Chapter 11

Parabola

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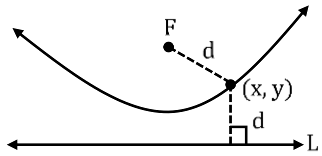
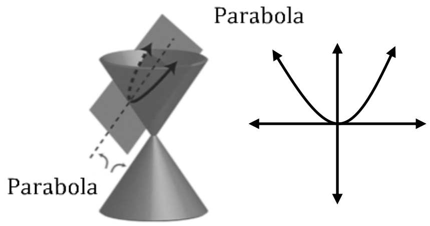
INTRODUCTION

The Parabola

A parabola can be defined as the collection of points within a plane that are equidistant from a specified line, referred to as the directrix, and a point that does not lie on that line, known as the focus. To put it simply, if you are given a line, denoted as L, as the directrix, and a point, denoted as F, as the focus, then a point (x, y) belongs to the parabola if the shortest distance from this point to the focus is the same as the shortest distance from it to the line, as illustrated below:



The vertex of the parabola is the location where the minimum distance to the directrix is achieved. Furthermore, a parabola is created when an oblique plane, parallel to the side of a cone, intersects the cone, resulting in the parabolic shape.

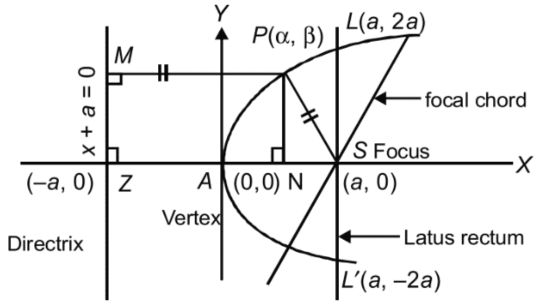


Remember that the graph of a quadratic function, which is a polynomial of degree 2, takes on a parabolic shape. We have the option to express the **equation of a parabola** in either its general form or its standard form.

STANDARD EQUATION OF PARABOLA

The locus of a point traversing a plane, wherein the equidistance from a stationary point equals that from a fixed straight line not inclusive of the said point, is denoted as a Parabola.

The immobile point is termed the Focus, while the fixed straight line assumes the appellation of Directrix. The perpendicular line extending from the Focus to its associated Directrix is recognized as the axis of the parabola. The juncture where the parabola intersects with its axis is referred to as the vertex of the parabola.



Consider the focus and directrix of the parabola represented by S(a, 0) and the equation $x + a = 0$, respectively. In accordance with the parabolic definition,.

$$SP = PM$$

$$(\alpha - a)^2 + \beta^2 = (\alpha + a)^2$$

$$\beta^2 = (\alpha + a)^2 - (\alpha - a)^2 = 4a\alpha$$

Locus of $P(\alpha, \beta)$ is $\quad y^2 = 4ax$

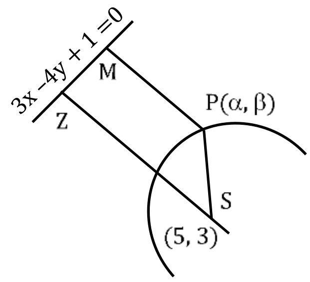
Thus $y^2 = 4ax$ represents a parabola with following specifications:

- Focus positioned at (a, 0)
- Vertex located at (0, 0)
- Directrix defined by $x + a = 0$
- Parabola's axis represented by $y = 0$
- Tangent equation at the vertex given by $x = 0$
- Latus rectum endpoints identified as (a, 2a) and (a, -2a)
- Latus rectum length measuring 4a
- Equation of the latus rectum expressed as $x = ax - a = 0$
- Parametric equations in the form $x = at^2$, $y = 2at$
- A point on the parabola $y^2 = 4ax$ can be specified as $(at^2, 2at)$.
- Any chord that passes through the focus S(a, 0) is termed a focal chord.
- The focal distance of a point P(x, y) is given by $x + a$, where $PS = PM = ZA + AN = a + x$.

Ex. **1.** Determine the equation of the parabola with a focus at (5, 3) and a directrix given by $3x - 4y + 1 = 0$.

2. Find the equation of the parabola whose axis is parallel to the y-axis, passing through the points (0, 4), (1, 9), and (-2, 6), and ascertain the length of its latus rectum.

