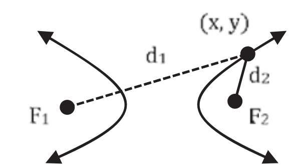
Chapter 16

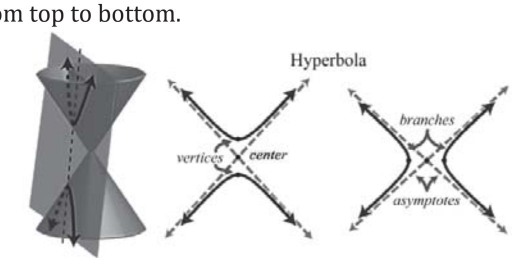
Hyperbola

INTRODUCTION

A **hyperbola** can be defined as the collection of points within a plane, where the absolute difference between their distances from two fixed points known as foci remains constant and positive. In simpler terms, if we have foci F_1 and F_2, and a given positive constant denoted as 'd,' a point (x, y) lies on the hyperbola if $d = |d_1 - d_2|$, as illustrated in the diagram below:



Furthermore, a hyperbola takes shape when a cone is intersected by a slanted plane that crosses its base. It comprises two distinct segments known as **branches**. The points on each branch where the distance is at its minimum are referred to as **vertices**. The center of a hyperbola is positioned at the midpoint between its vertices. Unlike a parabola, a hyperbola displays asymptotic behavior in relation to specific lines drawn through its center. In this section, our attention will be directed towards plotting hyperbolas that have an orientation either from left to right or from top to bottom.



The dashed lines represent the asymptotes and are not considered part of the actual graph; they merely depict the graph's behavior at its extremities.

STANDARD EQUATION OF THE HYPER BOLA

The trajectory of a point moving in a plane, where the ratio of its distance from a fixed point to a fixed straight line is consistently greater than unity, is termed a hyperbola. This fixed point is referred to as the focus, the fixed straight line is termed the directrix, and the constant ratio is known as the eccentricity of the hyperbola.

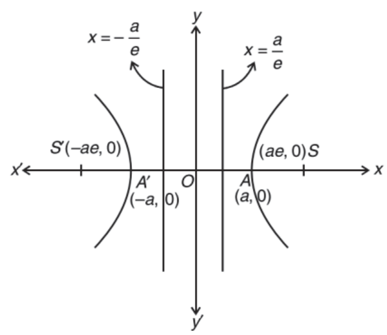
A hyperbola can also be defined as the path followed by a point in a plane, where the difference of its distances from two fixed points (ae, 0) and (-ae, 0) is consistently 2a.

Consider a moving point P(x, β). According to the given conditions,

$$\sqrt{(\alpha+ae)^2+\beta^2}-\sqrt{(\alpha-ae)^2+\beta^2}=2a$$
$$(\alpha+ae)^2+\beta^2=4a^2-4a\sqrt{(\alpha-ae)^2+\beta^2}+(\alpha-ae)^2+\beta^2$$
$$4ae\alpha-4a^2=-4a^2\sqrt{(\alpha-ae)^2+\beta^2}$$
$$e^2\alpha^2+a^2-2ae\alpha=\alpha^2+a^2e^2-2ae\alpha+\beta^2$$
$$(e^2-1)\alpha^2-\beta^2=a^2(e^2-1)$$
$$\frac{\alpha^2}{a^2}-\frac{\beta^2}{a^2(e^2-1)}=1$$
$$\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}=1b^2=a^2(e^2-1),\text{ say}$$

Locus of $P(\alpha,\beta)$ is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Thus $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, b^2=a^2(e^2-1)$ Represent a hyperbola with following sepcification.



The center of the hyperbola is O at the point (0, 0), with foci located at S (ae, 0) and S (–ae, 0), and the directrices are $x=\frac{a}{e}$ and $x=-\frac{a}{e}$ The vertices of the hyperbola are A(a, 0) and A'(-a, 0), and the line segment AA' is referred to as the transverse axis of the hyperbola.

