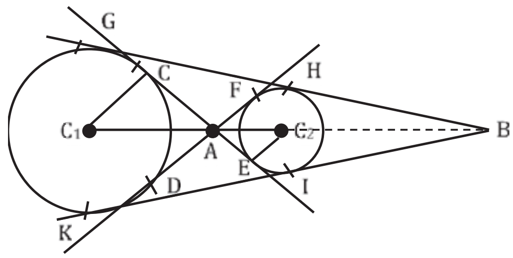
ANALYSIS OF TWO CIRCLE

1. POSITION OF TWO CIRCLE

We consider two circle with centers $C_1 \equiv (x_1, y_1)$ and $C_2 \equiv (x_2, y_2)$ having radii r_1 and r_2 repectively.

(a) One circle lies outside the order circle



In this case: $C_1C_2 > r_1 + r_2$

- A circle has four tangents in common, comprising two direct common tangents and two transverse common tangents.
- DF and CE serve as transverse common tangents, while GH and Ki act as direct common tangents.
- A is the intersection point of transverse common tangents, and B is the intersection point of direct common tangents.
- By using the property of similar triangle in C_1CA and C_2AE we get that $\frac{C_1A}{C_2A} = \frac{r_1}{r_2}$

  Hence $A \equiv \left(\frac{r_1x_2+r_2x_1}{r_1+r_2}, \frac{r_1y_2+r_2y_1}{r_1+r_2}\right)$

  Similarly B divides C_1C_2 in the ratio $r_1 : r_2$ externally.

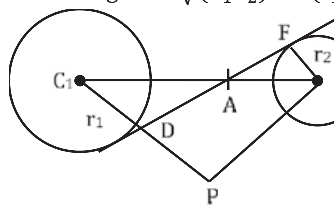
  $$B \equiv \left(\frac{r_1x_2-r_2x_1}{r_1-r_2}, \frac{r_1y_2-r_2y_1}{r_1-r_2}\right), (r_1 \neq r_2)$$

- Length of transverse common tangents

  From C_2 draw a line parallel to DF that is C_2P in triangle C_1C_2P.

  $$C_2P = \sqrt{(C_1C_2)^2 - (C_1P)^2} = \sqrt{(C_1C_2)^2 - (r_1 + r_2)^2}$$

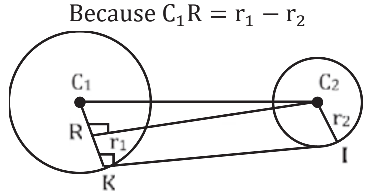
  Hence length $= \sqrt{(C_1C_2)^2 - (r_1 + r_2)^2}$



- Length of direct common tangents

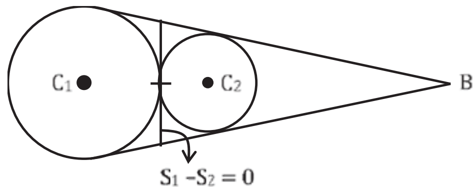
  $$\text{Length } = C_2R = \sqrt{(C_1C_2)^2 - (r_1 - r_2)^2}$$

  Because $C_1R = r_1 - r_2$



- With the assistance of points A and B, the equations for transverse and direct common tangents can be determined.

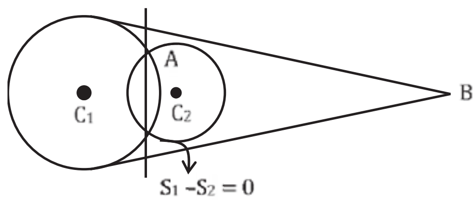
(b) Two circle touch each other externally



In this case: $C_1C_2 = r_1 + r_2$

- A circle has three tangents in common, consisting of two direct common tangents and one tangent at the point of contact.
- $$A = \left(\frac{r_1x_2+r_2x_1}{r_1+r_2}, \frac{r_1y_2+r_2y_1}{r_1+r_2}\right)$$ $$B = \left(\frac{r_1x_2-r_2x_1}{r_1-r_2}, \frac{r_1y_2-r_2y_1}{r_1-r_2}\right)$$
- The equation of direct common tangents can be find out by B by using $T^2 = SS_1$
- The equation of tangents at the point of contact is $S_1 - S_2 = 0$.

