PERIODIC FUNCTIONS

A function 'f' defined on its domain is said to be periodic function if their exist a positive number T such that $f(x+T) = f(x)"x \in D$

Additionally, both $x + T$ and $x - T$ should belong to D.

The smallest value of T, it exists is referred to the period of the function.

Ex. $f(x) = \sin x$

$$f(x) = \sin(x + 2\pi) = \sin(x + 4\pi) = \sin(x + 6\pi) \ldots$$

$$T = 2\pi, 4\pi, 6\pi \ldots$$

Least value of T is 2π, so time period of $\sin x$ is 2π

Some Standard Functions and their Period

Function	Period
$\sin x$	2π
$\cos x$	2π
$\tan x$	π
$\{x\}$	1

Some Special Point about Periodic Function

1. If period of $f(x)$ is 'T' then

(a) Period of $|f(x)|$ is. $\frac{T}{2}$.

(b) Period of $[f(x)]^n$ is, $\frac{T}{2}$, if n is even number $(n \in N)$

(c) Period of $[f(x)]^n$ is T, if n is odd number $(n \in N)$

(d) Period of $f(ax)$ and $f(ax + b)$ is. $\frac{T}{|a|}$.

(e) Period of $f(\frac{x}{a})$ is $|a|T$.

2. If the Period of $f(x)$ and $g(x)$ are both 'T' then the period of $f(x) \pm g(x)$is given by

(a) $\frac{T}{2}$ (if $f(x)$ and $g(x)$ both are even).

(b) T (If $f(x)$ is any function except even).

3. If period $f(x)$ is T_1 and $g(x)$ is, T_2. Then period of $f(x) \pm g(x)$ is given by L.C.M. of T_1 and T_2 (same for $\frac{f(x)}{g(x)}$)

Note

1. LCM of $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} = \frac{\text{LCM of a,c,e}}{\text{HCF of b,d,f}}$

2. Functions like $\text{Sin}\, x$ and $\sin x^2$ is not a periodic function because these can't be expressed in the form of $[f(x+T) = f(x)]$

3. It is not possible to find the least common multiple (LCM) of a rational number with an irrational number. For example, the LCM of $(\pi, 2{,}2\pi)$ is not achievable as $\pi, 2\pi \in$ irrational and $\in$ rational.

Ex. Determine the period of $f(x) = \sin 3x + \cos 2x$

Sol: Period of $\sin \mathbf{3x} = \frac{\mathbf{2\pi}}{\mathbf{3}}$

Period of $\cos 2x = \frac{2\pi}{2} = \pi$

So, Period of $f(x)$ is L.C.M. of $\frac{2\pi}{3}, \frac{\pi}{1} = 2\pi$

Ex. Identify the period of the function $\mathbf{f(x) = \sqrt{1 + \sin 2x}}$ if it is periodic.

Sol: $f(x) = \sqrt{1 + \sin 2x}$

$\sqrt{\sin^2 x + \cos^2 x + 2\sin x\cos x}$

$\sqrt{(\sin x + \cos x)^2}$

$f(x) = |\sin x + \cos x|$ (Remember $\sqrt{x^2} = |x|$)

Now, period of $\sin x + \cos x$ is 2π

So, period of $|\sin x + \cos x|$ is $\frac{2\pi}{2} = \pi$

Bounded and Unbounded Function

If f(x) is such that there exists a constant M where f(x) is always less than or equal to M for all x, then f(x) is considered bounded above. Similarly, if there exists a constant m such that f(x) is never less than m ($\mathbf{i.e.}, \mathbf{m} \leq \mathbf{f}(\mathbf{x}) \leq \mathbf{M}$ for all, then $\mathbf{f}(\mathbf{x})$ is said to be bounded below. If one or both of the upper and lower bounds (M and m) are infinite, then f(x) is termed unbounded.

Example:

The function $f(x) = 3 + \sin x$ is considered a bounded function since the maximum and minimum value of $\sin x$ are $+1$ and -1 respectively.

Consequently, $2 \leq f(x) \leq 4$ for all value of x.