Sol. **1.** Let $P(\alpha, \beta)$ be a point on the parabola whose focus is at (5, 3) and whose directrix is given by $\quad 3x - 4y + 1 = 0$



From the definition of the parabola we have

$$SP = PM$$

$$SP^2 = PM^2$$

$$(\alpha - 5)^2 + (\beta - 3)^2 = \frac{(3\alpha - 4\beta + 1)^2}{25}$$

Locus of (α, β) is

$25[x^2 + y^2 - 10x - 6y + 34] = (3x - 4y + 1)^2 = 9x^2 + 16y^2 - 24xy + 6x - 8y + 1$

$$16x^2 + 9y^2 + 24xy - 256x - 142y + 849 = 0$$

This represents the sought-after equation for the parabola.

2. The equation of the pearabola whose axis is prallel to y - axis may be written as.

$$y = ax^2 + bx + c$$

Which will pass through the points $(0,4), (1,9)$ and $(-2,6)$ if

$$4 = c, 9 = a + b + c \text{ and } 6 = 4a - 2b + c$$

$$a + b = 5 \text{ and } 4a - 2b = 2 \Rightarrow 2a - b = 1$$

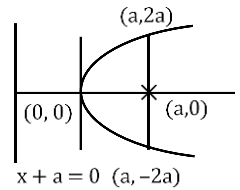
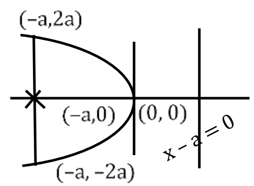
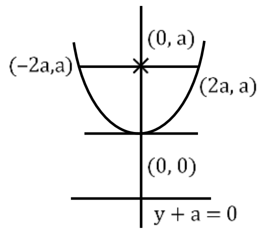
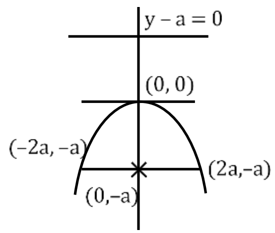
$$3a = 6, \Rightarrow a = 2, b = 3$$

Thus required equation of the parabola is

$$y = 2x^2 + 3x + 4 = 2(x + \frac{3}{4})^2 + \frac{23}{8}$$

Clearly length of the latus rectum is $\frac{1}{2}$

PARABOLAS WITH LATUS RECTUM 4a

Equation of Parabola	Vertex	Focus	Equation of the axis	Equation of the tangent at the vertex	Lengths of the latus rectum	End of latus rectum	Equation of the latus rectum	Paramatrics cordinates of a point on the Parabola	Equation of the directrix	Figure	Focal distane of a point (x, y)
$y^2 = 4ax, a > 0$	(0, 0)	(a, 0)	Y = 0	X = 0	4a	(a, 2a) and (a,–2a)	x –a =0	$(at^2, 2at)$	x + a =0		a + x
$y^2 = -4ax, a > 0$	(0, 0)	(–a, 0)	Y = 0	X = 0	4a	(–a, 2a) and (–a,–2a)	x + a = 0	$(-at^2, 2at)$	x – a = 0		a – x
$x^2 = 4ay, a > 0$	(0, 0)	(0, a)	X = 0	Y = 0	4a	(2a, a) and (–2a, a)	y – a = 0	$(2at, at^2)$	y + a =0		a + y
$r^2 = -4ay, a > 0$	(0, 0)	(0, –a)	X = 0	Y = 0	4a	(2a, –a) and (–2a, –a)	y + a = 0	$(2at, -at^2)$	y – a = 0		a – y

Ex. Determine the equation of the parabola with a focus at (3, 0) and a directrix given by x = -3.

Sol. The focal point of the parabola is located at (3, 0).

Given that the y-coordinate of the focus is '0', it resides on the x-axis.

Hence, the axis of the parabola coincides with the x-axis.

Therefore, the desired equation can take the form $y^2 = 4ax$ or $y^2 = -4ax$.

Given that the directrix is defined by x = -3 and the focus is located at (3, 0)

The equation for the parabola is $y^2 = 4(3)x$ as needed.

i.e., $y^2 = 12x$

Ex. Determine the equation of the parabola with the vertex at (0, 0) and the focus at (0, 5).

Sol. Given that the vertex of the parabola is located at (0, 0) and the focus is at (0, 5), situated on the y-axis, it follows that the axis of the parabola coincides with the y-axis.