(c) Intersecting Circle

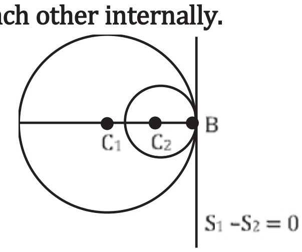


In this case: $|r_1 - r_2| < C_1C_2 < r_1 + r_2$

- Both the circle have only two direct common tangents.
- In this case both the circle have common chord whose equation. $S_1 - S_2 = 0$.
- Orthogonal circle: If angle between the circle is 90° the circle are called orthogonal and $r_1^2 + r_2^2 = (C_1C_2)^2$

$$S_1 = x^2 + y^2 + 2gx_1 + 2fy_1 + c_1 = 0$$
$$S_2 = x^2 + y^2 + 2gx_2 + 2fy_2 + c_2 = 0$$
$$(g_1^2 + f_1^2 - c_1) + (g_2^2 + f_2^2 - c_2) = (g_1 - g_2)^2 + (f_1 - f_2)^2$$
$2g_1g_2 + 2f_1f_2 = c_1 + c_2$. This is the condition for orthogonal circles

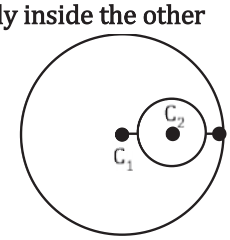
(d) Two circle touch each other internally.



In this case: $C_1C_2 = |r_1 - r_2|$.

- Both the circle have one common tangent whose equation is.$S_1 - S_2 = 0$.

(e) One circle is completely inside the other

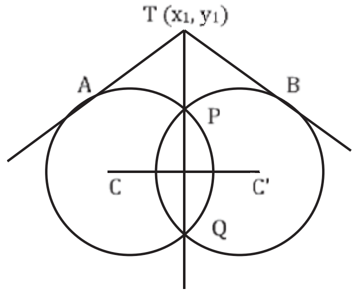


In this case: $C_1C_2 < |r_1 - r_2|$

- And circle do not have any common tangent.

2. Radical Axes

The radical axis of two circle is the locus of a point which move in a plane so that the length of tangents drawn from it two circle are equal.



They equation of radical axis and common chord are identical. The only difference is that the common chord exist only if the circle intersection in real point. While the radical axis exists for all pair (non-concentric) of circle irrespective of their position.

- The radical axis is perpendicular to the straight line joining the centers of the circle.
- If two circle intersect a third circle orthogonally, the radical axis of the first two passes through the centre of the third.
- Radical axis bisects common tangents of two circle.
- If two circle are concentric then they do not have radical axis.

3. Radical centre of three circle

Let three circle are. $S_1 = 0, S_2 = 0, S_3 = 0$ The point of intersection of radical axes of all three circle taken two at a time is called the radical centre of the circle.

Ex. Determine the equation of the circle passing through the origin and intersecting the circle $x^2 + y^2 - 4x + 6y + 10 = 0$ and $x^2 + y^2 + 12y + 6 = 0$ orthogonal.

Sol. Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Since it passes through the origin

$$0 + 0 + 2g \cdot 0 + 2f \cdot 0 + c = 0 \Rightarrow c = 0$$

(i) Becomes $x^2 + y^2 + 2gx + 2fy = 0$

The circle (ii) cuts the given circles

$$x^2 + y^2 + 12y + 6 = 0 \text{ and } x^2 + y^2 - 4x + 6y + 10 = 0 \text{ orthogonally}$$

$$2gg_1 + 2ff_1 = c + c_1$$

$$2g \cdot (0) + 2f \cdot 6 = 0 + 6 \text{ or } f = \frac{1}{2} \text{ and } 2g(-2) + 2f \cdot 3 = 0 + 10 \text{ or } g = -\frac{7}{4}$$

Substituting the values of g and f in (ii), we get

$$2(x^2 + y^2) - 7x + 2y = 0$$

This represents the needed equation of the circle.