LINE AND CIRCLE

A line can take various forms in relation to a circle, such as being a tangent, chord, or chord of contact.

1. Position of a line with respect to a circle:

Let d be the length of the perpendicular from the center to the line, and r be the radius of the circle.
If $d > r$, then the line is outside the circle.
If $d = r$, then the line becomes tangent to the circle.
If $d < r$, then the line becomes a chord of the circle.

2. Condition for tangency of a given line to the circle

1st Method :

Let the line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$

$$x^2 + (mx + c)^2 = a^2$$
$$x^2(1 + m^2) + 2mcx + c^2 - a^2 = 0$$

Line touches the circle
This equation should have equal roots.

$$D = 0$$
$$D = 4m^2c^2 - 4(1 + m^2)(c^2 - a^2)$$
$$= 4(a^2m^2 - c^2 + a^2) = 0$$
$$c^2 = a^2(1 + m^2)$$
$$c = \pm a\sqrt{1 + m^2}$$

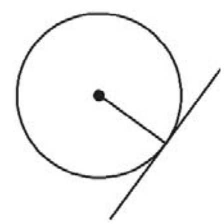
The line with the equation $y = mx + c$ is tangent to the circle.

$$x^2 + y^2 = a^2 \text{ if } c = \pm a\sqrt{1 + m^2}$$

Therefore, the equation of the tangent in terms of its slope m is $y = mx \pm a\sqrt{1 + m^2}$

2nd Method :

Let the line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$
The length of the perpendicular from the center of the circle to the line is equal to the radius of the circle.



$$\left|\frac{0-0+c}{\sqrt{1+m^2}}\right| = a$$
$$c = \pm a\sqrt{1 + m^2}$$

> The 1st method is applicable to determine the condition under which a line touches any second-degree curve (such as a circle, parabola, ellipse, and hyperbola). However, the 2nd method is limited to circles only as it relies on the properties specific to circles.

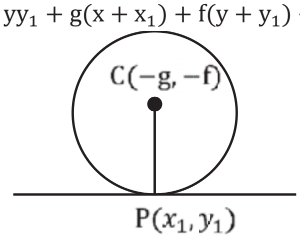
3. Equation of tangent to a circle

The equation of the tangent to the circle $x^2 + y^2 = a^2$ at a point $P(x, y_1)$on it is expressed as

$$xx + yy_1 = a^2.$$

The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at $P(x, y_1)$ on it is given by

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$



The equation of the tangent can be easily expressed using the following terminology.

$$S = x^2 + y^2 + 2gx + 2fy + c$$

$$T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$

T has been derived from S by substituting x^2 with x_1, $2x$ with $x + x_1$ and similarly replacing y^2 with yy_1 and $2y$ with, $y + y_1$.

The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x, y_1) is given by $T = 0$

The combined equation of a pair of tangents drawn from a point $P(x, y_1)$ to the circle...

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is}$$

$$T^2 = SS_1$$

$$[xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c]^2 = (x^2 + y^2 + 2gx + 2fy + c)(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)$$

The simultaneous equation of a pair of tangents from (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $(x_1^2 + y_1^2 - a^2)(x^2 + y^2 - a^2) = (xx_1 + yy_1 - a^2)^2$ which may be put in the form $SS_1 = T^2$

The condition that $y = mx + c$ may touch the circle $x^2 + y^2 = a^2$ is

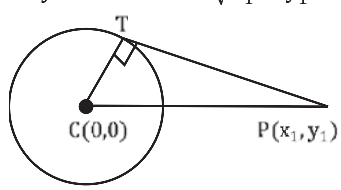
$c = \pm a\sqrt{1 + m^2}$ and then the point of contact is $(-\frac{a^2m}{c}, \frac{a^2}{c})$

The equation of any tangent to the circle $x^2 + y^2 = a^2$ may be taken as

$$y = mx + a\sqrt{1 + m^2} \text{ or} y = mx - a\sqrt{1 + m^2}$$

The length of tangent drawn from an external point $P(x_1, y_1)$ to the circle.

$$x^2 + y^2 = a^2 \text{ is } PT = \sqrt{x_1^2 + y_1^2 - a^2}$$



If we write $S \equiv x^2 + y^2 - a^2$ and $S_1 \equiv x_1^2 + y_1^2 - a^2$, then the length of tangent drawn from an external point(x_1, y_1) to the circle $S = 0$ is $\sqrt{S_1}$

In a similar fashion the length of tangent drawn from an external point (x_1, y_1) to the circle.