As the focus is situated on the positive side of the y-axis, the necessary equation takes the form $x^2 = 4ay$.

Here, a = 5

$x^2 = 20y$ is the required equation.

Ex. Determine the equation of the parabola that is symmetric about the x-axis, has its vertex at the origin, and passes through the point (-2, -3).

Sol. The information provided indicates that the parabola exhibits symmetry about the x-axis.

The equation needed can take the form $y^2 = 4ax$ or $y^2 = -4ax$.

Now, the indication is contingent upon whether the parabola opens towards the right or the left.

Since the parabola passes through the point (-2, -3) situated in the third quadrant, it follows that the parabola must open to the left.

Hence, the equation takes the form $y^2 = -4ax$.

Since, the parabola passes through (-2, -3), we have, (-3)2 = -4a(-2)

$$9 = 8a$$

$$a = \frac{9}{8}$$

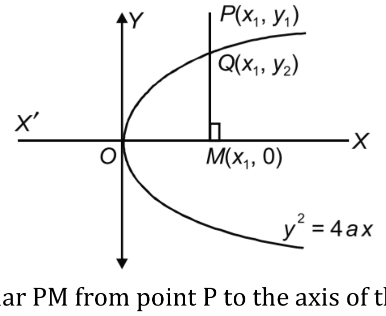
Therefore, the required equation of the parabola is

$$y^2 = -4 \times \left(\frac{9}{8}\right) \times x \text{ i.e., } y^2 = -\frac{9}{2}x$$

$$2y^2 = -9x$$

Position of a Point Relative to a Parabola

Let $P(x_1, y_1)$ be a point



Let us draw the perpendicular PM from point P to the axis of the OX of the parabola $y^2 = 4ax$.

Then $y_2^2 = 4ax_1$

Now, P will be outside, on or inside the parabola $y^2 = 4ax$ according as

$$PM \gtreqless QM$$

$$PM^2 \gtreqless QM^2$$

$$y_1^2 \gtreqless y_2^2$$

$$y_1^2 \gtreqless 4ax_1$$

$$y_1^2 - 4ax_1 \gtreqless 0$$

The point (x_1, y_1) is situated within, on, or outside $y^2 = -4ax$ according as $y_1^2 + 4ax_1 \lesseqgtr 0$
The point (x_1, y_1) is situated within, on, or outside $x^2 = 4\,ay$ according as $x_1^2 - 4ay_1 \lesseqgtr 0$.
The point (x_1, y_1) is situated within, on, or outside $x^2 = -4\,ay$ according as $x_1^2 + 4ay_1 \lesseqgtr 0$

Position of a Line With Respect to a Parabola

Let the line is $y = mx + c$ ... (i)
Parabola is $y^2 = 4ax$ ... (ii)
By (i) and (ii) $(mx + c)^2 = 4ax$

$$m^2x^2 + 2x(mc - 2a) + c^2 = 0$$

Discriminant $D = 4(mc - 2a)^2 - 4m^2c^2$

$$D = 16a(a - mc)$$

1. If the line is a chord, then$D > 0 \Rightarrow a > mc$
2. If the line is a tangent, then $Q = 0 \Rightarrow a = mc \Rightarrow c = \frac{a}{m}$
3. If the line lies outside the parabola, then $D < 0 \Rightarrow a < mc$

Thus the line $y = mx + c$ touches the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$.
Consequently the equation of the tangent to the parabola $y^2 = 4ax$ in slope form can be put in the form $y = mx + \frac{a}{m}$
If the line $lx + my + n = 0$ is a tangent to the parabola $y^2 = 4ax$, then $In = am^2$.

Equation of Tangent In Different From

1. Point form: The equation of the tangent at the point (x_1, y_1) is given by
   $yy_1 - 2a(x + x_1) = 0$
2. Parametric form: At the point $(at^2, 2at)$, the equation of the tangent is $yt = x + at^2$
3. Slope form: $y = mx + \frac{a}{m}$ at $(\frac{a}{m^2}, \frac{2a}{m})$

To enhance comprehension, we can create a chord.

