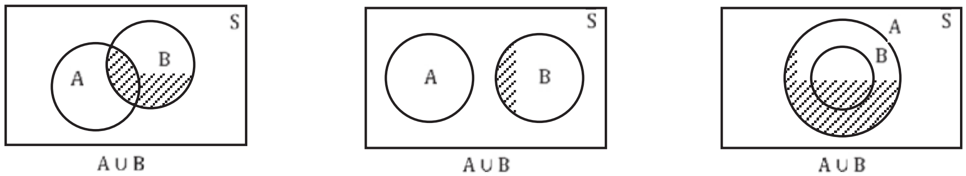
OPERATIONS ON SETS

UNION OF SETS:

The set A ∪ B, pronounced as 'A union B' or 'A cup B' or 'A join B,' encompasses all elements that belong to either A, or B, or both.

$$A \cup B = \{x : x \in A, x \in B\}$$



$$x \in A \cup B \Leftrightarrow x \in A \vee x \in B$$

$$x \notin A \cup B \Leftrightarrow x \notin A \text{ and } x \notin B \text{ 'v'denotes 'or'}$$

Example:

Q → Set of all rational numbers

Q′ → Set of all irrational numbers

$$R = Q \cup Q'$$

$$A_1 \cup A_2 \cup A_3 \ldots\ldots\ldots\ldots\ldots A_n = \cup_{i=1}^{n} A_i$$

$$A_1 \cup A_2 \cup A_3 \ldots\ldots\ldots\ldots\ldots \bigcup_{i=1}^{\infty} A_i$$

$$A \cup B = A + B$$

The combination of a finite number of finite sets forms a finite set, while the union of finite sets with an infinite set results in an infinite set.

$$A \cup A = A, A \cup f = A, A \cup S = S$$

$$S \cup f = S, f \cup f = f, \text{If } A \subset B \text{ then } A \cup B = B$$

Ex. Select the appropriate options from the following for sets A and B.

(a) P (A) ∪ P (B) may be equal to P (A)

(b) P (A) ∪ P (B) may be equal to P (A ∪ B)

(c) P (A) ∪ P (B) must be a subset of P(A ∪ B)

(d) P (A) ∪ P (B) must be equal to P (A ∪ B)

Sol. (a), (b), (c) are true.

If A ⊇ B then P (A) ⊇ P (B), hence option (a) is true.

If A = B then A ∪ B = A = B and P (A) ∪ P (B) = P (A ∪ B), option (b) is true.

Option (c) holds true in all cases for sets A and B.

Option (d) is incorrect when A ≠ B. For instance, A = {1, 2} and B = {1, 4}

$$A \cup B = \{1, 2, 4\}$$

$$P(A) = \{f, \{1\}, \{2\}, [1, 2]\}$$

$$P(B) = \{f, \{1\}, \{4\}, [1, 4]\}$$

$$P(A) \cup P(B) = \{f, \{1\}, \{2\}, \{4\}, \{1, 2,\}, \{1, 4\}\}$$

$$P(A \cup B) = \{f, \{1\}, \{2\}, \{4\}, \{1, 2,\}, \{1, 4\}, \{1, 2, 4\}\}$$

$$P(A) \cup P(B) \neq P(A \cup B).$$

INTERSECTION OF TWO SETS

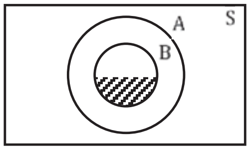
A ∩ B, pronounced as 'A intersection B' or 'A cap B' or 'A meet B,' is defined as a set comprising all elements that are common to both A and B.

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

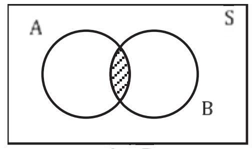
$$\{x : x \in A \text{ L } x \in B\}$$

$$x \in A \cap B \Leftrightarrow x \in A \text{ and } x \in B$$

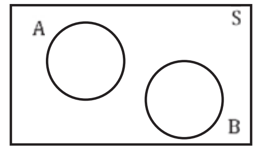
$$x \notin A \cap B \Leftrightarrow x \notin A \text{ or } x \notin B$$



$A \cap B$



$A \cup B$



$A \cap B = \phi$

$$A_1 \cap A_2 \cap A_2 \ldots \ldots \ldots \cap A_n = \bigcap_{i=1}^{n} A_i = \{x : x \in A_i \forall i\}$$

