CHAPTER-9
CIRCULAR MOTION

CIRCULAR MOTION :

When a particle moves in a plane such that its distance from a fixed (or moving) point remains constant, then its motion is known as circular motion with respect to that fixed (or moving) point. The fixed point is called centre, and the distance of particle from it is called radius.

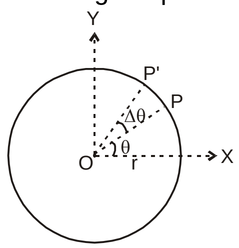
1. KINEMATICS OF CIRCULAR MOTION :

1.1 Variables of Motion :

(a) Angular Position : To decide the angular position of a point in space we need to specify (i) origin and (ii) reference line.

The angle made by the position vector w.r.t. origin, with the reference line is called angular position.

Clearly angular position depends on the choice of the origin as well as the reference line.



Circular motion is a two dimensional motion or motion in a plane. Suppose a particle P is moving in a circle of radius r and centre O.

The angular position of the particle P at a given instant may be described by the angle θ between OP and OX. This angle θ is called the **angular position** of the particle.

(b) Angular Displacement :

Definition : Angle through which the position vector of the moving particle rotates in a given time interval is called its angular displacement. Angular displacement depends on origin, but it does not depends on the reference line. As the particle moves on above circle its angular position θ changes. Suppose the point rotates through an angle $\Delta\theta$ in time Δt, then $\Delta\theta$ is angular displacement.

☞ **Important points :**

- Angular displacement is a dimensionless quantity. Its SI unit is radian, some other units are degree and revolution.
  2π rad = 360° = 1 rev
- Infinitesimally small angular displacement is a vector quantity, but finite angular displacement is a scalar, because while the addition of the Infinitesimally small angular displacements is commutative, addition of finite angular displacement is not.
  $d\vec{\theta}_1 + d\vec{\theta}_2 = d\vec{\theta}_2 + d\vec{\theta}_1$ but $\theta_1 + \theta_2 \neq \theta_2 + \theta_1$
- Direction of small angular displacement is decided by right hand thumb rule. When the fingers are directed along the motion of the point then thumb will represents the direction of angular displacement.

(c) Angular Velocity ω

(i) Average Angular Velocity

$$\omega_{av} = \frac{\text{Angular displacement}}{\text{Total time taken}}$$

$$\omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

where θ_1 and θ_2 are angular position of the particle at time t_1 and t_2. Since angular displacement is a scalar, average angular velocity is also a scalar.

(ii) Instantaneous Angular Velocity : It is the limit of average angular velocity as Δt approaches zero. i.e. $\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\theta}}{\Delta t} = \frac{d\vec{\theta}}{dt}$

Since infinitesimally small angular displacement $\overrightarrow{d\theta}$ is a vector quantity, instantaneous angular velocity $\vec{\omega}$ is also a vector, whose direction is given by right hand thumb rule.

☞ Important points :

- Angular velocity has dimension of $[T^{-1}]$ and SI unit rad/s.
- For a rigid body, as all points will rotate through same angle in same time, angular velocity is a characteristic of the body as a whole, e.g., angular velocity of all points of earth about earth's axis is $(2\pi/24)$ rad/hr.
- If a body makes 'n' rotations in 't' seconds then average angular velocity in radian per second will be

$$\omega_{av} = \frac{2\pi n}{t}$$

If T is the period and 'f' the frequency of uniform circular motion $\omega_{av} = \frac{2\pi}{T} = 2\pi f$

Example 1. If angular displacement of a particle is given by $\theta = a - bt + ct^2$, then find its angular velocity.

Solution : $\omega = \frac{d\theta}{dt} = -b + 2ct$

Example 2. Is the angular velocity of rotation of hour hand of a watch greater or smaller than the angular velocity of Earth's rotation about its own axis ?

