WAVY CURVE METHOD

This method is used to solve inequalities with a particular structure.

$$\frac{(x-\alpha_1)^{p_1}(x-\alpha_2)^{p_2}\dots(x-\alpha_m)^{p_m}}{(x-\beta_1)^{q_1}(x-\beta_2)^{q_2}\dots(x-\beta_n)^{q_n}} < 0 \text{ or } > 0 \text{ or } \le 0 \text{ or } \ge 0$$

Let's name the expression on the left-hand side as f(x). Hence, we can express it as "$f(x) > 0$" or "$f(x) \ge 0$" or " ... "
$f(x) < 0$ or$f(x) \le 0$ to solve, where $p_1, p_2, \dots, p_m, q_1, q_2, \dots, q_n \in N$ and $\alpha_1, \alpha_2, \dots, \alpha_m$, $\beta_1, \beta_2, \dots, \beta_n \in R$ such that $\alpha_i \ne \beta_j \forall i = 1,2,\dots, m$ and $j = 1,2,3,\dots, n$
We adopt the following stepwise procedure.

1. Determine the values of x where the numerator and denominator are zero individually. These values are...$\alpha_1, \alpha_2 \dots, \alpha_m$ and $\beta_1, \beta_2, \dots, \beta_n$. These are called critical points.

2. Plot them on the number line. Let us assume that $\alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_m < \beta_1 < \beta_2 < \cdots < \beta_n$

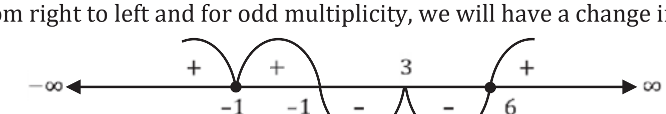
(i) For $f(x) > 0$ and $f(x) < 0$, plot the points $\alpha_1, \alpha_0, \dots, \alpha_m$ and $\beta_1, \beta_2, \dots, \beta_n$ with α_1, on the extreme left and β_n on the extreme right, e.g.,

(ii) For $f(x) \le 0$ and $f(x) \ge 0$, plot the points $\alpha_1, \alpha_2, \dots, \alpha m$ on the number line with darkened bubbles whereas $\beta_1, \beta_2, \dots, \beta_n$ as undarkened bubbles.

3. If we are solving an expression of the type with plus sign before f(x) like $+\frac{(x-\alpha_1)^{p_1}(x-\alpha_2)^{p_2}\dots(x-\alpha_m)^{p_m}}{(x-\beta_1)^{q_1}(x-\beta_2)^{q_2}\dots(x-\beta_n)^{q_n}}$ then the wave will start from above the line to the right of β_n as shown below.

And if the expression has a negative sign before f(x) like $-\frac{(x-\alpha_1)^{p_1}(x-\alpha_2)^{p_2}\dots(x-\alpha_m)^{p_m}}{(x-\beta_1)^{q_1}(x-\beta_2)^{q_2}\dots(x-\beta_n)^{q_n}}$ then the wave will start from below the line as shown below.

4. If some of the αi 's or β_j 's are the repeated roots with even multiplicity, then the sign will remain the same from right to left and for odd multiplicity, we will have a change in sign, e.g.,



The wave bounced back at '3' because $x = 3$ is a root with multiplicity 10 (even).

5. The solution to "$f(x) > 0$" or "$f(x) \ge 0$" is the combination of all intervals where we observe positive signs, while the solution to "$f(x) < 0$" or "$f(x) \le 0$" is the combination of all intervals with negative signs.

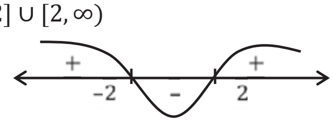
Ex. Solve $x^2 - 4 \ge 0$

Sol. $x^2 - 4 \ge 0$

$(x + 2)(x - 2) \ge 0$

The critical values are -2 and 2

$\therefore$ The solution set is $(-\infty, -2] \cup [2, \infty)$



Ex. Evaluate $\frac{x-1}{x+3} \leq 3$

Sol. $\frac{x-1}{x+3} - 3 \leq 0$

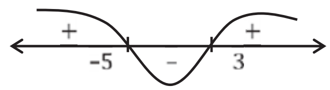
$\frac{x-1-3x-9}{x+3} \leq 0$

$\frac{-2x-10}{x+3} \leq 0$

$\frac{x+5}{x+3} \geq 0$

The critical points are $-5, -3$

$\therefore x \in (-\infty, -5] \cup (-3, \infty)$



Ex. If $x \in [-5,5]$ then find the interval in which $\frac{1}{x}$ lies.

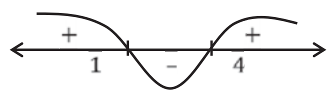
Sol. $\because x \in [-5,5]$ then $-5 \leq x \leq 5$

$\therefore$ Then $\frac{1}{x} \leq -\frac{1}{5}$ or $\frac{1}{x} \geq \frac{1}{5}$

$\therefore \frac{1}{x} \in (-\infty, -\frac{1}{5}] \cup [\frac{1}{5}, \infty)$

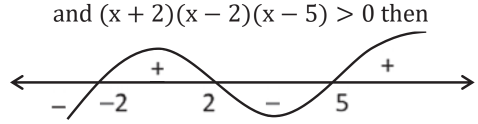
Ex. Find the common solution of the inequalities $(x - 1)(x - 4) > 0$ and $(x + 2)(x - 2)(x - 5) > 0$.

Sol. $\because (x - 1)(x - 4) > 0$ then



$\therefore x \in (-\infty, 1) \cup (4, \infty)$ .... (i)

and $(x + 2)(x - 2)(x - 5) > 0$ then



$\therefore x \in (-2,2) \cup (5, \infty)$

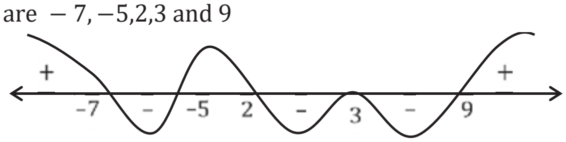
From (i) and (ii),

$x \in (-2,1) \cup (5, \infty)$

Ex. Solve for x, $\frac{(x-3)^2(x+5)(x-2)^3}{(x+7)(x-9)} > 0$

Sol. $\because \frac{(x-3)^2(x+5)(x-2)^3}{(x+7)(x-9)} \geq 0$

The critical value are $-7, -5, 2, 3$ and 9



Solution set is $(-\infty, -7) \cup [-5,2] \cup (9, \infty) \cup \{3\}$