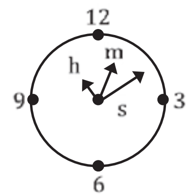
$$A \cap B = AB$$

The intersection of a finite number of finite sets results in a finite set.
The intersection of a finite set with an infinite set yields a finite set.
The intersection of two or more infinite sets may be finite or infinite.

$$A \cap A = A;\ A \cap f = f;\ A \cap S = A;\ S \cap f = f$$

$$f \cap f = f;\ \text{if } A \supseteq B$$

$$A \cap B = B;\ (A \cup B) \cap A = A;\ (A \cap B) \cup A = A$$



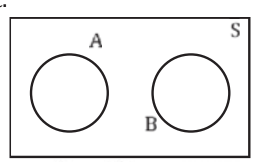
Ex. Given the wall clock depicted in the figure,
S = Set of all points in area covered by second's hand in 12 hours.
M = Set of all points in area covered by minute's hand in 12 hours.
H = Set of all points in area covered by hour's hand in 12 hours.
Then pick the correct statement among following.

(a) $S \cup M \cup H = S$ (b) $(S \cap M) \cup H = S$
(c) $S \cap M \cap H = H$ (d) All are correct

Sol. (a) and (c).
$H \subset M \subset S$ (since hours hand is smallest in length)
Option (b) is wrong, since $S \cap M = M$ and $(S \cup M) \cup H = M \cup H = M$.

DISJOINT SETS

Two sets A and B, having no common elements, are considered disjoint or mutually exclusive.
$A \cap B = f \Leftrightarrow$ A and B are disjoint.

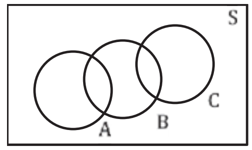


A and B are disjoint

Example:

- z+ and z– are mutually exclusive.
- The set of all boys and the set of all girls are distinct.
- The set of Hindi alphabet and the set of English alphabet are separate.
- The set of years of birth of adults and the set of years of birth of minors do not overlap.
- Q and Q’ are not connected.
- A family of various sets is pairwise disjoint if no two members of this family share a common element.

If $A_1, A_2 \ldots\ldots\ldots . . A_n$ are pairwise disjoint then $A_1 \cap A_2 \ldots\ldots\ldots \cap A_n = \phi$,
But if $A_1 \cap A_2 \ldots\ldots\ldots . \cap A_n = \phi, A_1, A_2 \ldots\ldots\ldots\ldots . A_n$ may not be pairwise disjoint.

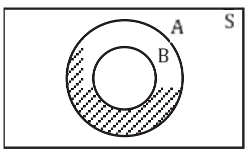


$A \cap B \cap C = \phi$; but A and B, B and C are not disjoint.
Therefore, A, B and C are not pairwise disjoint.

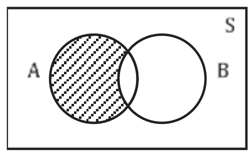
DIFFERENCE OF TWO SETS

A – B, read as 'A minus B' or the relative complement of B in A, represents the set comprising all elements of A that are not elements of B.

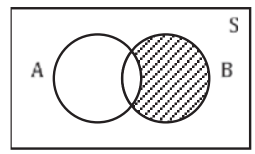
$$A - B = \{x : x \in A \text{ and } x \notin B\}$$
$$B - A = \{x : x \in B \text{ and } x \notin A\}$$



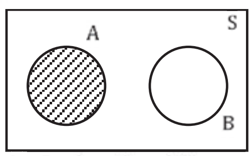
A – B when B ⊂ A



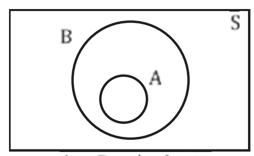
A–B



B –A



A – B when A and B are disjoint A – B = A



A – B = ϕ when A = B or A ⊂ B

$$x \in A - B \Leftrightarrow x \in A \text{ and } x \notin B$$

[Difference of two sets is not commutative]

$$A - B \neq B - A$$

Remove the elements of set B from set A, and the resulting set is denoted as A – B.

(A – B), (B – A) and $(A \cap B)$ are disjoint sets.

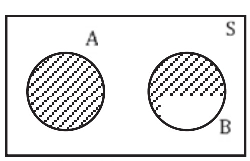
$$A - B \subseteq A;\ B - A \subseteq B,\ A - f = A$$
$$A - A = f$$

$A - B = A \sim B = A/B = C_A B$ (complement of B in A)

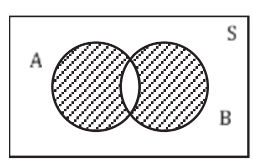
SYMMETRIC DIFFERENCE OF TWO SETS

Represented as $A \Delta B$ or $A \oplus B$, it is referred to as the direct sum of A and B.