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is}$$

$$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}, \text{ i.e. } \sqrt{S_1}$$

4. Director circle

The locus of the point of intersection of perpendicular tangents to any second-degree curve is the director circle.

Let the line $y = mx \pm a\sqrt{1 + m^2}$ be a tangent. It passes through (h, k)

$$(k - mh)^2 = a^2(1 + m^2)$$

$$m^2(h^2 - a^2) - 2khm + (k^2 - a^2) = 0$$

This is a quadratic equation in m. Let m_1 and m_2 be the roots of this equation.

$$m_1m_2 = \frac{k^2-a^2}{h^2-a^2}$$

Tangents are perpendicular $\quad m_1m_2 = -1$

$$\frac{k^2 - a^2}{h^2 - a^2} = -1$$

$$h^2 + k^2 = 2a^2$$

Locus of (h, k) is, $x^2 + y^2 = 2a^2$ this is the equation of director circle

5. Equation of the normal to a circle

Since the normal is perpendicular to the tangent, once we have determined the equation of the tangent, we can also find the equation of the normal.

The equation of the normal to the circle $x^2 + y^2 = a^2$ at (x_1, y_1) is $\frac{x}{x_1} = \frac{y}{y_1}$

The equation of the normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is $\frac{x-x_1}{x_1+g} = \frac{y-y_1}{y_1+f}$

6. Equation of chord in terms of its middle point

The equation of the chord of the circle $x^2 + y^2 = a^2$ whose middle point is (x_1, y_1) is
$xx_1 + yy_1 = x_1^2 + y_1^2$.

The equation can be rewritten as.

$$xx_1 + yy_1 - a^2 = x_1^2 + y_1^2 - a^2$$

In terms of $S \equiv x^2 + y^2 - a^2, T \equiv xx_1 + yy_1 - a^2$ and $S_1 \equiv x_1^2 + y_1^2 - a^2$ the equation of chord whose middle point is (x_1, y_1) is $T = S_1$

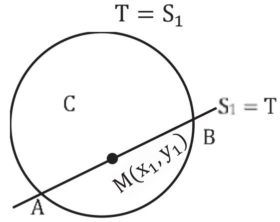
The equation of the chord of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

In terms of its middle point (x_1, y_1) is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$$T = S_1$$



. **Equation of Chord of Contact of Tangents drawn from an External Point**

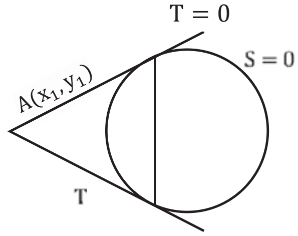
If tangents are drawn to the circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

If tangents are drawn to the circle from an external point, $A(x_1, y_1)$, then the equation of the chord of contact is:

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

$$T = 0$$



8. Equation of pair of tangents

$$S = x^2 + y^2 + 2gx + 2fy + c = 0$$

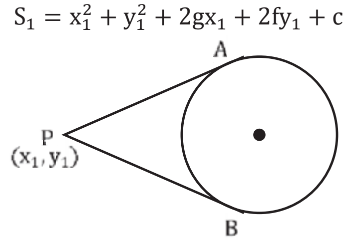
The equation of the pair of tangents from point P to the circle with equation $S = 0$ is:

$$T^2 = SS_1$$

$$T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$

$$S = x^2 + y^2 + 2gx + 2fy + cz$$

$$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$



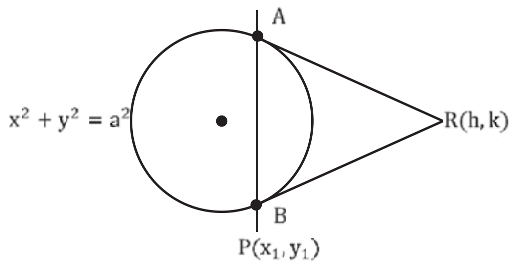
9. Equation of Polar

Consider a point $P(x_1, y_1)$ located inside, outside, or on the given circle. An infinite number of chords can be drawn from (x_1, y_1) to the circle.

