ALGEBRA OF LIMITS

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exists then

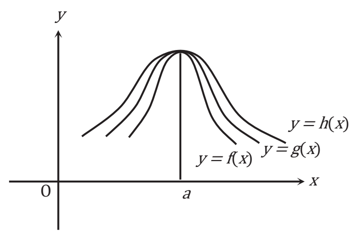
1. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
2. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}\ (\lim_{x\to a} g(x) \neq 0)$
3. $\lim_{x\to a}[Kf(x)] = K\lim_{x\to a} f(x)$
4. $\lim_{x\to a}[f(x)]^{g(x)} = [\lim_{x\to a} f(x)]^{\lim_{x\to a} g(x)}$
5. $\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)}$
6. $\lim_{x\to a} \log f(x) = \log[\lim_{x\to a} f(x)]$
7. $\lim_{x\to a} f(x) \times g(x) = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$
8. $\lim_{x\to a} f \circ g(x) = f[\lim_{x\to a} g(x)]$(Provided f(x) is contineous at $x = \lim_{x\to a} g(x)$)
9. Sandwich theorem:- If there exists a function h(x) such that
   $f(x) \le h(x) \le g(x) \forall x$ and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ Then $\lim_{x\to a} h(x) = \lim_{x\to a} g(x)$ or $\lim_{x\to a} f(x)$

Theorem 2 (Sandwich Theorem)

Let f, g and h be real functions such that $f(x) \le g(x) \le h(x)$ for all x in the common domain of definition. For some real member a,

$$\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$$
$$\lim_{x\to a} g(x) = 1.$$

This is illustrated in figure:

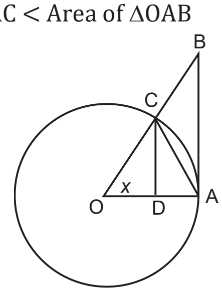


Here is a geometric demonstration of the following significant inequality involving trigonometric functions.

$$\cos x < \frac{\sin x}{x} < 1 \text{ for } 0 < |x| < \frac{\pi}{2} \quad \dots (i)$$

Proof: We are aware that sin (–x) and cos (–x) = cos x. Therefore, it is adequate to establish the inequality for $0 < x < \frac{\pi}{2}$. In the illustration, O is the centre of the unit circle, with angle AOC measuring x radians, and $0 < x < \frac{\pi}{2}$. Perpendicular line segments BA and CD are drawn to OA. Additionally, we connect AC.

Area of ΔOAC < Area of sector OAC < Area of ΔOAB



$$\frac{1}{2} OA \cdot CD < \frac{x}{2\pi} \cdot \pi \cdot (OA)^2 < \frac{1}{2} OA \cdot AB$$
$$CD < x.0A < AB$$
$$\frac{1}{2} OA \cdot CD < \frac{x}{2\pi} \cdot \pi \cdot (OA)^2 < \frac{1}{2} OA \cdot AB$$

$$CD < x.0\,A < AB$$

From ΔOCD $\qquad \sin x = \frac{CD}{OA}$

(Since OC = OA) and hence CD = OA sin x

$$\tan x = \frac{AB}{OA}$$

$$AB = 0A \cdot \tan x$$

$$OA\sin x < 0A \cdot x < 0A \cdot \tan x$$

Since length OA is positive, we have

$$\sin x < x < \tan x$$

Since $0 < x < \frac{\pi}{2}$, sin x is positive and thus by dividing throughout by sin x, $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$

Taking reciprocals throughout, we have $\cos x < \frac{\sin x}{x} < 1$, which complete the proof.

$$\lim_{x\to 0}\cos x < \lim_{x\to 0}\frac{\sin x}{x} < \lim_{x\to 0}1$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

Ex. Evaluate: $\lim_{x\to 2}\frac{x-3}{x+4}$

Sol. $\lim_{x\to 2}\frac{x-3}{x+4} = \frac{2-3}{2+4} = -\frac{1}{6}$

Ex. Evaluate: $\lim_{x\to 2}\frac{x^5-32}{x^3-8}$

Sol. $\lim_{x\to 2}\frac{x^5-32}{x^3-8} = \lim_{x\to 2}(\frac{x^5-32}{x-2}) \div (\frac{x^3-8}{x-2})$

$$\lim_{x\to 2}(\frac{x^5-2^5}{x-2}) \div \lim_{x\to 2}(\frac{x^3-2^3}{x-2})$$

$$5(2)^4 \div 3(2)^2 = \frac{20}{3}$$

$$[\text{ As }\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}]$$