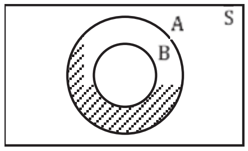
$$A \Delta B = (A - B) \cup (B - A)$$
$$= (A \cup B) - (A \cap B)$$



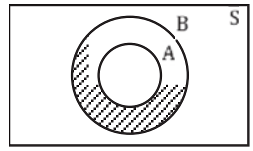
AΔB = A ∪ B when A and B are disjoint



AΔB = (A ∪ B) – (A ∩ B)



$A \Delta B = (A - B)$
when $B \subseteq A$



$A \Delta B = (B - A)$
when $A \subseteq B$

$A \,\Delta\, B = B \,\Delta\, A$ commutative

$A \,\Delta\, B = \{x : x \in A \text{ and } x \notin B\} \cup \{x : x \notin A \text{ and } x \in B\}$

COMPLEMENT OF A SET

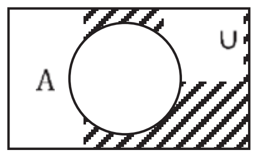
Given a set 'A' and a universal set U such that A is a subset of U ($A \subset U$) then complement of set A is denoted by A′ or A^C or C (A) or U – A

$$A' = A^C = C(A) = U - A = \{x : x \in U \text{ and } x \notin A\}$$

$$x \in A \Leftrightarrow x \notin A'$$

$$x \in A' \Leftrightarrow x \notin A$$

$$U' = \phi;\ \phi' = U;\ A \cup A' = U,\ A \cap A' = \phi$$



U – A, or A^c or A′ or C(A)

Ex. If A = {5, 6, 7, 8}; B = {3, 9, 8, 10} and S = {1, 2, 3, 4....... 10} then,

(a) $(A \cup B)' = \{1, 2, 3, 4\}$ (b) $(A \cap B)' = \{8\}$

(c) $(A' \cap B)' = (A \cap B)'$ (d) None of these is true

Sol. (d)

$A \cup B = \{3, 5, 6, 7, 8, 9, 10\}$ hence $(A \cup B)' = \{1, 2, 4\}$, (a) is wrong.

$A \cap B = \{8\}$ hence $(A \cap B)' = \{1, 2, 3, 4, 5, 6, 7, 9, 10\}$, (b) is wrong.

$(A' \cap B)' = \{1, 2, 4\} \neq (A \cap B)'$, (c) is wrong.

Hence answer (d) is correct.

Ex. If $A = \{(x, y) : y = e^x; x \in R\}$

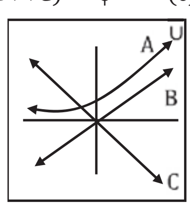
$U = \{(x, y) : x, y \in R\}$

$B = \{(x, y) : y = x; x \in R\}$

$C = \{(x, y) : y = -x; x \in R\}$

Select the correct statement(s) from the following:

(a) $(A \cap B)' = \phi$ (b) $(A \cap B \cap C)' = \phi$ (c) $A - B = \phi$ (d) $A \,\Delta\, B = A \cup B$



Sol. (d)

Points A, B, and C lie on the curves depicted in the adjacent diagram.

Clearly $A \cap B = \phi$

So $(A \cap B)' = U$, option (a) is wrong.

Similarly $A \cap B \cap C = \phi$ as there are no common points to all the three curves.

$\therefore$ $(A \cap B \cap C)' = U$.

Option (b) is wrong.

From figure it is clear that A – B = A, as A and B are disjoint sets.

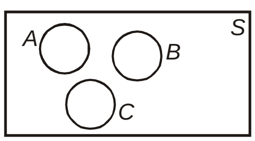
Option (c) is wrong.

