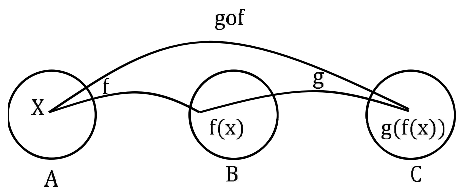
COMPOSITE FUNCTION

Let $f: A \to B$ and $g: B \to C$ then the composition of g and f is denoted by gof and is defined as gof: $A \to C$ given by gof $(x) = g(f(x))$

Similarly fog is defined. Note that, gof is defined only if Range $f \subseteq$ Dom g and fog is defined only if Range $\subseteq$ dom f.

The domain of fog is given by $\{x \in \text{dom } g: g(x) \in \text{dom } f\}$



Note

(a) Function gof will exist only when range of f is the subset of domain of g.
(b) Gof (x) is simply the g-image of f(x), where f(x) is f-image of elements $x \in A$
(c) Fog does not exist here because range of g is not a subset of domain of f.

Properties of composite function:

1. If f and g are two functions then for composite of two functions fog ^{1} gof.
2. Composite functions obeys the property of associativity fo (goh) = (fog) oh.
3. Composite function of two one-one onto functions if exist, will also be a one-one onto function.

Ex. Let $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$. Since the domain g is,$[0, \infty)$, and the domain f is R

Sol. We have $fog(x) = f(g(x))$
$f(\sqrt{x}) = (\sqrt{x})^2 + 3 = x + 3$
So, dom fog $= \{x \in [0, \infty): g(x) \in R\} = [0, \infty)$
Now, let's determine the composition gof,
We have $(gof)(x) = g(f(x)) = g(x^2 + 3) = \sqrt{x^2 + 3}$,
dom gof $= \{x \in R: f(x) \in [0, \infty)\} = R$.

Ex. Describe fog and gof wherever is possible for the following functions
1. $f(x) = \sqrt{x + 3}, g(x) = 1 + x^2$ **2.** $f(x) = \sqrt{x}, g(x) = x^2 - 1$.

Sol: **1.** Domain of f is $[-3, \infty)$, range of f is $[0, \infty)$.
Domain of gR, range of g is $[1, \infty)$.
For gof(x)
Since range of f is a subset of domain of g,
Domain of gof is $[-3, \infty)$ {equal to the domain of f}
$gof(x) = g\{f(x)\}$
$g(x) = 1 + \sqrt{x + 3}(x + 3) = x + 4$.
Range of gof is $[1, \infty)$.
For fog(x)
Since range of g is a subset of domain of f,
Domain of fog is R {equal to the domain of g}
fog $(x) = f\{g(x)\}$
$f(1 + x^2) = \sqrt{x^2 + 4}$
Range of fog is $[2, \infty)$.

2. $f(x) = \sqrt{x}, g(x) = x^2 - 1$.
Domain of f is $[0, \infty)$, range of f is $[0, \infty)$.
Domain of g is R, range of g is $[-1, \infty)$.
For gof(x)

Since range of f is a subset of the domain of g,

Domain of gof is $[0,\infty)$ and $g\{f(x)\} = g(\sqrt{x}) = x - 1$. Range of gof is $[-1,\infty)$ For fog(x)
Since range of g is not a subset of the domain of f i.e. $[-1,\infty) \not\subset [0,\infty)$ fog is not defined on whole of the domain of g.
Domain of fog is $\{x \in R$, the domain of g: $g(x) \in [0,\infty)$, the domain of f}..
Thus the domain of fog is $D = \{x \in R: 0 \le g(x) < \infty\}$
$D = \{x \in R: 0 \le x^2 - 1\}$
$\{x \in R: x \le -1 \text{ or } x \ge 1\}$
$(-\infty,-1] \cup [1,\infty)$
$fog(x) = f\{g(x)\}$
$f(x^2 - 1) = \sqrt{x^2 - 1}$
Its range is $[0,\infty)$

Ex. Find fog(x) If $f(x) = \cos x + x$ and $g(x) = x^2$.
Sol: $fog(x) = \cos g(x) + g(x)$
$= \cos x^2 + x^2$
Ex. If $f(x) = ||x-3| - 2|;\ 0 \le x \le 4$ and $g(x) = 4 - |2 - x|;\ -1 \le x \le 3$ Then find fog (2)
Sol: $fog(2) = f(4)$ $(\because g(2) = 4)$
$fog\ (2) = 1$

Inverse Function

Two functions f and g are inverse of each other if $f\big(g(x)\big) = x$ for $x \in$ domain of g and $g\big(f(x)\big) = x$ for $x \in$ dom f, i.e., $gof = I_{dom}\, f$ and $fog = I_{dom}\, g$ where I_{dom} fis identity function on dom f and I_{dom} g is identity function on dom g. We denote g by
f^{-1} or f by g^{-1}.. To find the inverse of f, write down the equation $y = f(x)$ and then solve x as a function of y. The resulting equation is $x = f^{-1}(y)$

Note

For an inverse function to exist, the original function should be both one-to-one and onto

Properties:

1. Inverse of a bijection is also a bijection function.
2. Inverse of a bijection is unique.
3. $(f-1)^{-1} = f$
4. If f and g are two bijections such that (gof) exists then $(\, gof\,)^{-1} = f^{-1}og^{-1}$
5. If $f: A \to B$ is a bijection then $f^{-1}: B \to A$ is an inverse function of f.

$$f^{-1}of = I_A \text{ and } fof^{-1} = I_B.$$

HereI_A, represents the identity function on set A, and I_B, represents the identity function on set B.

Ex. To determine the inverse of $f(x) = \frac{e^x - e^{-x}}{2}$
Sol. We write $y = \frac{e^x - e^{-x}}{2}$
$2y = \frac{e^{2x}-1}{e^x}$
$e^{2x} - 2ye^x - 1 = 0$
$e^x = \frac{2y \pm \sqrt{4y^2+4}}{2}$
$e^x = y \pm \sqrt{y^2 + 1}$
e^{X^3} 0 so $e^x = y + \sqrt{y^2 + 1}$
$x = \log(y + \sqrt{y^2 + 1})$
$f^{-1}(x) = \log(x + \sqrt{x^2 + 1})$