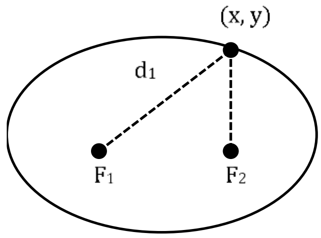
Chapter 12

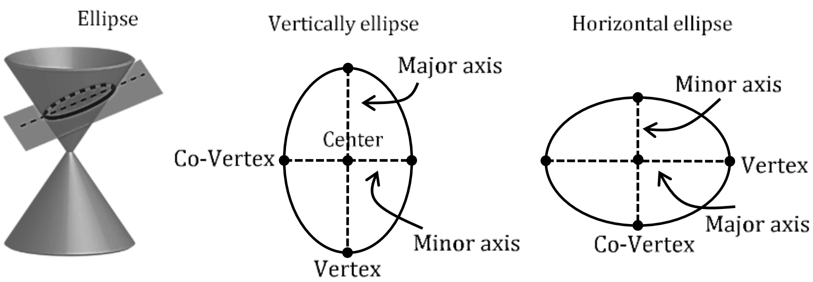
Ellipse

INTRODUCTION

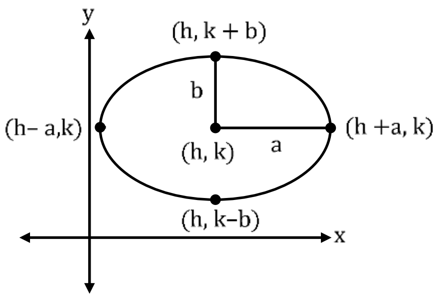
An **ellipse** can be defined as the collection of points within a plane where the combined distances from two designated fixed points, referred to as foci, always sum up to a constant positive value. In simpler terms, if we denote the two foci as F_1 and F_2 and set a given positive constant as d, then any point (x, y) is considered part of the ellipse if the equation $d = d_1 + d_2$ holds true, as depicted in the illustration below:



Furthermore, an ellipse can be generated by the intersection of a cone with a slanting plane that doesn't align with the cone's side and doesn't cross the cone's base. Within this elliptical shape, the points where the distance between them is greatest are termed vertices, and these points delineate the major axis. The center of an ellipse is located at the midpoint between these vertices. The **minor axis** is the line segment that passes through the ellipse's center, marked by two points on the ellipse where the distance between them is at its minimum. The extremities of the minor axis are referred to as **co-vertices.**



When the major axis of an ellipse aligns parallel to the x-axis in a Cartesian coordinate plane, we refer to it as a horizontal ellipse. Conversely, if the major axis aligns parallel to the y-axis, we describe it as a vertical ellipse. For the scope of this section, our focus is primarily on sketching these two types of ellipses. Nevertheless, it's important to recognize that ellipses find extensive use in various real-world applications, and further exploration of this fascinating subject is encouraged. In a rectangular coordinate plane, when the center of a horizontal ellipse is denoted as (h, k), we have the following:

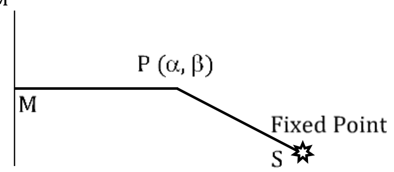


As depicted, when a is greater than b, a being half the length of the major axis, it's referred to as the **major radius.** Similarly, b, which is half the length of the minor axis, is known as the **minor radius**

STANDARD EQUATION OF ELLIPSE

An ellipse is the locus of a point in a plane where the ratio of its distances from a fixed point to a fixed straight line remains constant and is consistently less than one.

$$\frac{SP}{PM} = e < 1, \text{ then locus of S is an ellipse.}$$



The focus is the designated fixed point, and the directrix is the fixed straight line of the ellipse. Additionally, an ellipse is defined as the path of a point in a plane where the sum of its distances from two fixed points, namely (ae, 0) and (-ae, 0), consistently equals 2a.