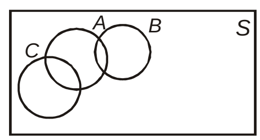
$$A \Delta B = (A - B) \cup (B - A)$$
$$A \cup B$$

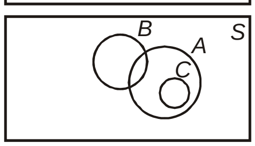
ALGEBRA OF SETS

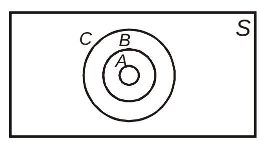
1. **Idempotent Laws:** For any set A, we have
   (a) $A \cup A = A$ (b) $A \cap A = A$
2. **Identity laws:** For any set A, we have
   (a) $A \cup \phi = A$ (b) $A \cap \phi = \phi$ (c) $A \cup U = U$ (d) $A \cap U = A$
3. **Commutative laws:** For any two sets A and B, we have
   (a) $A \cup B = B \cup A$ (b) $A \cap B = B \cap A$
4. **Associative laws:** For any three sets A, B and C, we have
   (a) $(A \cup B) \cup C = A \cup (B \cup C)$ (b) $(A \cap B) \cap C = A \cap (B \cap C)$
5. **Distributive laws:** For any three sets A, B and C, we have
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
6. **De Morgan's laws:** For any three sets A, B and C, we have
   (a) $(A \cup B)' = A' \cap B'$ (b) $(A \cap B)' = A' \cup B'$
   (c) $A - (B \cup C) = (A - B) \cap (A - C)$ (d) $A - (B \cap C) = (A - B) \cup (A - C)$
   $(A')' = A$ (for any set A)
   $P(A) \cap P(B) = P(A \cap B)$
   $P(A) \cup P(B) \subseteq P(A \cup B)$

Ex. Shade $(A \cup B) \cap (A \cup C)$ in the following diagrams

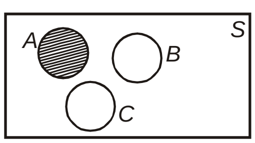
(a)


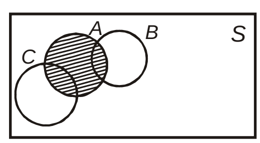
(b)


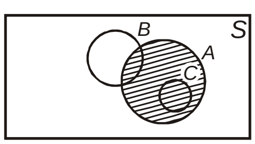
(c)


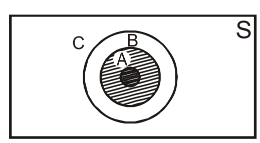
(d)


Sol.

(a)


(b)


(c)


(d)


Some Theorems

For any sets A, B and C, we have

1. $A - B = A \cap B'$
2. $A \cup B = B \Leftrightarrow A \subseteq B$
3. $A \cap B = A \Leftrightarrow A \subseteq B$
4. $(A - B) \cup B = A \cup B$
5. $A - (B \cup C) = (A - B) \cap (A - C)$
   $A - (B \cap C) = (A - B) \cup (A - C)$

Some More Operations On Sets.

$$A \subseteq A \cup B;\ A \cap B \subseteq A;\ (A - B) \cap B = \phi$$
$$A \subseteq B \Leftrightarrow B' \subseteq A';\ A - B = B' - A'$$
$$(A \cup B) \cap (A \cup B') = A;\ A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$$
$$A - (A - B) = A \cap B;\ A - B = B - A \Leftrightarrow A = B$$
$$A \cup B = A \cap B \Leftrightarrow A = B;\ A \cap (B\ \Delta\ C) = (A \cap B)\ \Delta\ (A \cap C)$$

Some Basic Results About Cardinal Numbers

If A, B and C are finite sets and U is finite universal set, then we have

1. $n(A') = n(U) - n(A)$
2. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
3. If $n(A \cup B) = n(A) + n(B) \Leftrightarrow A \cap B = \phi$ i.e. A and B are disjoint.
4. $n(A \cap B') = n(A) - n(A \cap B)$
5. $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$
6. $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$
7. $n(A - B) = n(A) - n(A \cap B)$
8. $n(A \cap B) = n(A \cup B) - n(A \cap B') - n(A' \cap B)$
9. $n(A\ \Delta\ B) = n(A \cup B) - n(A \cap B)$
10. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$
11. If $A_1, A_2....A_n$ are pairwise disjoint sets then, $n(A_1 \cup A_2 \cup .... \cup A_n) = n(A_1) + n(A_2) + ... + n(A_n)$
12. Number of elements belonging to exactly two of sets A, B and C $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3.\, n(A \cap B \cap C)$
13. Number of elements belonging to at least two of sets A, B and C $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 2.\, n(A \cap B \cap C)$
14. Number of elements belonging to at most two of sets A, B and C $= n(A \cup B \cup C) - n(A \cap B \cap C)$
15. Numbers of elements belonging to exactly one of the sets A, B and C $= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(A \cap C) - 2n(B \cap C) + 3\, n(A \cap B \cap C)$