The $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ hyperbola does not intersect the y-axis. Nevertheless, we select two points, B and B', on the y-axis, positioned on opposite sides of the center O of the hyperbola in a manner such that.

$$OB=OB'=b$$

B'B is denoted as the conjugate axis of the hyperbola. The endpoints of the latera recta are.

$$(ae,\frac{b^2}{a}),(ae,-\frac{b^2}{a})$$

$(-ae,\frac{b^2}{a})$ and $(-ae,-\frac{b^2}{a})$ And the length of latus rectum is $\frac{2b^2}{a}$.the equation of latera recta are

$$x=ae \text{ and } x=-ae$$

The constant difference between the focal distances of any point on a hyperbola is equal to the length of the transverse axis of the hyperbola.

When the transverse axis of one hyperbola coincides with the conjugate axis of another, and vice versa, these hyperbolas are termed conjugate to each other.

The equation of the hyperbola that is conjugate to the hyperbola is. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$$
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$$
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$$

Represents a hyperbola with the following specifications

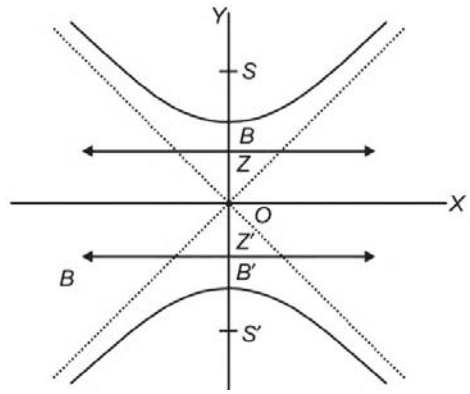
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

where $a^2 = b^2(e^2 - 1)$

Centre O: (0,0)

Vertices B and B′: $(0, \pm b)$

Foci S and S′$(0, \pm be)$



Equations of the directrices : $y = \pm \frac{b}{e}$

Equation of the transverse axis : $x = 0$

Equation of the conjugate axis : $y = 0$

Length of the transverse axis $= 2b$

Length of the conjugate axis $= 2a$

Equation of the latus rectum : $y = \pm\, be$

End points of the latus rectum :

$\left(\pm \frac{a^2}{b}, be\right), \left(\pm \frac{a^2}{b}, -be\right)$

Length of the latus rectum $= \frac{2a^2}{b}$

Ex. If $\frac{x^2}{36} - \frac{y^2}{64} = 1$If the equation \(x^2 - y^2 = 1\) represents a hyperbola, determine the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum.

Sol. By comparing the provided equation. $\frac{x^2}{36} - \frac{y^2}{64} = 1$with the standard equation of the hyperbola, that is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$$a = 6 \text{ and } b = 8 \Rightarrow c = \sqrt{a^2 + b^2} = 10$$

The coordinates of the foci are $(\pm 10,\ 0)$, and the vertices are $(\pm 6,\ 0)$. The eccentricity can be expressed as.

$$e = \frac{c}{a}$$

$$e = \frac{10}{6} = \frac{5}{3}$$

Length of the latus rectum $= \frac{2b^2}{a} = \frac{2(64)}{6} = \frac{64}{3}$

Ex. Determine the equation of the hyperbola given the foci at $(\pm 5,\ 0)$ and vertices at $(\pm 3,\ 0)$.

Sol. As the foci lie on the x-axis, the hyperbola's equation takes the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Since, vertices are $(\pm 3,0)$, therefore $a = 3$ i.e., $a^2 = 9$

Also, since foci are $(\pm 5,0)$, therefore $c = 5$

Now, $b^2 = c^2 - a^2$

$$b^2 = 25 - 9$$

$$b^2 = 16$$

Therefore, the equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$

Ex. Determine the equation of the hyperbola with foci at $(0, \pm 10)$ and a latus rectum length of 30.

Sol. As the foci are located on the y-axis, the equation of the hyperbola can be expressed in the following form:

Now, co-ordinates of the foci are $(0, \pm 10) \Rightarrow c = 10$

Length of the latus rectum $= \frac{2b^2}{a} = 30 \Rightarrow b^2 = 15a$

We, know that $c^2 = a^2 + b^2$

$$(10)^2 = a^2 + 15a$$

$$a^2 + 15a - 100 = 0$$

$$a^2 + 20a - 5a - 100 = 0$$

$$a(a+20)-5(a+20)=0$$
$$a=5 \quad \text{... (2)[As a cannot be negative]}$$

Equations (1) and (2) implies $c^2=25+b^2$

$$(10)^2=25+b^2$$
$$b^2=100-25=75$$

$\frac{y^2}{25}-\frac{x^2}{75}=1$ is the required equation of the hyperbola.