Let one of the chords be AB. The point of intersection of tangents at A and B is denoted as R. The locus of R will be the polar of P, with P being the pole of the polar. AB is the chord of contact with respect to R. Its equation is:

$$xh + yk - a^2 = 0$$

It passes through (x_1, y_1). Hence $\quad x_1h + y_1k - a^2 = 0$
Locus of R is $xx_1 + yy_1 - a^2 = 0$



10. Properties of pole and polar

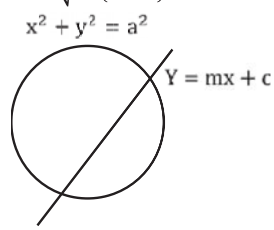
(a) Two points in relation to the given circle are termed conjugate points if the polar of one point passes through the other point.

(b) The distance of two points from the center of a circle is proportional to the distance of each from the polar of the other.

11. Length of intercept cut off by a Line (Length of a chord)

The length of the intercept formed by the circle $x^2 + y^2 = a^2$ on the straight line $y = mx + c$ is.

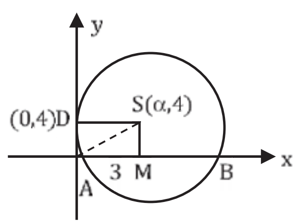
$$2\left|\sqrt{\frac{a^2(1+m^2)-c^2}{(1+m^2)}}\right| \text{ units}$$



Ex. Determine the equation of the circle that touches the y-axis at the point (0, 4) and creates an intercept of length 6 on the x-axis.

Sol. **1st Method:**

Consider S as the center of the circle...



Since SD is perpendicular to the y-axis, the y-coordinate of S equals the y-coordinate of D, which is 4. Therefore, the coordinates of S are (α, 4).

Radius = SA = SD = |x|

$\triangle$ SMA we have

$$3^2 + 4^2 = |\alpha|^2 \Rightarrow |\alpha|^2 = 25 \Rightarrow |\alpha| = 5 \therefore \alpha = \pm 5$$

Thus, there are two circles, one as illustrated and another identical circle to the left of the y-axis. Their equations are:

$$(x-5)^2 + (y-4)^2 = 25 \text{ and } (x+5)^2 + (y-4)^2 = 25$$
$$x^2 + y^2 - 10x - 8y + 16 = 0 \text{ and } x^2 + y^2 + 10x - 8y + 16 = 0$$

2nd Method:

Let the equation of the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

This will intersect the y-axis at points determined by the equation.

$y^2 + 2fy + c = 0 \ldots$ (i) (i.e., setting $x = 0$ in the equation)

Since the circle touches the y-axis at (0, 4), both roots of the above equation are equal to 4. This implies that (i) must be equivalent to $(y\text{-}4)^2 = 0$.

$$y^2 - 8y + 16 = 0$$

$$\text{on comparison } f = -4, c = 16$$

Equation (i) now become

$$x^2 + y^2 + 2gx - 8y + 16 = 0$$

This will meet x-axis in point $(x_1, 0)$ and $(x_2, 0)$ given by $x^2 + 2gx + 16 = 0$ (i.e., setting $y = 0$ in the equation)

$$x_1 + x_2 = -2g \text{ and } x_1x_2 = 16$$

Distance between the points $= |x_1 - x_2| = \sqrt{(x_1 + x_2)^2 - 4x_1x_2} = \sqrt{(2g)^2 - 64}$

$$\sqrt{(2g)^2 - 64} = 6$$

$$4g^2 - 64 = 36$$

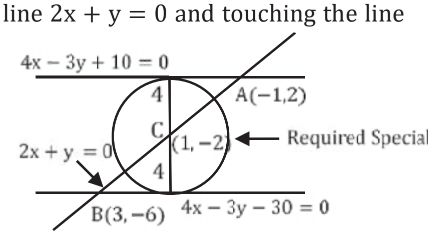
$$4g^2 = 100 \therefore g = \pm 5$$

Thus the equation are

$$x^2 + y^2 + 10x - 8y + 16 = 0 \text{ and } x^2 + y^2 - 10x - 8y + 16 = 0$$

Ex. Determine the equation of the circle with its center on the line $2x + y = 0$ and touching the line $4x - 3y + 10 = 0$ or $4x - 3y = 30$.

Sol. The touching lines are parallel.



The distance between them. $= \frac{|10+30|}{\sqrt{16+9}}$

$= \frac{40}{5} = 8$

Radius of the circle $= 4$

The line $2x + y = 0$ intersects the parallel lines at points.

A (–1, 2) and B (3, –6). The midpoint of the line AB is the center C (1, –2).

Therefore, the desired equation of the circle is.

$$(x - 1)^2 + (y + 2)^2 = 4^2$$

$$x^2 + y^2 - 2x + 4y - 11 = 0$$