Parabola	Tangent in Point Form	Tangent In Paramatric form	Tangent In Slope Form
$y^2 = 4ax$	$yy_1 - 2a(x + x_1) = 0$ at (x_1, y_1)	$yt = x + at^2$ at $(at^2, 2at)$	$y = mx + \frac{a}{m}$ at $(\frac{a}{m^2}, \frac{2a}{m})$
$y^2 = -4ax$	$yy_1 + 2a(x + x_1) = 0$ at (x_1, y_1)	$yt = -x + at^2$ at $(-at^2, 2at)$	$y = mx - \frac{a}{m}$ at $(\frac{-a}{m^2}, \frac{-2a}{m})$
$x^2 = 4ay$	$xx_1 = 2a(y + y_1) = 0$ at (x_1, y_1)	$xt = y + at^2$ at $(2at, at^2)$	$y = mx - am^2$ at $(2am, am^2)$
$x^2 = -4ay$	$xx_1 = -2a(y + y_1)$ at (x_1, y_1)	$xt = -y + at^2$ at $(2at, -at^2)$	$y = mx + am^2$ at $(-2am, -am^2)$

Point Of Intersection Of Tangent at t_1, t_2

At $(at_1^2, 2at_1)$ tangent is $yt_1 = x + at_1^2$ ... (1)
At $(at_2^2, 2at_2)$ tangent is $yt_2 = x + at_2^2$ ... (2)
By (1) and (2), $x = at_1t_2$
$y = a(t_1 + t_2)$
Here, we determine that the x-coordinate of the point of intersection is the geometric mean of at_1^2 and at_2^2while y coordinates is the arthematic mean of $2at_1$ and $2at_2$.

Equation Of Director Circle

The director circle of the parabola is the path traced by the point of intersection of perpendicular tangents.
Let us take a tangent of $y^2 = 4ax$ is $y = mx + \frac{a}{m}$

It passes through (h,k) hence $k = mh + \frac{a}{m}$

$$mk = m^2h + a$$
$$m^2h - mk + a = 0$$

This equation has two roots, denoted as m_1 and m_2, which correspond to the slopes of tangents drawn from the point (h, k). Hence,

$$m_1 + m_2 = \frac{k}{h}$$
$$m_1m_2 = \frac{a}{h}$$

If tangents are perpendicular, then $m_1m_2 = -1$

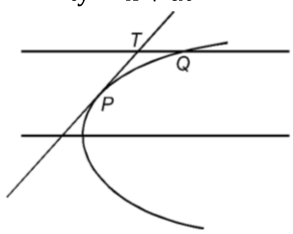
$h = -a$ locus of (h,k) is $x + a = 0$

Therefore, the director circle of any parabola coincides with its directrix.

Ex. Prove that tangents drawn to the parabola $y^2 = 4ax$ intersect at points with abscissas in the ratio $\mu:1$ that they intersect on the curve $y^2 = (\mu^{\frac{1}{4}} + \mu^{-\frac{1}{4}})^2ax$.

Sol. Consider two points, P and Q, on the given parabola $y^2 = 4ax$. The abscissa of P is μat^2, and that of Q is at^2. The corresponding ordinates are $2at\sqrt{\mu}$ and $2at$. The equations of the tangents at $P(\mu at^2, 2at\sqrt{\mu})$ and $Q(at^2, 2at)$ are as follows:

$$ty\sqrt{\mu} = (x + \mu at^2) \quad \dots (1)$$
$$ty = x + at^2 \quad \dots (2)$$



Eliminating the parameter t from the above equation will yield the desired locus of the point of intersection.

Now, $\mu(2) -$ (1) gives

$$ty\sqrt{\mu}(1 - \sqrt{\mu}) = x(1 - \mu)$$
$$t = \frac{x}{y} \cdot \left(\frac{1+\sqrt{\mu}}{\sqrt{\mu}}\right)$$

Subsituting for t in (1) and multiplaying by μy^2, we get

$$xy^2\sqrt{\mu} + xy^2\mu = xy^2\mu + ax^2(1 + \sqrt{\mu})^2$$
$$y^2\sqrt{\mu} = ax(1 + \mu + 2\sqrt{\mu})$$
$$y^2 = ax\left[\frac{1}{\sqrt{\mu}} + \sqrt{\mu} + 2\right]$$
$$y^2 = [\mu^{\frac{1}{4}} + \mu^{-\frac{1}{4}}]^2 \cdot ax$$

Which is the required locus.