Solution : Hour hand completes one rotation in 12 hours while Earth completes one rotation in 24 hours. So, angular velocity of hour hand is double the angular velocity of Earth. $\left(\omega = \frac{2\pi}{T}\right)$.

(d) Angular Acceleration α :

(i) Average Angular Acceleration : Let ω_1 and ω_2 be the instantaneous angular speeds at times t_1 and t_2 respectively, then the average angular acceleration α_{av} is defined as $\vec{\alpha}_{av} = \frac{\vec{\omega}_2 - \vec{\omega}_1}{t_2 - t_1} = \frac{\Delta\vec{\omega}}{\Delta t}$

(ii) Instantaneous Angular Acceleration : It is the limit of average angular acceleration as Δt approaches zero, i.e., $\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{\Delta t} = \frac{d\vec{\omega}}{dt}$

since $\vec{\omega} = \frac{d\vec{\theta}}{dt}$, $\quad \therefore \vec{\alpha} = \frac{d\vec{\omega}}{dt} = \frac{d^2\vec{\theta}}{dt^2}$, $\quad$ Also $\vec{\alpha} = \omega\frac{d\vec{\omega}}{d\theta}$

☞ Important points :

- Both average and instantaneous angular acceleration are axial vectors with dimension $[T^{-2}]$ and unit rad/s^2.
- If $\alpha = 0$, circular motion is said to be uniform.

1.2 Motion with constant angular velocity

$\theta = \omega t$, $\alpha = 0$

1.3 Motion with constant angular acceleration

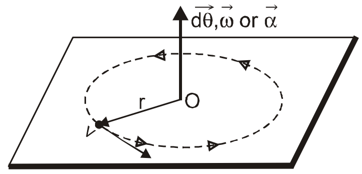
$\omega_0 \Rightarrow$ Initial angular velocity

$\omega \Rightarrow$ Final angular velocity

$\alpha \Rightarrow$ Constant angular acceleration

$\theta \Rightarrow$ Angular displacement

Circular motion with constant angular acceleration is analogous to one dimensional translational motion with constant acceleration. Hence even here equation of motion have same form.



$\omega = \omega_0 + \alpha t \quad ; \quad \theta = \omega_0 t + \frac{1}{2}\alpha t^2$

$\omega^2 = \omega_0^2 + 2\alpha\theta \quad ; \quad \theta = \left(\frac{\omega + \omega_0}{2}\right) t$

$\theta_{n^{th}} = \omega_0 + \frac{\alpha}{2}(\theta_n - \theta_{n-1})$

2. RELATION BETWEEN SPEED AND ANGULAR VELOCITY :

$\vec{v} = \vec{\omega} \times \vec{r}$

Here, $\vec{v}$ is velocity of the particle, $\vec{\omega}$ is angular velocity about centre of circular motion and '$\vec{r}$' is position of particle w.r.t. center of circular motion.

Since $\vec{\omega} \perp \vec{r}$

v = ω r for circular motion.

Example 3. A particle is moving with constant speed in a circular path. Find the ratio of average velocity to its instantaneous velocity when the particle describes an angle $\theta = \frac{\pi}{2}$

Solution : Time taken to describe angle θ, $t = \frac{\theta}{\omega} = \frac{\theta R}{v} = \frac{\pi R}{2v}$

Average velocity = $\frac{\text{Total displacement}}{\text{Total time}} = \frac{\sqrt{2}\;R}{\pi R/2v} = \frac{2\sqrt{2}}{\pi} v$

Instantaneous velocity = v

The ratio of average velocity to its instantaneous velocity = $\frac{2\sqrt{2}}{\pi}$ **Ans.**

Example 4. A fan is rotating with angular velocity 100 rev/sec. Then it is switched off. It takes 5 minutes to stop. (a) Find the total number of revolution made before it stops. (Assume uniform angular retardation) (b) Find the value of angular retardation (c) Find the average angular velocity during this interval.

