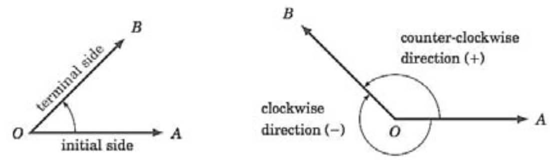
TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

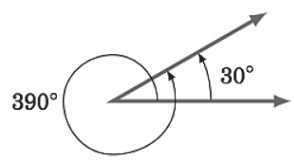
To define trigonometric functions for any angle, including those less than 0° or greater than 360°, we need a more general definition of an angle. We define an angle as being formed by rotating a ray.$\overrightarrow{OA}$ around the endpoint O (called the vertex), so that the ray moves to a new position, referred to as the ray $\overrightarrow{OB}$. The ray $\overrightarrow{OA}$ is known as the initial side of the angle, and $\overrightarrow{OB}$ is called the terminal side of the angle.



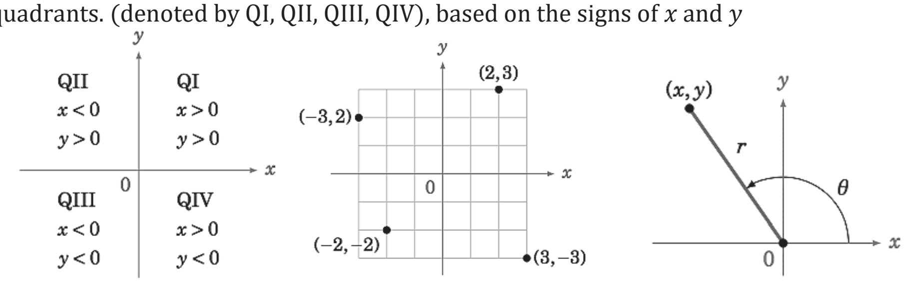
(a) angle ∠AOB (b) positive and negative angles

We represent the angle formed by this rotation as ∠AOB, or simply ∠O, or even just O. If the rotation is counterclockwise, the angle is considered positive; if the rotation is clockwise, the angle is considered negative.

A complete counterclockwise rotation of $\overrightarrow{OA}$ back onto itself (called a revolution), so that the terminal side aligns with the initial side, forms an angle of 360°. In the clockwise direction, this would be -360°. Not rotating $\overrightarrow{OA}$ constitutes an angle of 0° Any rotation exceeding one full turn results in an angle greater than 360°. For example, notice that 30° and 390° since $30 + 360 = 390$.



Now, we can define trigonometric functions for any angle using Cartesian coordinates. In the xy-coordinate plane, each point is represented by a pair of real numbers, (x, y). Here, x represents the horizontal position (x-coordinate) and y represents the vertical position (y-coordinate) of the point. These coordinates determine the exact location of the point in the plane, dividing it into four quadrants. (denoted by QI, QII, QIII, QIV), based on the signs of x and y



(a) Quadrants I-IV (b) Points in the plane (c) Angle θ in the plane

Now, consider any angle θ. We define θ to be in standard position if its initial side coincides with the positive x-axis and its vertex is at the origin (0,0). Let (x, y) be any point on the terminal side of θ, located at a distance $r > 0$ from the origin.

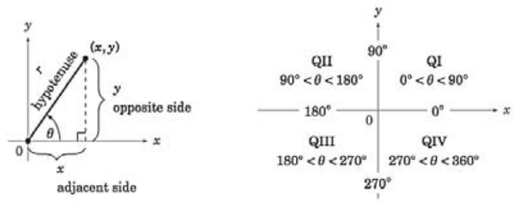
(Note that $r = \sqrt{x^2 + y^2}$. Why?) We then define the trigonometric functions of θ as follows:

$$\sin\theta = \frac{y}{r} \cos\theta = \frac{x}{r} \tan\theta = \frac{y}{x} \quad \dots (1)$$

$$\csc\theta = \frac{r}{y} \sec\theta = \frac{r}{x} \cot\theta = \frac{x}{y} \quad \dots (2)$$

