Since range of f is a subset of the domain of g,

Domain of gof is $[0,\infty)$ and $g\{f(x)\} = g(\sqrt{x}) = x - 1$. Range of gof is $[-1,\infty)$ For fog(x)
Since range of g is not a subset of the domain of f i.e. $[-1,\infty) \not\subset [0,\infty)$ fog is not defined on whole of the domain of g.
Domain of fog is $\{x \in R$, the domain of g: $g(x) \in [0,\infty)$, the domain of f$\}$..
Thus the domain of fog is $D = \{x \in R: 0 \le g(x) < \infty\}$
$D = \{x \in R: 0 \le x^2 - 1\}$
$\{x \in R: x \le -1 \text{ or } x \ge 1\}$
$(-\infty,-1] \cup [1,\infty)$
$fog(x) = f\{g(x)\}$
$f(x^2 - 1) = \sqrt{x^2 - 1}$
Its range is $[0,\infty)$

Ex. Find fog(x) If $f(x) = \cos x + x$ and $g(x) = x^2$.
Sol: $fog(x) = \cos g(x) + g(x)$
$= \cos x^2 + x^2$
Ex. If $f(x) = ||x - 3| - 2|; 0 \le x \le 4$ and $g(x) = 4 - |2 - x|; -1 \le x \le 3$ Then find fog (2)
Sol: $fog(2) = f(4)$ $(\because g(2) = 4)$
$fog\ (2) = 1$

Inverse Function

Two functions f and g are inverse of each other if $f\big(g(x)\big) = x$ for $x \in$ domain of g and $g\big(f(x)\big) = x$ for $x \in$ dom f, i.e., $gof = I_{dom}$ f and $fog = I_{dom}$ g where I_{dom} fis identity function on dom f and I_{dom} g is identity function on dom g. We denote g by
f^{-1} or f by g^{-1}.. To find the inverse of f, write down the equation $y = f(x)$ and then solve x as a function of y. The resulting equation is $x = f^{-1}(y)$

Note

For an inverse function to exist, the original function should be both one-to-one and onto

Properties:

1. Inverse of a bijection is also a bijection function.
2. Inverse of a bijection is unique.
3. $(f - 1)^{-1} = f$
4. If f and g are two bijections such that (gof) exists then $(\ gof\ )^{-1} = f^{-1}og^{-1}$
5. If $f: A \to B$ is a bijection then $f^{-1}: B \to A$ is an inverse function of f.

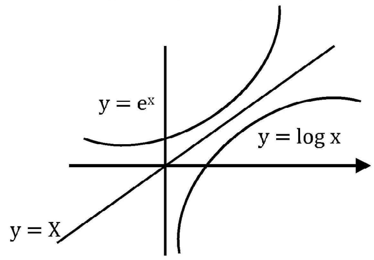
$$f^{-1}of = I_A \text{ and } fof^{-1} = I_B.$$

HereI_A, represents the identity function on set A, and I_B, represents the identity function on set B.

Ex. To determine the inverse of $f(x) = \frac{e^x - e^{-x}}{2}$
Sol. We write $y = \frac{e^x - e^{-x}}{2}$
$2y = \frac{e^{2x} - 1}{e^x}$
$e^{2x} - 2ye^x - 1 = 0$
$e^x = \frac{2y \pm \sqrt{4y^2+4}}{2}$
$e^x = y \pm \sqrt{y^2 + 1}$
e^{X^3} 0 so $e^x = y + \sqrt{y^2 + 1}$
$x = \log(y + \sqrt{y^2 + 1})$
$f^{-1}(x) = \log(x + \sqrt{x^2 + 1})$

The relationship between the graphs of f and f^{-1} is such that if the point (x, y) is on the graph of f, then the point (y, x) is on the graph of f^{-1} and vice versa. Consequently, the graph of f^{-1} is a reflection of the graph of f across the line y = x.

This can be illustrated with the example of y = log x and which are inverses of each other.



Existence of inverse function

Not all functions possess an inverse. For instance, the function $f(x) = x^2$ lacks an inverse when the domain of f is R. For a function to have an inverse, it must satisfy the conditions of being both one-one and onto, making it bijective.

Ex. **1.** Determine the objectiveness of $f(x) = \frac{2x+3}{4}$ forf: R → R, If bijective. Find its inverse function $f^{-1}(x)$.

2. Consider, $f(x) = x^2 + 2x; x \geq -1$. Sketch the graph of $f^{-1}(x)$ and determine the number of Solutions of the equation, $f(x) = f^{-1}(x)$

3. If $y = f(x) = x^2 - 3x + 1$ for $x \geq 2$. Calculate the value of g¢ (1) where g is inverse of f.

Sol: **1.** The function is bijective, indicating that it has a unique inverse.

