EVALUATION OF ALGEBRAIC LIMITS

Direct Substitution Method

Limits using Direct Substitution

The substitution rule for evaluating limits is an approach to determine limits by directly replacing the variable x with the specific value at which the limit is being calculated. Let's consider a function f(x), where the objective is to find the limit of the function at x = a. In this method, x is straightforwardly replaced with "a" in the expression for the function f(x).

$$\lim_{x\to a} f(x) = f(a)$$

Ex. Considering $f(x) = x^2$

$$\lim_{x\to 1} f(x) = f(1)$$

$$\lim_{x\to 1} x^2 = f(1)$$

$$1^2 = f(1)$$

$$1 = f(1)$$

Frequently, it is feasible to compute the limits for the function using this rule; to express this formally,

If f(x) is an expression constructed from polynomials, roots, absolute values, exponentials, logarithms, trigonometric functions, and/or inverse trigonometric functions through the composition of functions and the operations of addition, subtraction, multiplication, and division, then for any a for which f(a) is defined,

$$\lim_{x\to a} f(x) = f(a)$$

Undefined limits by direct substitution

There exist certain limits that cannot be computed using this approach. For instance, let's consider a function. $f(x) = \frac{x}{\ln(x)}$ Determine the limit of this function as x approaches 1.

$$\lim_{x\to 1} f(x)$$

$$\lim_{x\to 1} \frac{x}{\ln(x)}$$

$$\lim_{x\to 1} \ln(x)$$

$$\frac{1}{0}$$

This limit lacks a defined value. In such situations, the direct substitution method.

Limits of Trigonometric Function

The direct substitution method is occasionally applicable for computing limits in functions that involve trigonometric functions. For instance, consider a function f(x), and we aim to determine the limits of this function as x approaches 0. Let's illustrate this with an example.

Ex. Solve the $\lim_{x\to 0} f(x)$

$$f(x) = \sin(x) + \sin(x)\cos(x)$$

Sol. $\lim_{x\to 0} f(x)$

$$\lim_{x\to 0}(\sin(x) + \sin(x)\cos(x))$$

$$\lim_{x\to 0}(\sin(x)(1 + \cos(x))$$

$$\lim_{x\to 0}\sin(x) \times \lim_{x\to 0}(1 + \cos(x))$$

$$0 \times (1 + 1) \Rightarrow 0$$

Limits of Piecewise Function

When dealing with piecewise functions, the substitution rule typically encounters challenges where the definition of the function undergoes changes. For such functions, it is applied in a slightly modified manner. Let's solve an illustrative problem to enhance our comprehension of this concept.

Ex. Solve the value of $\lim_{x\to 1} f(x)$.

$$f(x) = \begin{cases} x^2 - 1, & \text{if } x \geq 1 \\ x, & \text{otherwise} \end{cases}$$

Sol. At $x = 1$, there is a change in the function definition. Hence, it is not recommended to directly apply the rule. In such functions, it is advisable to evaluate the limit from both sides.

Left-hand Side Limit $\lim_{x\to 1^-} f(x)$

$\lim_{x\to 1^-} x \Rightarrow 1$

Right-hand Side Limit $\lim_{x\to 1^+} f(x)$

$\lim_{x\to 1^+} x^2 - 1 \Rightarrow 0$

In this case, limits from both sides are different.

Ex. Calculate the $\lim_{x\to 0} f(x)$

$$f(x) = x^2 + x + 1$$

Sol. $\lim_{x\to 0} f(x)$

$$\lim_{x\to 0}(x^2 + x + 1)$$

$$(0^2 + 0 + 1) \Rightarrow 1$$

Ex. Calculate the $\lim_{x\to 1} f(x) = \frac{x^2+x+1}{x+1}$

Sol. $\lim_{x\to 1} \frac{x^2+x+1}{x+1}$

$$\frac{\lim_{x\to 1} x^2+x+1}{\lim_{x\to 1} x+1}$$

$$\frac{1^2+1+1}{1+1} \Rightarrow \frac{3}{2}$$

Ex. Calculate the $\lim_{x\to 1} f(x)$

$$f(x) = \begin{cases} \sin(x), & \text{if } x \geq 1 \\ \cos(x), & \text{otherwise} \end{cases}$$

Sol. At $x = 1$, the function definition is changing. So it is not advised to directly apply the rule.

In such functions, one should look for the limit from both sides

Left-hand Side Limit $\lim_{x\to 1^-} \cos(x)$

$\cos(1)$

Right-hand Side Limit $\lim_{x\to 1^+} \sin(x)$

$\sin(1)$

In this case, limits from both sides are different.