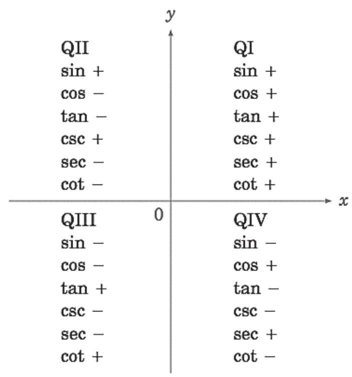
Similarly in the acute case, these definitions are well-defined using similar triangles (i.e., they do not depend on the specific point (x, y) chosen on the terminal side of θ). Also, observe that $|\sin\theta| \le 1$ and $|\cos\theta| \le 1$, since $|y| \le r$ and $|x| \le r$ in the above definitions.

Note that for an acute angle, these definitions align with our previous definitions involving right triangles: construct a right triangle where angle θ $x =$ adjacent side, $y =$ opposite side and $r =$ hypotenuse. For example, this would give us $\sin\theta = \frac{y}{r} = \frac{\text{opposite}}{\text{hypotenuse}}$ and $\cos\theta = \frac{x}{r} = \frac{\text{adjacent}}{\text{hypotenuse}}$ just as previously explained.

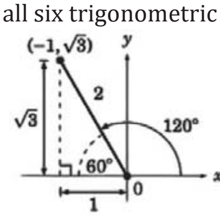


(a) Acute angle θ (b) Angles by quadrant

In which quadrants or along which axes the terminal side of an angle $0^\circ \le \theta < 360^\circ$ may fall and from equations (1) and (2), it's evident that trigonometric functions can yield negative values. For instance, $\sin\theta < 0$ when $y < 0$. summarizes the signs (positive or negative) for the trigonometric functions based on the angle's quadrant:



Ex. Determine the exact values for all six trigonometric functions of 120°.

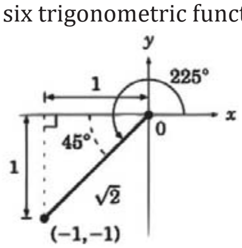


Sol. We know $120^\circ = 180^\circ - 60^\circ$. we see that we can use the point $(-1, \sqrt{3})$ on the terminal side of the angle 120° in the second quadrant, as we observed in that example, a fundamental right triangle with a 60° angle has an adjacent side of length 1 and an opposite side of $\sqrt{3}$, and hypotenuse of length 2, as illustrated in the diagram on the right. Drawing this triangle in the second quadrant (QII), ensuring the hypotenuse lies on 120° makes $r = 2, x = -1$, and $y = \sqrt{3}$.Hence:

$\sin 120^\circ = \frac{y}{r} = \frac{\sqrt{3}}{2} \cos 120^\circ = \frac{x}{r} = \frac{-1}{2} \tan 120^\circ = \frac{y}{x} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$

$\csc 120^\circ = \frac{r}{y} = \frac{2}{\sqrt{3}} \sec 120^\circ = \frac{r}{x} = \frac{2}{-1} = -2 \cot 120^\circ = \frac{x}{y} = \frac{-1}{\sqrt{3}}$

Ex. Determine the exact values for all six trigonometric functions of 225°.



Sol. We know that 225° = 180° + 45°. We can use the point (-1, -1) on the terminal side of the angle 225° in the third quadrant (QIII). As observed in that example, a fundamental right triangle with a 45° angle has an adjacent side of length 1, opposite side of length 1, and a hypotenuse of length $\sqrt{2}$, as shown in the diagram on the right. Drawing that triangle in the third quadrant (QIII), ensuring the hypotenuse lies on the terminal side 225° makes $r = \sqrt{2}, x = -1$, and $y = -1$. Hence:

$\sin 225^\circ = \frac{y}{r} = \frac{-1}{\sqrt{2}} \cos 225^\circ = \frac{x}{r} = \frac{-1}{\sqrt{2}} \tan 225^\circ = \frac{y}{x} = \frac{-1}{-1} = 1$

$\csc 225^\circ = \frac{r}{y} = -\sqrt{2} \sec 225^\circ = \frac{r}{x} = -\sqrt{2} \cot 225^\circ = \frac{x}{y} = \frac{-1}{-1} = 1$