SUM OF SINE OR COSINE OF N ANGLES IN A.P.

$$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\cdots+\sin(\alpha+(n-1)\beta)=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\times\sin\left(\alpha+(n-1)\frac{\beta}{2}\right)$$

Proof: Let $S=\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\cdots+\sin(\alpha+(n-1)\beta)$

Here angles are in A.P and common difference of angles in β

$$=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\times\sin\left(\frac{\text{First angle }+\text{ last angle}}{2}\right).$$

multiplying both sides by $2\sin\left(\frac{\beta}{2}\right)$, we get

$$2\sin\left(\frac{\beta}{2}\right)S=\sin\alpha 2\sin\left(\frac{\beta}{2}\right)+\sin(\alpha+\beta)\,2\sin\left(\frac{\beta}{2}\right)+\sin(\alpha+2\beta)\,2\sin\left(\frac{\beta}{2}\right)+\cdots+\sin(\alpha+(n-1)\beta)2\sin(\frac{\beta}{2})$$

(Since $2\sin A\sin B=\cos(A-B)-\cos(A+B)$)

$$2\sin\alpha\sin\left(\frac{\beta}{2}\right)=\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{\beta}{2}\right),$$

$$2\sin(\alpha+\beta)\sin\left(\frac{\beta}{2}\right)=\cos\left(\alpha+\beta-\frac{\beta}{2}\right)-\cos\left(\alpha+\beta+\frac{\beta}{2}\right)$$

$$=\cos(\alpha+\frac{\beta}{2})-\cos(\alpha+\frac{3\beta}{2}),$$

$$2\sin(\alpha+2\beta)\sin(\frac{\beta}{2})=\cos(\alpha+2\beta-\frac{\beta}{2})-\cos(\alpha+2\beta+\frac{\beta}{2}),$$

$$=\cos(\alpha+\frac{3\beta}{2})-\cos(\alpha+\frac{5\beta}{2}),$$

From above equations,

$$2\sin(\frac{\beta}{2})S=\cos(\alpha-\frac{\beta}{2})-\cos(\alpha+(2n-1)\frac{\beta}{2}),$$

$$2\sin(\frac{\beta}{2})S=2\sin(\alpha+(n-1)\frac{\beta}{2})\sin(\frac{n\beta}{2}),$$

$$S=\frac{\sin(\frac{n\beta}{2})}{\sin(\frac{\beta}{2})}\sin(\alpha+(n-1)\frac{\beta}{2}),$$

In the above result, replacing α by $\frac{\pi}{2}+\alpha$, we get

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\cdots+\cos(\alpha+(n-1)\beta)=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos(\alpha+(n-1)\frac{\beta}{2})$$

Hence proved.

Ex. Find the value of $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}$.

Sol.: Given that $\cos(\frac{2\pi}{7})+\cos(\frac{4\pi}{7})+\cos(\frac{6\pi}{7})$,

$$S=\frac{\sin(\frac{n\beta}{2})}{\sin(\frac{\beta}{2})}\sin(\alpha+(n-1)\frac{\beta}{2})$$

$$=\frac{\sin(3\frac{\pi}{7})}{\sin(\frac{\pi}{7})}\cos(\frac{\pi}{7}+\frac{3\pi}{7})$$

$$=\frac{2\sin(\frac{3\pi}{7})\cos(\frac{4\pi}{7})}{2\sin(\frac{\pi}{7})}=\frac{\sin(\frac{7\pi}{7})-\sin(\frac{\pi}{7})}{2\sin(\frac{\pi}{7})}=-\frac{1}{2}.$$