Consider a moving point P(α, β) and fixed points S(ae, 0) and S'(-ae, 0) such that

$$PS + PS' = 2a$$

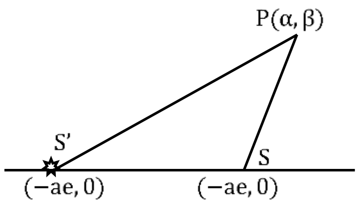
$$\sqrt{(\alpha - ae)^2 + \beta^2} + \sqrt{(\alpha + ae)^2 + \beta^2} = 2a$$

$$(\alpha + ae)^2 + \beta^2 = 4a^2 - 4a\sqrt{(\alpha - ae)^2 + \beta^2} + (\alpha - ae)^2 + \beta^2$$

$$4ae\alpha - 4a^2 = -4a\sqrt{(\alpha - ae)^2 + \beta^2}$$

$$e^2\alpha^2 + a^2 - 2ae\alpha = \alpha^2 + a^2e^2 - 2ae\alpha + \beta^2$$

$$a^2(1 - e^2) = \alpha^2(1 - e^2) + \beta^2$$

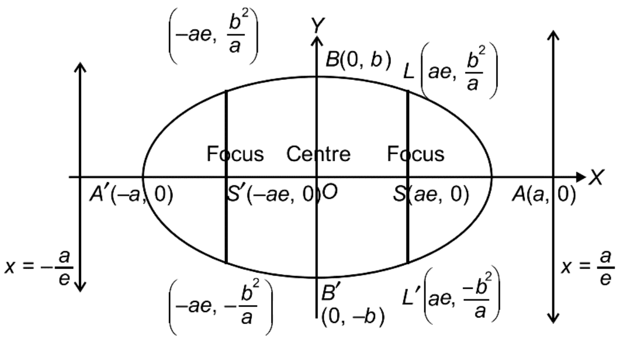


$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{a^2(1-e^2)} = 1$$

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1, b^2 = a^2(1 - e^2)$$

Locus of (α, β) is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2(1 - e^2)$

Thus $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > b)$ represents an ellipse with following specifications :



1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$

where $b^2 = a^2(1 - e^2)$

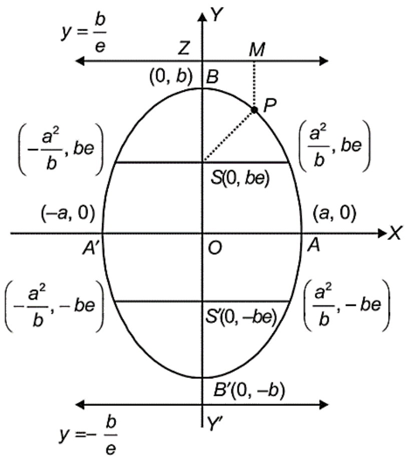
Centre O: (0,0)

Vertices A and A′: $(\pm a, 0)$

Foci S and S′: $(\pm ae, 0)$

Equations of the directrices : $x = \pm \frac{a}{e}$
Equation of the major axis $y = 0$
Equation of the minor axis $x = 0$
Length of the major axis $= 2a$
Length of the minor axis $= 2b$
End points of the minor axis B and $B' = (0, \pm b)$
Equation of the latus rectum : $x = \pm ae$
End points of the latus rectum : $(ae, \pm \frac{b^2}{a}), (-ae, \pm \frac{b^2}{a})$
Length of the latus rectum $= \frac{2b^2}{a}$
Focal distance of a point $(x, y) = a \pm ex$

2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a < b$
where $a^2 = b^2(1 - e^2)$
Centre O: (0,0)
Vertices B and B': $(0, \pm b)$
Foci S and S': $(0, \pm be)$
Equations of the directrices : $y = \pm \frac{b}{e}$
Equation of the major axis : $x = 0$
Equation of the minor axis : $y = 0$
Length of the major axis $= 2b$
Length of the minor axis $= 2a$
End points of the minor axis A and A': $(\pm a, 0)$
Equation of the latus rectum : $y = \pm be$



Ex. For the ellipse defined by the equation $x^2 + 4y^2 = 16$, determine the lengths of the major and minor axes, eccentricity, foci, and vertices.

Sol. Initially, we'll transform the equation into standard form by dividing by 64, resulting in:

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$
$$\frac{x^2}{(4)^2} + \frac{y^2}{(2)^2} = 1$$
$$a = 4 \text{ and } b = 2$$

As the denominator of x^2 is greater, the major axis aligns with the x-axis, and the minor axis aligns with the y-axis.

1. The lengths of the major and minor axes are 8 and 4, respectively.

2. Eccentricity i.e., $e = \frac{c}{a}$

$$c = \sqrt{a^2 - b^2} \quad \dots (1)$$

Dividing (1) by ' a ', we get

$$e = \frac{\sqrt{a^2 - b^2}}{a}$$
$$e = \frac{\sqrt{16-4}}{4}$$
$$e = \frac{2\sqrt{3}}{4}$$
$$e = \frac{\sqrt{3}}{2}$$

3. The co-ordinates of the foci are (c, 0) and (-c, 0) where
$c = \sqrt{a^2 - b^2} = \sqrt{16 - 4} = \sqrt{12} = 2\sqrt{3}$ $\therefore$ $(2\sqrt{3}, 0)$ and $(-2\sqrt{3}, 0)$ are the required co-ordinates.