Ex. Solve the $\lim_{x\to 1} f(x)$

Sol. $f(x) = \frac{e^x+x+1}{\log(x)+1}$

$$\lim_{x\to 1} \frac{e^x+x+1}{\log(x)+1}$$

$$\lim_{x\to 1} e^x + x + 1$$

$$\frac{e^1+1+1}{\log(1)+1}$$

$$\frac{e^1+2}{0+1}$$

$$e + 2$$

Ex. Solve this $\lim_{x\to1} f(x)$

$$f(x) = \frac{e^{\sin(x)} + \tan(x) + 1}{\log(x) + 1}$$

Sol. $\lim_{x\to1} \frac{e^{\sin(x)} + \tan(x) + 1}{\log(x) + 1}$

$\frac{\lim_{x\to1} e^{\sin(x)} + \tan(x) + 1}{\lim_{x\to1} \log(x) + 1}$

$$\frac{e^{\sin(1)} + 1 + 1}{\log(1) + 1}$$

$$\frac{e^{\sin(1)} + 2}{0 + 1}$$

$$e^{\sin(1)} + 2$$

Ex. Calculate the $\lim_{x\to0} f(x)$ using substitution rule.

$$f(x) = \frac{\sin^2(x) + \cos(x) - 1}{1 - c\quad(x)}$$

Sol. $\lim_{x\to0} \frac{\sin^2(x) + \cos(x) - 1}{1 - \cos(x)}$

$$\lim_{x\to0} \sin^2(x) + \cos(x) - 1$$

$$\frac{\sin^2(0) + \cos(0) - 1}{1 - \cos(0)}$$

This expression is in the indeterminate form of 0/0, leading to an undefined limit. The substitution rule is not applicable in this case.

Ex. Calculate the value of $\lim_{x\to1} f(x)$.

$$f(x) = \begin{cases} e^x, & \text{if } x \geq 1 \\ e^{-x}, & \text{otherwise} \end{cases}$$

Sol. When $x = 1$, there is a shift in the function definition. Hence, it is not recommended to directly apply the rule. In such cases, one should examine the limit from both sides.

Left-hand Side Limit $\lim_{x\to1^-} e^{-x}$

Right-hand Side Limit $\lim_{x\to1^+} e^x$

In this case also, limits from both sides are different.

Factorization Method

In this approach, factorize (if feasible) both the numerator and denominator, and eliminate terms that result from $\frac{0}{0}$ the factorization.

Ex. $\lim_{x\to3} \frac{x^3 - 4x - 15}{x^3 + x^2 - 6x - 18}$

Sol. $\lim_{x\to3} \frac{(x-3)(x^2+3x+\ )}{(x-3)(x^2+4x+\ )}$

$$\lim_{x\to3} \frac{(x-3)(x^2+3x+5)}{(x-3)(x^2+4x+6)}$$

$$\lim_{x\to3} \frac{x^2+3x+5}{x^2+4x+6} = \frac{23}{27}$$

Ex. $\lim_{x\to0} \frac{\cos^2 x + \cos x - 2}{\sin^2 x}$

Sol. $\lim_{x\to0} \frac{\cos^2 x + 2co\ \ x - \cos x - 2}{1 - \cos^2 x}$

$$\lim_{x\to0} \frac{(\cos x + 2)(\cos x - 1)}{(1 + \cos x)(1 - \cos x)}$$

$$\lim_{x\to0} \frac{-(\cos x + 2)}{1 + \cos x} = -\frac{3}{2}$$

Ex. $\lim_{x\to\frac{\pi}{2}} \frac{2x-\pi}{\cos x}$

Sol. $\lim_{x\to\frac{\pi}{2}} \frac{2x-\pi}{\cos x}$, Put $x = \frac{\pi}{2} + h$ when $x \to \frac{\pi}{2}, h \to 0$

$$\lim_{h\to 0} \frac{2(\frac{\pi}{2}+h)-\pi}{\cos(\frac{\pi}{2}+h)}$$

$$\lim_{h\to 0} \frac{2h}{\sin h} = 2$$

Evaluation of Algebraic Limits at Infinity

Limits at infinity are employed to characterize the behavior of a function as the independent variable increases or decreases without bound. If the function approaches a numerical value L, in either of these scenarios, express it as:

$$\lim_{x\to+\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L$$

The function f(x) is acknowledged to possess a horizontal asymptote at $y = L$.A function might exhibit distinct horizontal asymptotes in each direction, contain a horizontal asymptote in just one direction, or lack horizontal asymptotes altogether.

Ex. Evaluate $\lim_{x\to+\infty} \frac{2x^2+3}{x^2-5x-1}$

Factor out the highest power of "x" from each term in the numerator and the highest power of "x" from each term in the denominator.

Sol. It is determined that

$$\lim_{x\to+\infty} \frac{2x^2+3}{x^2-5x-1} = \lim_{x\to+\infty} \frac{x^2(2+\frac{3}{x^2})}{x^2(1-\frac{5}{x}-\frac{1}{x^2})}$$

$$\lim_{x\to+\infty} \frac{2+\frac{3}{x^2}}{1-\frac{5}{x}-\frac{1}{x^2}}$$

$$\frac{2+0}{1-0-0}$$

$$\lim_{x\to+\infty} \frac{2x^2+3}{x^2-5x-1} = 2$$

The function has a horizontal asymptote at $y = 2$.

Ex. Evaluate $\lim_{x\to+\infty} \frac{9x^2}{x+2}$

Factor out x^2 from each term in the numerator and "x" from each term in the denominator, resulting in:

$$\lim_{x\to+\infty} \frac{9x^2}{x+2} = \lim_{x\to+\infty} \frac{x^2(9)}{x(1+\frac{2}{x})}$$

$$\lim_{x\to+\infty} x(\frac{9}{1+\frac{2}{x}})$$

$$[\lim_{x\to+\infty} (x)][\frac{9}{1+0}]$$

$$[\lim_{x\to+\infty} (x)][9]$$

$$\lim_{x\to+\infty} \frac{9x^2}{x+2} = +\infty$$

As this limit does not converge to a real number value, the function does not exhibit a horizontal asymptote as x increases indefinitely.