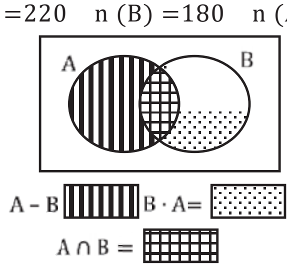
PRACTICAL PROBLEMS ON UNION AND INTERSECTION OF TWO SETS:

Ex. In a school with 500 students, 220 students have an interest in the science subject, 180 students like math, and 40 students like both science and math. Determine the number of students who have an interest in each subject.

- Science but not math
- Math but not science
- Either math or science

Sol. Consider the total number of students as U, representing the universal set.
Let A denote the set of students who have an interest in science.
Let B denote the set of students who have an interest in math.

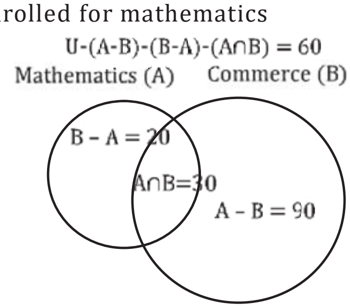
Here, $n(U) = 500$ $n(A) = 220$ $n(B) = 180$ $n(A \cap B) = 40$



- Here, we need to determine the number of students who have an interest in science but not in math. Symbolically, we have to find A-B.
  As illustrated in the Venn diagram,
  $$A = (A-B) \cup (A\cap B)$$
  $$n(A) = n(A-B) + n(A\cap B)$$
  $$n(A-B) = n(A) - n(A\cap B)$$
  $$= 220 - 40 = 180$$
  Therefore, the count of students who have an interest in science only, not math, is 180.
- In this case, we aim to determine the number of students who have an interest in math but not science. Symbolically, we have to find B-A.
  $$B = (B-A) \cup (A\cap B)$$
  $$n(B) = n(B-A) + n(A\cap B)$$
  $$n(B-A) = n(B) - n(A\cap B)$$
  $$=180-40=140$$
  Therefore, the count of students who have an interest in math only, not science, is 140.
- In this case, our objective is to determine the number of students who have an interest in either math or science.
  $$n(A\cup B) = n(A) + n(B) - n(A\cap B)$$
  $$= 220 + 180 - 40 = 360$$

Ex. In a college, 20 students are exclusively enrolled for commerce, 90 students are exclusively enrolled for mathematics, 30 students are enrolled for both commerce and mathematics, and 60 students are enrolled for other courses. Determine

- The overall student count in the college
- Total students enrolled in either commerce or mathematics
- Total students enrolled for commerce
- Total students enrolled for mathematics



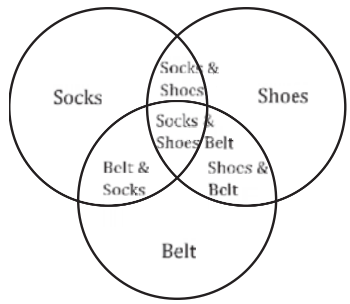
Sol. As illustrated in the Venn diagram, consider the total number of students as U.
Refer to students in Commerce as B.
Refer to students in Mathematics as A.
$$B - A = 20 \qquad A - B = 90 \qquad A\cap B = 30$$
As depicted in Venn diagram,
$$n(U)-n(A-B)-n(B-A)-n(A\cap B) = 60$$
$$n(U) = 60+ n(A-B) + n(B-A) + n(A\cap B)$$
$$= 60 + 90 + 20 + 30 = 200$$
Therefore, the total number of students in the college is 200.
$n(A\cup B) = n(A-B) + n(A\cap B) + n(B-A)$ that is, the formula of difference of sets
$$= 90 + 30 + 20 = 140$$
Therefore, there are 140 students enrolled in either commerce or mathematics.
$$n(B) = n(B-A) + n(A\cap B)$$
$$= 20 + 30 = 50$$

Therefore, there are 50 students enrolled in commerce.

$$n(A) = n(A-B) + n(A \cap B)$$
$$= 90 + 30 = 120$$

Therefore, there are 120 students enrolled in mathematics.

Ex. In a store, 380 individuals purchase socks, 150 individuals purchase shoes, and 200 individuals purchase belts. If there are a total of 580 individuals who bought either socks, shoes, or belts, and only 30 people bought all three items, how many people purchased exactly two items?