Parametric Form And Auxiliary Circle

The parametric equation of the hyperbola is expressed as follows:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ are } x=a\sec\theta, y=b\tan\theta$$

The circle formed with the transverse axis of the hyperbola as the diameter is termed the auxiliary circle of the hyperbola. Its equation is given by: $x^2+y^2=a^2$

Ex. **1.** Determine the equation of the hyperbola whose axes coincide with the axes of coordinates, given that the distance between the foci is 16 and the eccentricity is $\sqrt{2}$.

2. For the hyperbola represented by the equation $4x^2$ - $9y^2$ = 36, ascertain the axes, coordinates of the foci, eccentricity, and the length of the latus rectum.

Sol. **1.** According to the equation:

$$2ae=16$$
$$2a\cdot\sqrt{2}=16$$
$$a=4\sqrt{2}$$
$$b^2=a^2(e^2-1)=a^2(2-1)=a^2=32$$

Hence the required equation of the hyperbola is:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$
$$x^2-y^2=32$$

2. The given equation is:

$$4x^2-9y^2=36$$
$$\frac{x^2}{9}-\frac{y^2}{4}=1$$

Comparing the given equation with the standard equation of the hyperbola:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

We get: $a^2=9, b^2=4 \Rightarrow a=3, b=2$

Length of transverse axis $=2a=6$

Length of conjugate axis $=2b=4$

$$\text{Eccentricity }=e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{4}{9}}=\frac{\sqrt{13}}{3}$$

The coordinate of the foci are $(\pm ae, 0)$ hence the foci of the given hyperbola are:

$$\left(3\frac{\sqrt{13}}{3}, 0\right) \text{ and } \left(-3\frac{\sqrt{13}}{3}, 0\right)$$
$$(\sqrt{13}, 0) \text{ and } (-\sqrt{13}, 0)$$

Length of latus rectum $=\frac{2b^2}{a}=2\cdot\frac{4}{3}=\frac{8}{3}$

Ex. If e and e' represent the eccentricities of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and its conjugate, respectively, demonstrate $\frac{1}{e^2}+\frac{1}{e'^2}=1$ the following relation.

Sol. The eccentricity e of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is given by $b^2=a^2(e^2-1)$ and the eccenticity e' of the conjugate hyperbola $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ is given by $a^2=b^2(e'^2-1$ multiplaying then we get.

$$b^2a^2=a^2(e^2-1)\cdot b^2(e'^2-1)$$
$$1=e^2e'^2-e^2-e'^2+1$$

$$e^2 + e'^2 = e^2e'^2$$
$$\frac{1}{e^2} + \frac{1}{e'^2} = 1$$

Point Of Intersection Of A Straight Line And A Hyperbola

Consider the straight line given by the equation $y = mx + c$, which intersects the hyperbola.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

At the point of intersection, we obtain:

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$$
$$(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(b^2 + c^2) = 0 \quad \text{... (1)}$$

This forms a quadratic equation in x, yielding two values for the x-coordinates of the points of intersection.

$$x_1 + x_2 = \frac{2a^2mc}{b^2-a^2m^2} \text{ and } x_1x_2 = -\frac{a^2(b^2+c^2)}{b^2-a^2m^2}$$

The corresponding y-coordinates, y_1 and y_2, can be determined by substituting x_1 and x_2 into the equation $mx + c$. The straight line becomes a tangent to the given hyperbola if the roots of the quadratic equation are equal.

$$4a^4m^2c^2 + 4(b^2 + c^2)a^2(b^2 - a^2m^2) = 0$$
$$b^2c^2 + b^4 - a^2b^2m^2 = 0$$
$$c^2 = a^2m^2 - b^2$$
$$c = \pm\sqrt{a^2m^2 - b^2}$$

Therefore, the equation of the tangent in slope form to the hyperbola is as follows: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be written us.