Solution : (a) $\theta = \left(\frac{\omega + \omega_0}{2}\right) t = \left(\frac{100+0}{2}\right) \times 5 \times 60 = 15000$ revolution.

(b) $\omega = \omega_0 + \alpha t \Rightarrow 0 = 100 - \alpha\,(5 \times 60) \Rightarrow \alpha = \frac{1}{3}$ rev./sec^2

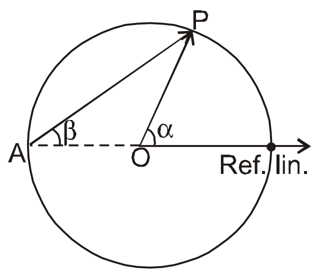
(c) $\omega_{av} = \frac{\text{Total Angle of Rotation}}{\text{Total time taken}} = \frac{15000}{50 \times 60} = 50$ rev./sec.

3. RELATIVE ANGULAR VELOCITY

Just as velocities are always relative, similarly angular velocity is also always relative. There is no such thing as absolute angular velocity. Angular velocity is defined with respect to origin, the point from which the position vector of the moving particle is drawn.

☞ Consider a particle P moving along a circular path shown in the figure given below.

Here angular velocity of the particle P w.r.t. 'O' and 'A' will be different



Angular velocity of a particle P w.r.t. O, $\omega_{PO} = \frac{d\alpha}{dt}$

Angular velocity of a particle P w.r.t. A, $\omega_{PA} = \frac{d\beta}{dt}$

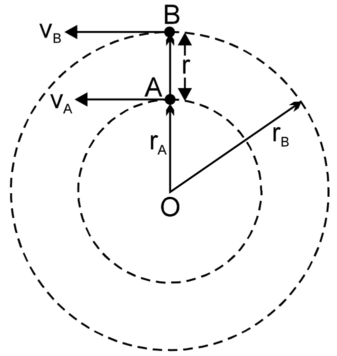
Definition **:**

Angular velocity of a particle 'A' with respect to the other moving particle 'B' is the rate at which position vector of 'A' with respect to 'B' rotates at that instant. (or it is simply, angular velocity of A with origin fixed at B). Angular velocity of A w.r.t. B, ω_{AB} is mathematically define as

$$\omega_{AB} = \frac{\text{Component of relative velocity of A w.r.t. B, perpendicular to line}}{\text{seperation between A and B}} = \frac{(V_{AB})_{\perp}}{r_{AB}}$$

Important points:

- If two particles are moving on two different concentric circles with different velocities then angular velocity of B as observed by A will depend on their positions and velocities. Consider the case when A and B are closest to each other moving in same direction as shown in figure. In this situation

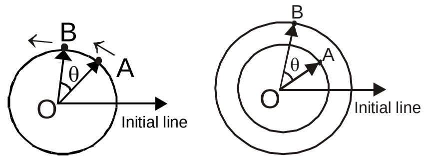


$(V_{AB})_\perp = v_{rel} = |\vec{v}_B - \vec{v}_A| = v_B - v_A$

Separation between A and B is $r_{BA} = r_B - r_A$

so, $\omega_{AB} = \dfrac{(V_{AB})_\perp}{r_{AB}} = \dfrac{v_B - v_A}{r_B - r_A}$

- If two particles are moving on the same circle or different coplanar concentric circles in same direction with different uniform angular speed ω_A and ω_B respectively, the rate of change of angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is



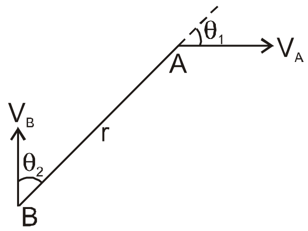
$$\frac{d\theta}{dt} = \omega_B - \omega_A$$