Sol. Let S, H, and B denote the sets representing the number of people who purchased socks, shoes, and belts, respectively.

$$n(S) = 380,\ n(H) = 150,\ n(B) = 200$$
$$n(S \cup H \cup B) = 580,\ n(S \cap H \cap B) = 30$$

Therefore, $n(S \cup H \cup B)$

$= n(S) + n(H) + n(B) - n(S \cap H) - n(H \cap B) - n(B \cap S) + n(S \cap H \cap B)$,

Now, we will substitute the given values into the formula.

$$580 = 380 + 150 + 200 - n(S \cap H) - n(H \cap B) - n(B \cap S) + 30$$

This gives that,

$$n(S \cap H) + n(H \cap B) + n(B \cap S) = 180$$

However, this encompasses the number of people who bought all three items as well. Therefore, we need to subtract this number of people from it.

Let, $n(S \cap H \cap B) = a$

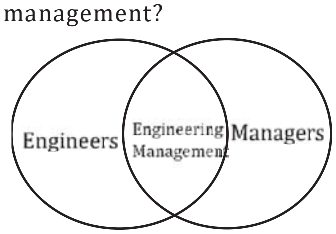
As evident from the Venn diagram,

$n(S \cap H) - a + n(H \cap B) - a + n(B \cap S) - a =$ the required number

$$n(S \cap H) + n(H \cap B) + n(B \cap S) - 3a$$
$$180 - 90 = 90$$

Therefore, there are 90 individuals who purchased exactly two items.

Ex. In a school, there are 25 teachers who instruct in either engineering or management. Among them, 15 teach engineering, and 6 teach both engineering and management. How many teachers instruct in management?



Sol. Consider M as the set of teachers who teach management, and E as the set of teachers who teach engineering. Here, 'or' represents union, and 'and' represents intersection.

Therefore, we have: $n(E \cup M) = 25$, $n(E) = 15$ and $n(E \cap M) = 6$

We have to calculate $n(M)$.

According to formula $n(E \cup M) = n(E) + n(M) - n(E \cap M)$, We obtain $25 = 15 + n(M) - 6$

Thus $n(M) = 16$

Therefore, 16 teachers are engaged in teaching management.

Ex. If there are 400 elements under consideration, set A comprises 100 elements, set B comprises 150 elements, and 75 are common to both sets. Determine the number of elements that are neither in set A nor in set B. Provide a Venn diagram to illustrate your answer.

Sol. Let U represent the set comprising all elements.

Given $n(U) = 400$, $n(A) = 100$, $n(B) = 150$ and $n(A \cap B) = 75$.

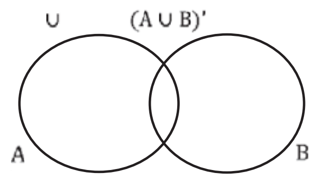
We need to determine the elements that are not in set A nor in set B, and in accordance with De Morgan's law of complement,

We know that $n(A' \cap B') = n(A \cup B)'$

$= n(U) - n(A \cup B)$

In other words, the complement of any set is U – A, and in this case, we need to calculate the complement of the union of A and B.

$$n(A \cup B)' = n(U) - n(A \cup B)$$
$$= n(U) - n(A) - n(B) + n(A \cap B)$$
$$= 400 - 100 - 150 + 75 = 225$$



Therefore, the count of elements that are neither in set A nor in set B is 225.

The Venn diagram of $n(A \cup B)'$ is illustrated in the figure on the right:

Ex. An examination was conducted on 500 families with pets. Out of these, 400 families had dogs, and 200 had cats, with 50 owning both cats and dogs as pets. Now, determine if this data is accurate and provide justification for your answer.

Sol. Consider U as the set of families investigated, D as the set of families with dogs as pets, and C as the set of families with cats as pets.

Given, $n(U) = 500$, $n(D) = 400$, $n(C) = 200$ and $n(D \cap C) = 50$.

Now, we will substitute the values into the formula for the union of two sets.

$$n(D \cup C) = n(D) + n(C) - n(D \cap C)$$
$$= 200 + 400 - 50 = 550$$

In this case, the union of two sets is 550, and the total number of families investigated is 500.

$$D \cup C \subset U \text{ implies } n(D \cup C) \leq n(U).$$

The union of two sets must be a subset of the universal set, meaning it should be less than or equal to the universal set. Since there is a contradiction, the provided data is inaccurate.