## TRIGONOMETRIC RATIOS FOR COMPOUND ANGLES

A compound angle is what we call an angle that results from adding up two or more angles. We can represent compound angles using their corresponding trigonometric identities. By using the concept of compound angles, we can perform basic math operations like adding and subtracting functions. This helps in calculating more complex angles or expressions involving trigonometric functions.

**A. Trigonometric Ratios of the Sum and Difference of Two Angles Addition formulas**

1. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
2. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
3. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$
4. $\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot A + \cot B}$

$\cos(A+B) = \sin\left[\frac{\pi}{2} - (A+B)\right]$

$= \sin[(\frac{\pi}{2} - A) + (-B)]$

$= \sin[\frac{\pi}{2} - A]\cos(-B) + \cos(\frac{\pi}{2} - A)\sin(-B)$

$= \cos A\cos B - \sin A\sin B$ [ Applying $\cos(-B) = \cos B, \sin(-B) = -\sin B$]

**Proof**

5. $\tan(A+B) = \frac{(\sin(A+B))}{(\cos(A+B))} = \frac{(\sin A\cos B + \cos A\sin B)}{(\cos A\cos B - \sin A\sin B)}$

   Dividing the numerator and denominator by $\cos A \cdot \cos B$, we get

   $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

6. $\cot(A+B) = \frac{(\cos(A+B))}{(\sin(A+B))} = \frac{(\cos A\cos B - \sin A\sin B)}{(\sin A\cos B + \cos A\sin B)}$

   Dividing the numerator and denominator by sin A .sinB, we get

   $\cot(A+B) = \frac{\cot A\cot B - 1}{\cot A + \cot B)}$

**Difference Formulas:**

1. $\sin(A-B) = \sin[A + (-B)]$

   $\sin A\cos(-B) + \cos A\sin(-B)$

   $\sin A\cos B - \cos A\sin B$

2. $\cos(A-B) = \cos[A + (-B)]$

   $\cos A\cos(-B) - \sin A\sin(-B)$

   $\cos A\cos B + \sin A\sin B$

3. $\tan(A-B) = \tan[A + (-B)]$

   $\frac{\tan A + \tan(-B)}{1 - \tan A\tan(-B)}$

   $\frac{\tan A - \tan B}{1 + \tan A\tan B}$

4. $\cot(A-B) = \cot[A + (-B)]$

   $\frac{\cot A\cot(-B) - 1}{\cot A + \cot(-B)}$

   $\frac{1 + \cot A\cot B}{\cot B - \cot A}$

**Special Results:**

1. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

**Proof:**

LHS $= \sin(A+B)\sin(A-B)$

$= \sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$

$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$

$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$

$= \sin^2 A - \sin^2 B$

$= (1-\cos^2 A) - (1-\cos^2 B)$

$= (1-\cos^2 A) - (1-\cos^2 B)$

$= \cos^2 B - \cos^2\ A$

$= (1 - \cos^2 A) - (1 - \cos^2 B)$

2. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

**Proof:**

LHS $= \cos(A+B)\cos(A-B) = (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$

$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$

$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$

$= \cos^2 A(1 - \sin^2 B) - (1 - \cos^2 A)\sin^2 B$

$= \cos^2 A - \sin^2 B = (1 - \sin^2 A) - (1 - \cos^2 B)$

$= \cos^2 B - \sin^2 A$

3. $\tan(45° + A) = \frac{\tan 45° + \tan A}{1 - \tan 45°.\tan A} = \frac{1 + \tan A}{1 - \tan A}$

4. $\tan(45° - A) = \frac{1 - \tan A}{1 + \tan A}$

**Ex.** If $\sin A = \frac{3}{5}, 0 < A < \frac{\pi}{2}$ and $\cos B = \frac{-12}{13}, \pi < B < \frac{3\pi}{2}$, then find the values of the following:

1. $\sin(A-B)$
2. $\cos(A+B)$
3. $\tan(A-B)$

**sol.** We have

$$\sin A = \frac{3}{5}, \text{ where } 0 < A < \frac{\pi}{2}$$

$$\cos A = \pm\sqrt{1-\sin^2 A}$$

$$\cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$$

[ ∵ Cosine is positive in the first quadrant]

In the first quadrant, tangent function is positive

$$\tan A = \frac{\sin A}{\cos A} = \frac{3}{4}$$

It is given that, $\cos B = -\frac{12}{13}$ where $\pi < B < \frac{3\pi}{2}$

$$\sin B = \pm\sqrt{1-\cos^2 B}$$

$$\sin B = -\sqrt{1-\cos^2 B}$$

[∵ Sine is negative in the third quadrant]

$$\sin B = -\sqrt{1-(\frac{-1}{13})^2} = -\frac{5}{13}$$

In the third quadrant, tangent function is positive

$$\therefore \tan B = \frac{\sin B}{\cos B} = \frac{5}{12}$$

1. $\sin(A-B) = \sin A\cos B - \cos A\sin B = \frac{3}{5}\times\left(\frac{-12}{13}\right) - \frac{4}{5}\times\left(\frac{-5}{13}\right) = \frac{-16}{65}$
2. $\cos(A+B) = \cos A\cos B - \sin A\sin B = \frac{4}{5}\times\left(\frac{-12}{13}\right) - \frac{3}{5}\times\left(\frac{-5}{13}\right) = \frac{-33}{65}$
3. $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B} = \frac{\frac{3}{4}-\frac{5}{12}}{1+\frac{3}{4}\times\frac{5}{12}} = \frac{16}{63}$