4. The coordinates of the vertices are (a, 0) and (-a, 0), which are (4, 0) and (-4, 0) respectively.

Ex. Determine the equation of the ellipse with vertices at $(\pm 5, 0)$ and foci at $(\pm 4,0)$.

Sol. As the vertices lie on the x-axis, the equation will take the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Where a is the semi - major axis.

Given that $a = 5$ and $c = 4$

From the relation $c^2 = a^2 - b^2$, we get

$$16 = 25 - b^2$$
$$b^2 = 9$$

Hence, the equation of the ellipse is $\frac{x^2}{25}+\frac{y^2}{9}=1$

Ex. Determine the equation of the ellipse with foci at $(0, \pm 4)$ and eccentricity $\frac{4}{5}$.

Sol. Given that the foci are located on the y-axis, the major axis aligns with the y-axis. Thus, the equation of the ellipse takes the form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$.

Given that, $c = 4$ and $e = \frac{4}{5}$

We know that $e = \frac{c}{a}$

$$a = 5 \text{ i.e., } a^2 = 25$$
$$c^2 = a^2 - b^2$$
$$(4)^2 = (5)^2 - b^2$$
$$b^2 = 25 - 16$$
$$b^2 = 9$$

The required equation of the ellipse is $\frac{x^2}{9}+\frac{y^2}{25}=1$

Ex. If the eccentricity of an ellipse is $\frac{4}{9}$ and the separation between its foci is 8 units, determine the length of the latus rectum of the ellipse.

Sol. It is specified that $e = \frac{4}{9}$ and the distance between the foci of the ellipse, denoted as 2c, is 8, which implies $c = 4$.

$$4 = a(\tfrac{4}{9})$$
$$a = \frac{36}{4}$$
$$a = 9$$
$$b^2 = a^2(1 - e^2)$$
$$b^2 = 81(1 - \tfrac{16}{81}) = 81 \times \frac{65}{81} = 65$$

Now the length of the latus rectum is given by. $\frac{2b^2}{a}$ i.e., $\frac{2(65)}{9} = 14.4$ units

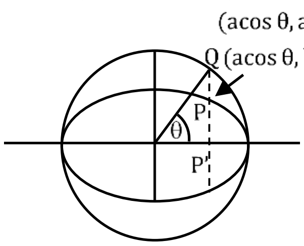
Parametric form and Auxiliary Circle

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \ldots (1)$$

$x = a\cos\theta, y = b\sin\theta$ is the parametric form of the ellipse (1)

Assume a circle is being drawn on the major axis of the ellipse with its diameter; thus, its equation becomes. $x^2 + y^2 = a^2$

Let three be a point on this circle $(a\cos\theta, a\sin\theta)$



A perpendicular is extended from Q to the major axis, intersecting the ellipse at points P and P'. The coordinates of point P are (acosθ, k), and it resides on the ellipse.

$$\frac{a^2\cos^2\theta}{a^2}+\frac{k^2}{b^2}=1$$

$$k=\pm b\sin\theta$$

Hence a general point on ellips $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$is described as $(a\cos\theta, b\sin\theta)$.

The circle with the equation $x^2+y^2=a^2$ is referred to as the auxiliary circle, and θ is termed the eccentric angle of the ellipse.

Ex. Determine the equation of the ellipse with foci at (2, 3) and (-2, 3) and a semi-minor axis of$\sqrt{5}$.

Sol. **1st Method**

Here $S\equiv(2,3), S'\equiv(-2,3), b=\sqrt{5}$

$$SS'=4\Rightarrow 2ae=4$$

$$ae=2$$

$$b^2=a^2(1-e^2)\Rightarrow 5=a^2-4\Rightarrow a=3$$

Let $P(\alpha,\beta)$ be any point on the ellipse, then

$$SP+S'P=2a=6$$

$$\sqrt{(\alpha-2)^2+(\beta-3)^2}+\sqrt{(\alpha+2)^2+(\beta-3)^2}=6$$

Upon simplification, it can be condensed to $5\alpha^2+9\beta^2-54y+36=0$

Hence locus of (α,β) is

$$5x^2+9y^2-54y+36=0$$

This represents the desired equation of the ellipse.