So the time taken by one to complete one revolution around O w.r.t. the other

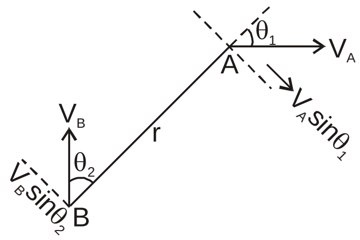
$$T = \frac{2\pi}{\omega_{rel}} = \frac{2\pi}{\omega_2 - \omega_1} = \frac{T_1T_2}{T_1 - T_2}$$

- $\omega_B - \omega_A$ is rate of change of angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. This is not angular velocity of B w.r.t. A. (Which is rate at which line AB rotates)

Example 5. Find the angular velocity of A with respect to B in the figure given below:



Solution : Angular velocity of A with respect to B ;



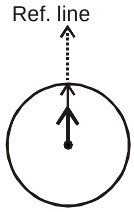
$\omega_{AB} = \dfrac{(V_{AB})_\perp}{r_{AB}}$

$\Rightarrow (V_{AB})_\perp = V_A \sin\theta_1 + V_B \sin\theta_2$

$\Rightarrow r_{AB} = r$

$$\omega_{AB} = \frac{v_A \sin\theta_1 + v_B \sin\theta_2}{r}$$

Example 6. Find the time period of meeting of minute hand and second hand of a clock.



Solution : $\omega_{min} = \frac{2\pi}{60}$ rad/min., $\omega_{sec} = \frac{2\pi}{1}$ rad/min

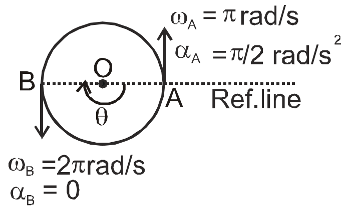
$\theta_{sec} - \theta_{min} = 2\pi$ (for second and minute hand to meet again)

$(\omega_{sec} - \omega_{min})\, t = 2\pi$

$2\pi(1 - 1/60)\, t = 2\pi \quad \Rightarrow \quad t = \frac{60}{59}$ min.

Example 7. Two particle A and B move on a circle. Initially Particle A and B are diagonally opposite to each other. Particle A move with angular velocity π rad/sec., angular acceleration $\pi/2$ rad/sec^2 and particle B moves with constant angular velocity 2π rad/sec. Find the time after which both the particle A and B will collide.

Solution : Suppose angle between OA and OB = θ then, rate of change of θ,



$\dot{\theta} = \omega_B - \omega_A = 2\pi - \pi = \pi$ rad/sec

$\ddot{\theta} = \alpha_B - \alpha_A = -\frac{\pi}{2}$ rad/sec^2

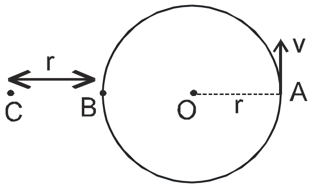
If angular displacement is $\Delta\theta$,

$\Delta\theta = \dot{\theta}t + \frac{1}{2}\ddot{\theta}t^2$

for A and B to collide angular displacement $\Delta\theta = \pi$

$\Rightarrow \quad \pi = \pi t + \frac{1}{2}\left(\frac{-\pi}{2}\right)t^2 \quad \Rightarrow \quad t^2 - 4t + 4 = 0 \quad \Rightarrow \quad t = 2$ sec. **Ans.**

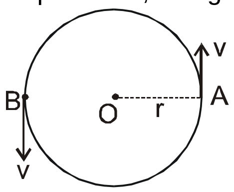
Example 8. A particle is moving with constant speed in a circle as shown, find the angular velocity of the particle A with respect to fixed point B and C if angular velocity with respect to O is ω.



Solution : Angular velocity of A with respect to O is ; $\omega_{AO} = \frac{(v_{AO})_\perp}{r_{AO}} = \frac{v}{r} = \omega$

$\therefore \quad \omega_{AB} = \frac{(v_{AB})_\perp}{r_{AB}} = \frac{v}{2r} = \frac{\omega}{2}$ and $\omega_{AC} = \frac{(v_{AC})_\perp}{r_{AC}} = \frac{v}{3r} = \frac{\omega}{3}$

Example 9. Particles A and B move with constant and equal speeds in a circle as shown, find the angular velocity of the particle A with respect to B, if angular velocity of particle A w.r.t. O is ω.