Ex. Determine the equation(s) of the common tangent(s) to the parabola $y^2 - 4x - 2y + 5 = 0$ and$y^2 = -4x$.

Sol. The equation of tangent to $y^2 = -4x$ is

$$y = mx - \frac{1}{m} \quad \dots (1)$$

The parabola $y^2 - 4x - 2y + 5 = 0$ can be written as

$$y^2 - 2y + 1 = 4x - 4$$
$$(y - 1)^2 = 4(x - 1)$$

The equation tangent to it be

$$y - 1 = m(x - 1) + \frac{1}{m}$$
$$y = mx - m + \frac{1}{m} + 1 \quad \dots (2)$$

The equation for the common tangent to the parabolas (1) and (2) corresponds to the same line.

$$-\frac{1}{m} = -m + \frac{1}{m} + 1$$
$$m - \frac{2}{m} - 1 = 0$$
$$m^2 - m - 2 = 0$$
$$m = 2, -1$$

The required common tangents are

$$y = 2x - \frac{1}{2} \text{ and } 4x - 2y - 1 = 0 \text{ and}$$
$$y = -x + 1$$
$$x + y = 1$$

Normal

1. **Point Form**

The equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

2. **Parametric Form**

By putting $x_1 = at^2, y_1 = 2at$ in (i) we get

$$y - 2at = -\frac{2at}{2a}(x - at^2)$$
$$y + xt = 2at + at^3$$

3. **Slope Form**

Let the slope of (ii) is m, then $m = -t \Rightarrow t = -m$

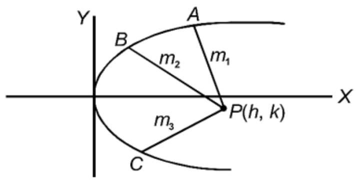
Hence equation (ii) becomes

$$y = mx - 2am - am^3, \text{ at } (am^2, -2am)$$

from (iii) we may conclude that, if $y = mx + c$ is normal to $y^2 = 4ax$, then $c = -2am - am^3$.

4. **Conormal Point**

If normals at some points with respect to parabola are concurrent, then the point are called conormal points.



Let $y = mx - 2am - am^3$, which passes through (h, k)

$$k = mh - 2am - am^3$$
$$am^3 + 2am - mh + k = 0$$
$$am^3 + m(2a - h) + k = 0$$

The equation is cubic in m, allowing for a maximum of three normals (at least one) to be drawn from any point to the parabola. The roots of this cubic equation represent the slopes of the normals, thus.

$$m_1 + m_2 + m_3 = 0$$
$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$$
$$m_1m_2m_3 = \frac{-k}{a}$$

5. **Proparties of Conormal Point**

   a. The sum of the slopes of three co-normal points is algebraically zero.

   b. The algebraic sum of the ordinates of the feet of three normals drawn to a parabola is zero.

   c. If $h > 2a$, then three normals can be drawn from the point (h, k) to the parabola $y^2 = 4ax$.

   d. The centroid of a triangle formed by co-normal points lies on the axis of the parabola.

6. Intersecion of Normals

Let
$$y + xt_1 = 2at_1 + at_1^3$$
$$y + xt_2 = 2at_2 + at_2^3$$

Are two normal at the point t_1 and t_2.

By solving (i) and (ii), we get
$$x = 2a + a(t_1^2 + t_2^2 + t_1t_2)$$
$$y = -at_1t_2(t_1 + t_2)$$

The formula for the chord that connects $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is $y(t_1 + t_2) = 2(x + at_1t_2)$

The measurement of the focal chord that connects t_1 and t_2 is $a(t_1 - t_2)^2$

If the chord $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ pass through focus $(a, 0)$, then $t_1t_2 = -1$. i.e. if one end of a focal chord of $y^2 = 4ax$ is $(at^2, 2at)$ then other and will be $(\frac{a}{t^2}, \frac{-2a}{t}), t \neq 0$.

Consider the chord that is perpendicular at $(at_1^2, 2at_1)$ intersect with the parabola at $(at_2^2, 2at_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$

The semi-latus rectum serves as the harmonic mean of the focal radii. If l_1 and l_2 represent the focal radii, then the semi-latus rectum is given by $= \frac{2l_1l_2}{l_1+l_2}$.

The equation of the chord of contact formed by tangents drawn from (x_1, y_1) to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$.