**Ex.** Prove that $\cos\left(\frac{\pi}{4}-A\right)\cos\left(\frac{\pi}{4}-B\right)-\sin\left(\frac{\pi}{4}-A\right)\sin\left(\frac{\pi}{4}-B\right)=\sin(A+B)$

**Sol.** We have,

LHS $=\cos(\frac{\pi}{4}-A)\cos(\frac{\pi}{4}-B)-\sin(\frac{\pi}{4}-A)\sin(\frac{\pi}{4}-B)$

$=\cos\{(\frac{\pi}{4}-A)+(\frac{\pi}{4}-B)\}$

$=\cos\{\frac{\pi}{2}-(A+B)\}$

$=\sin(A+B)=$ RHS

Hence proved

## Greatest and least values of a cos θ + b sin θ

$S = a\cos\theta + b\sin\theta$

$r(\frac{a}{r}\cos\theta + \frac{b}{r}\sin\theta)$: $r = \sqrt{a^2 + b^2}$

$r(\sin(\theta + \alpha))$; $\sin\alpha = \frac{a}{r}$; $\cos\alpha = \frac{b}{r}$

Since $-1 \leq \sin(\theta + \alpha) \leq 1$, therefore, $-r \leq S \leq r$

**Ex.** Demonstrate that $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$ is within the range of –4 to 10.

**Sol.** $5\cos\theta + 3\cos(\theta + \frac{\pi}{3}) + 3$

$5\cos\theta + 3(\frac{\cos\theta}{2} - \frac{\sqrt{3}\sin\theta}{2}) + 3$

$\frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3$

Since $-7 \leq \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \leq 7$

$-4 \leq S \leq 10$

## Conditional Identities

When three angles A, B, and C adhere to a given relationship, various identities can be derived that establish connections between the trigonometric ratios of these angles in a triangle.

ABC, $A + B + C = \pi$

$\sin(A + B) = \sin(\pi - C) = \sin C$

And $\cos(A + B) = \cos(\pi - C) = -\cos C$

Also, $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$

Hence $\sin\left(\frac{A+B}{2}\right) - \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) - \cos\frac{C}{2}$

$\cos\left(\frac{A+B}{2}\right) - \cos(\frac{\pi}{2} - \frac{C}{2}) - \sin\frac{C}{2}$

**Remember :**

If $A + B + C = \pi$, then

(i) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

(ii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$

(iii) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(iv) $\sin A + \sin B + \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(v) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(vi) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(vii) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(viii) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

**Ex.** If $A = \cos^2\theta + \sin^4\theta$, then this expression holds true for all values of θ.

(1) $1 \leq A \leq 2$ (2) $\frac{13}{16} \leq A \leq 1$ (3) $\frac{3}{4} \leq A \leq \frac{13}{16}$ (4) $\frac{3}{4} \leq A \leq 1$

**Sol.** $A = \cos^2\theta + \sin^2\theta \sin^2\theta$

$A \leq \cos^2\theta + \sin^2\theta.1$ ($\because \sin^2\theta \leq 1$)

$A \leq 1$

Again, $A = (1 - \sin^2\theta) + \sin^4\theta$

$A = (\sin^2\theta - \frac{1}{2})^2 + (1 - \frac{1}{4})$

$A \geq \frac{3}{4}$

Hence, $\frac{3}{4} \leq A \leq 1$

**Ex.** Solve $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$

(1) 0 (2) $\frac{1}{2}$ (3) $\frac{1}{3}$ (4) $-\frac{1}{8}$

**Sol.** $\frac{1}{2\sin\frac{\pi}{7}}\times(2\sin\frac{\pi}{7}\cos\frac{\pi}{7})\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$

$\frac{1}{2\sin\frac{\pi}{7}}\times\frac{1}{2}(2\sin\frac{2\pi}{7}\cos\frac{2\pi}{7})\cos\frac{4\pi}{7}$

$\frac{1}{4\sin\frac{\pi}{7}}\times\frac{1}{2}(2\sin\frac{4\pi}{7}\cos\frac{4\pi}{7})$

$\frac{1}{8}\times\frac{\sin\frac{8\pi}{7}}{\sin\frac{\pi}{7}}=\frac{\sin(\pi+\frac{\pi}{7})}{8\sin\frac{\pi}{7}}$

$\frac{-\sin\frac{\pi}{7}}{8\sin\frac{\pi}{7}}=-\frac{1}{8}$

**Ex.** The period of the function, f (x) = 3 sin (2x + 1) measured in radians is

(1) $2\pi$ (2) $\pi$ (3) $\frac{\pi}{2}$ (4) $-\pi$

**Sol.** Period of sin x is $2\pi$, the period of

$f(x) = 3\sin(2x + 1)$ is $\frac{2\pi}{2}=\pi$

Alternatively we have

$f(x) = 3\sin(2x + 1) = 3\sin(2\pi + 2x + 1)$

$3\sin\{2(\pi + x) + 1\} = f(\pi + x)$

Period of f (x) is $\pi$