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

Equation of Tangent

1. The equation for the tangent at the point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is expressed as:
$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$
2. The equation in parametric form, namely the equation for the tangent at the point $(a\sec\theta, b\tan\theta)$ is expressed as:
$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$$
3. The equation representing the tangent in slope form to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} =$ is expressed as:
$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

Equation Of Normal

1. The equation for the normal at the point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by:
$$\frac{x-x_1}{\frac{x_1}{a^2}} = -\frac{y-y_1}{\frac{y_1}{b^2}}$$
2. The equation in parametric form for the normal at the point $(a\sec\theta, b\tan\theta)$ is expressed as:
$$ax\cos\theta + by\cot\theta = a^2 + b^2$$

Direct Circle of Hyperbola

The set of points where mutually perpendicular tangents intersect on a hyperbola forms a circle known as the director circle of the hyperbola.

The equation for any tangent to the hyperbola is: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$y - mx = \sqrt{a^2m^2 - b^2}$$

Let it passes through (h, k)

$$(k - mh)^2 = a^2m^2 - b^2$$

$$m^2 \cdot (h^2 - a^2) - 2khm + (k^2 + b^2) = 0$$

which is quadratic equation in m

Let roots of this equation be m_1, m_2.

$$m_1m_2 = \frac{k^2+b^2}{h^2-a^2} \quad \text{... (1)}$$

Tangents are perpendicular

$$m_1m_2 = -1$$

From (1) and (2), we get

$$\frac{k^2+b^2}{h^2-a^2} = -1$$

$$h^2 + k^2 = a^2 - b^2$$

The locus of (h, k) is $x^2 + y^2 = a^2 - b^2$

In hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $a > b$ only then real director circle can be drawn. If $a < b$ then angel between tangents in always acute.

Equation Of Chord Whose Middle Point Is (x_1, y_1)

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$

$$T = S_1$$

Equation Of The Pair Of Tanhents From (x_1, y_1)

$$SS_1 = T^2$$

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2$$

Diameter and conjugate diameter of a hyperbola

The path traced by the midpoints of a set of parallel chords on a hyperbola is referred to as a diameter of the hyperbola. Two such diameters, each bisecting chords parallel to the other, are termed conjugate diameters $y = \frac{b^2}{a^2m}x$ The equation $x = m$ represents a diameter with m as the slope of the system of parallel chords. Two diameters, $y = m_1x$ and $y = m_2x$, are conjugate diameters with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $m_1m_2 = \frac{b^2}{a^2}$.

Ex. Demonstrate that the curve formed by the intersection points of tangents to a hyperbola, meeting at a constant angle β, is given by: $(x^2 + y^2 + b^2 - a^2)^2 = 4\cot^2\beta(a^2y^2 - b^2x^2 + a^2b^2)$.

Sol. Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The equation of a tangent to the hyperbola may be taken as

$$y = mx + \sqrt{a^2m^2 - b^2}$$

Which will pass through a point (x_1, y_1), if

$$y_1 = mx_1 + \sqrt{a^2m^2 - b^2}$$

$$m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 + b^2) = 0$$

Which is a quadratic equation in m, giving two values of m, say m_1 and m_2 where

$$m_1 + m_2 = \tan\theta_1 + \tan\theta_2 = \frac{2x_1y_1}{x_1^2-a^2}$$

$$m_1m_2 = \tan\theta_1\tan\theta_2 = \frac{y_1^2+b^2}{x_1^2-a^2}$$

If the two tangents meet at an angle β, then

$$\beta = \theta_1 - \theta_2$$

$$\cot\beta = \cot(\theta_1 - \theta_2)$$

$$= \frac{1+\tan\theta_1\tan\theta_2}{\tan\theta_2-\tan\theta_1}$$

$$= \frac{1+\tan\theta_1\tan\theta_2}{\sqrt{(\tan\theta_1+\tan\theta_2)^2-4\tan\theta_1\tan\theta_2}}$$

$$\cot^2\beta = \frac{(x_1^2+y_1^2+b^2-a^2)^2}{4(a^2y_1^2-b^2x_1^2+a^2b^2)}$$

Hence the required locus of (x_1, y_1) is

$$(x^2+y^2+b^2-a^2)^2 = 4\cot^2\beta(a^2y^2-b^2x^2+a^2b^2)$$