2nd Method

The centre C of the ellipse $=$ mid point of $S\&S'=(0,3)$

$$b=\sqrt{5}, 2ae=4\Rightarrow ae=2b^2=a^2(1-e^2) \text{ given } a=3$$

The slope of SS' is zero, indicating that the major axis of the ellipse is parallel to the x-axis, and the minor axis is parallel to the y-axis.

Hence the required equation of the ellipse is

$$\frac{(x-0)^2}{9}+\frac{(y-3)^2}{5}=1$$

$$5x^2+9y^2-54y+36=0$$

Position Of A Point

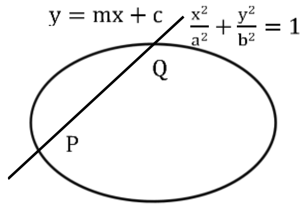
Let $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

The point (x_1,y_1) will be outside, on, or inside the ellipse, respectively, if $S_1>0,=0,<0$.

Point Of Intersection Of A Straight Line And An Ellipse

Consider the equation of a given straight line as $y=mx+c$, where $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the ellipse.

The intersection points of the line and the ellipse can be determined by solving the two equations simultaneously.



Thus $\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$

$$(b^2+a^2m^2)x^2+2a^2mcx+a^2(c^2-b^2)=0$$

Being a quadratic equation in x, it provides the abscissae of the points of intersection. The given line will intersect, touch, or neither touch nor cut the given ellipse depending on whether the discriminant of the above equation is greater, equal to, or less than zero.

$$4a^4m^2c^2 - 4(a^2)(c^2 - b^2)(b^2 + a^2m^2) > \text{ or } =< 0$$
$$a^2m^2c^2 - b^2c^2 + b^4 - a^2m^2c^2 + a^2m^2b^2 \gtreqless 0$$
$$c^2 \gtreqless a^2m^2 + b^2$$

Therefore, the line $y = mx + c$ will touch the given ellipse if $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$c^2 = a^2m^2 + b^2$$
$$c = \pm\sqrt{a^2m^2 + b^2}$$

Therefore, the equations of the tangents to the ellipse in slope form can be expressed as

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

Equation Of Tangent

1. Point Form:

The equation for the tangents to the ellipse can be expressed as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

2. Parametric Form:

The equation for the tangents to the ellipse can be expressed as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a\cos\theta, b\sin\theta)$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

3. Slope Form:

The equation of the tangents to the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ For a slope m it can be expressed as.

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

Direct Circle Of An Ellipse

The path followed by the point where mutually perpendicular tangents to an ellipse intersect is a circle known as the director circle of the ellipse.

The equation for any tangent to the ellipse is.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

In slope form, it can be expressed as.

$$y - mx = \sqrt{a^2m^2 + b^2} \qquad \text{... (1)}$$

And the eqaution of tangent perpendicular to (i) is.

$$x + my = \sqrt{a^2 + b^2m^2} \text{ (Replacing m by } -\frac{1}{m} \text{ in equation (1))} \qquad \text{... (2)}$$

The desired locus of the point of intersection is determined by eliminating m between equations (1) and (2).

Squaring (1) and (2) and then adding we get

$$(1 + m^2)(x^2 + y^2) = (1 + m^2)a^2 + (1 + m^2)b^2$$
$$x^2 + y^2 = a^2 + b^2$$

Which is the equation of the director circle of given ellipse.

Equation Of The Normal

1. Point Form:

Normal at (x_1, y_1) is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

2. Parametric Form:

Normal at $(a\cos\theta, b\sin\theta)$ is $ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$

$$\tan^4\frac{\theta}{2} + 2(ax + a - b)\tan^3\frac{\theta}{2} + 2(ax - a + b)\tan\frac{\theta}{2} - by = 0,$$

Which will be a biquadratic equation in $\tan\frac{\theta}{2}$ for fixed x and y. hence from a fixed point atmost four real normals can be drawn to the ellipse.

Ex. Determine the locus of the midpoint of the segments of tangents enclosed between the axes.

Sol. Consider $M(\alpha, \beta)$ as the midpoint of the segment of the tangent at $P(a\cos\theta, b\sin\theta)$ on the ellipse. The equation of the tangent is then given by.

