-4x^2+2

$=y$

$\frac{dy}{dx}=\frac{d}{dx}(x^4)-\frac{d}{dx}(4x^2)+\frac{d}{dx}(2)$

$4x^3-8x+0$

$4x(x^2-2)$