$y = \frac{2x+3}{4}$

$x = \frac{4y-3}{2}$

$f^{-1}(x) = \frac{4x-3}{2}$

2. The equation $f(x) = f^{-1}(x)$ is equivalent to $f(x) = x$

$x^2 + 2x = x$

$x(x + 1) = 0$

$x = 0, -1$

3. Therefore, there are two solutions for $f(x) = f^{-1}(x)$

$y = 1$

$x^2 - 3x + 1 = 1$

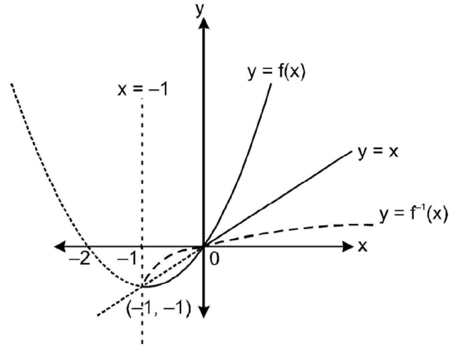
$x(x - 3) = 0$

$x = 0,3$

$x \geq 2$

$x = 3$

$g(f(x)) = x$



Differentiating both sides w.r.t. x

$g'(f(x)) \cdot f'(x) = 1$

$g'(f(x)) = \frac{1}{f'(x)}$

$g'(f(3)) = \frac{1}{f'(3)}$

$g'(1) = \frac{1}{6-3} = \frac{1}{3} = (\text{As}(x) = 2x - 3)$

Alternate Method

$y = x^2 - 3x + 1$

$x^2 - 3x + 1 - y = 0$

$x = \frac{3 \pm \sqrt{9-4(1-y)}}{2} = \frac{3 \pm \sqrt{5+4y}}{2}$

$x \geq 2$

$x = \frac{3+\sqrt{5+4y}}{2}$

$g(x) = \frac{3+\sqrt{5+4x}}{2}$

$g'(x) = 0 + \frac{1}{4\sqrt{5+4x}} 4$

$g'(1) = \frac{1}{\sqrt{5+4}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$

Domain and Range of Some Standard Function

Function	Domain	Range
Polynomial function	R	R
Identity function x	R	R
Constant function c	R	$\{c\}$
Reciprocal fn $1/x$	R_0	R_0
Signum function	R	$\{-1,0,1\}$
$ax + b; a, b \in R$	R	R
$ax^3 + b; a, b \in R$	R	R
$x^2, \|x\|$	R	$R^+ \cup \{0\}$
$x^3, x\|x\|$	R	R
$x + \|x\|$	R	$R^+ \cup \{0\}$
$x - \|x\|$	R	$R^- \cup \{0\}$
$[x]$	R	Z
$x - [x]$	R	$[0,1)$
$\frac{\|x\|}{x}$	R_0	$\{-1,1\}$
$\sqrt{x}$		$[0,\infty)[0,\infty)$
a^x	R	R^+
log x	R^+	R
sin x	R	$[-1,1]$
cos x	R	$[-1,1]$
tan x	$R - \{\frac{(2n+1)\pi}{2}\}$	$R \mid n \in z\}$
cot x	$R - \{n\pi \mid n \in z\}$	R
sec x	$R - \{\frac{(2n+1)\pi}{2}$	$R - (-1,1) \mid n \in z\}$
cosec x	$R - \{n\pi \mid n \in z\}$	$R - (-1,1)$
$\sin^{-1}x$	$[-1,1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\cos^{-1}x$	$[-1,1]$	$[0,\pi]$
$\tan^{-1}x$	R	$\left(-\frac{\pi}{2},+\frac{\pi}{2}\right)$
$\cot^{-1}x$	R	$(0,\pi)$
$\sec^{-1}x$	$R-(-1,1)$	$[0,\pi]-\{\frac{\pi}{2}\}$
$\text{cosec}^{-1}x$	$R-(-1,1)$	$(-\frac{\pi}{2},\frac{\pi}{2}]-\{0\}$

Ex. Determine the value of $\tan^{-1}(\tan\frac{3\pi}{4})$.

Sol. $\tan^{-1}(\tan x)=x$

$x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

$\tan^{-1}\left(\tan\left(\pi-\frac{\pi}{4}\right)\right)$

$\tan^{-1}(-\tan\frac{\pi}{4})$

$\tan^{-1}(-\tan\frac{\pi}{4})$

$-\tan^{-1}(\tan\frac{\pi}{4})$ (using property 1)

$-\tan^{-1}(\tan\frac{\pi}{4})=-\frac{\pi}{4}$ (using property 3)