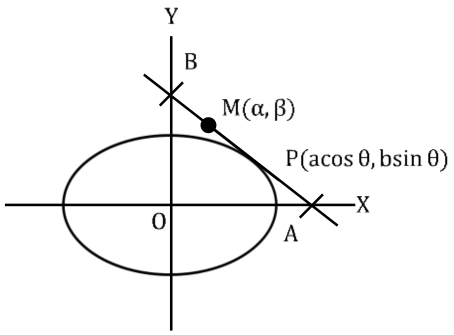
$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$
$$\frac{x}{a\sec\theta} + \frac{y}{b\operatorname{cosec}\theta} = 1$$

Whose intercepts on x - axis and y - axis are a $\sec\theta$ and $\operatorname{cosec}\theta$ respectively.

$$\alpha = \frac{a\sec\theta}{2} \Rightarrow \cos\theta = \frac{a}{2\alpha}$$
$$\beta = \frac{b\operatorname{cosec}\theta}{2} \Rightarrow \sin\theta = \frac{b}{2\beta}$$
$$\cos^2\theta + \sin^2\theta = \frac{a^2}{4\alpha^2} + \frac{b^2}{4\beta^2} = 1$$

Locus of (α, β) is

$$\frac{a^2}{x^2} + \frac{b^2}{y^2} = 4$$



Equation Of Chord Passing Through $(a\cos\alpha, b\sin\alpha)$, $(a\cos\beta, b\sin\beta)$

The equation for the chord is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

If a chord passes through the foci, then the chord constitutes the major axis of the ellipse.

Equation Of Chord With Middle Point (x_1, y_1)

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$$
$$T = S_1$$

Equation Of Pair Of Tangents From External Point (x_1, y_1)

Equation is $T^2 = SS_1$

$$\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right)$$

Diameter And Conjuget Diameter Of An Ellipse

The path followed by the midpoint of a set of parallel chords of an ellipse is known as a diameter of the ellipse. Two diameters of an ellipse are considered conjugate diameters if each one bisects the chords parallel to the other.

Consider the midpoint of a chord parallel to the given chord $y = mx$, be (h, k).

The equation for the chord is

$$\frac{xh}{a^2} + \frac{yk}{b^2} - 1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$$

Slope of this chord $= -\frac{h}{a^2}\cdot\frac{b^2}{k} = m$

Locus of (h, k) is $y = -\frac{b^2}{a^2 m}x$

if $y = m_1x$ and $y = m_2x$ are the conjugate diameters then

$$m_2 = -\frac{b^2}{a^2m_1}$$

$$m_1m_2 = \frac{-b^2}{a^2}$$

Pole and Polar

The equation for the polar of a point (x, y) with respect to the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ the point (x_1, y_1) is called the pole of ths polar.

Equation of the chord of contact

The equation for the chord of contact of tangents drawn from the point (x_1, y_1) to the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

Ex. Determine the eccentricity of the ellipse when $y = x$ and $2x + 3y = 0$ represent the equations of a pair of its conjugate diameters.

Sol. We are aware that if m_1 and m_2 are the slopes of a pair of conjugate diameters of an ellipse, where If the semi-major and semi-minor axes are a and b, respectively, then $m_1m_2 = -\frac{b^2}{a^2}$.

Here we have $m_1 = 1$ and $m_2 = -\frac{2}{3}$ Thus $-\frac{2}{3} = -\frac{b^2}{a^2}$

$$2a^2 = 3b^2 = 3a^2(1 - e^2)$$

$$2 = 3 - 3e^2$$

$$3e^2 = 3 - 2 = 1$$

$$e = \frac{1}{\sqrt{3}}$$

The eccentricity of the given ellipse $= \frac{1}{\sqrt{3}}$.

Ex. Determine the locus of the poles of the normal chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol. Consider P(h, k) as the pole. The equation for the polar of P is.

$$\frac{xh}{a^2} + \frac{yk}{b^2} = 1$$

1. Is a normal chord.

It should be same as.

$$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$$

2. (The equation of a normal at $(a\cos\theta, b\sin\theta)$)

Comparing co-efficent in (i) \& (ii), we get

$$\frac{h/a^2}{a\sec\theta} = \frac{k/b^2}{-b\operatorname{cosec}\theta} = \frac{1}{a^2-b^2}$$

$$\cos\theta = \frac{a^3}{h(a^2-b^2)}, \sin\theta = \frac{-b^3}{k(a^2-b^2)}$$

Squaring and adding $\quad 1 = \frac{a^6}{h^2(a^2-b^2)^2} + \frac{b^6}{k^2(a^2-b^2)^2}$

Locus of (h, k) is $\quad \frac{a^6}{x^2(a^2-b^2)^2} + \frac{b^6}{y^2(a^2-b^2)^2} = 1$