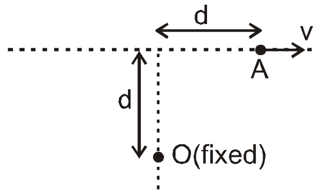


Solution : Angular velocity of A with respect to O is ; $\omega_{AO} = \frac{(v_{AO})_\perp}{r_{AO}} = \frac{v}{r} = \omega$

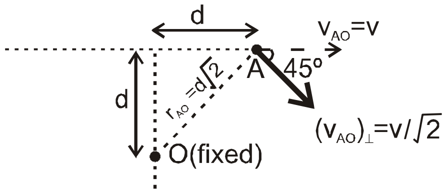
Now, $\omega_{AB} = \frac{(v_{AB})_\perp}{r_{AB}} \Rightarrow v_{AB} = 2v$, Since v_{AB} is perpendicular to r_{AB},

$\therefore \quad (v_{AB})_\perp = v_{AB} = 2v$; $\quad r_{AB} = 2r \Rightarrow \quad \omega_{AB} = \frac{(v_{AB})_\perp}{r_{AB}} = \frac{2v}{2r} = \omega$

Example 10. Find angular velocity of A with respect to O at the instant shown in the figure.



Solution : Angular velocity of A with respect to O is ;



$$\omega_{AO} = \frac{(v_{AO})_\perp}{r_{AO}}$$

$$v_{AO} = v,\ (v_{AO})_\perp = \frac{v}{\sqrt{2}}$$

$$r_{AO} = d\sqrt{2}$$

$$\omega_{AO} = \frac{(v_{AO})_\perp}{r_{AO}} = \frac{v/\sqrt{2}}{d\sqrt{2}} = \frac{v}{2d}$$

4. RADIAL AND TANGENTIAL ACCELERATION

There are two types of acceleration in circular motion ; Tangential acceleration and centripetal acceleration.

(a) Tangential acceleration : Component of acceleration directed along tangent of circle is called tangential acceleration. It is responsible for changing the speed of the particle. It is defined as,

$$a_t = \frac{dv}{dt} = \frac{d|\vec{v}|}{dt} = \text{Rate of change of speed.}$$

$$a_t = \alpha r$$

☞ **IMPORTANT POINT**

(i) In vector form $\vec{a}_t = \vec{\alpha} \times \vec{r}$

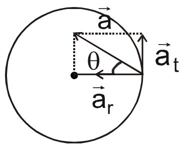
(ii) If tangential acceleration is directed in direction of velocity then the speed of the particle increases.

(iii) If tangential acceleration is directed opposite to velocity then the speed of the particle decreases.

(b) Centripetal acceleration : It is responsible for change in direction of velocity. In circular motion, there is always a centripetal acceleration.
Centripetal acceleration is always variable because it changes in direction.
Centripetal acceleration is also called radial acceleration or normal acceleration.

(c) Total acceleration : Total acceleration is vector sum of centripetal acceleration and tangential acceleration.

$$\vec{a} = \frac{d\vec{v}}{dt} = \vec{a}_r + \vec{a}_t$$



$$a = \sqrt{a_t^2 + a_r^2}$$

$$\tan\theta = \frac{a_r}{a_t}$$

☞ **IMPORTANT POINT**

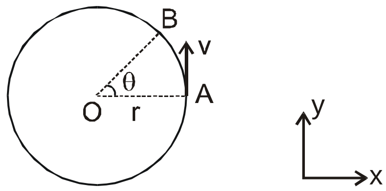
(i) Differentiation of speed gives tangential acceleration.

(ii) Differentiation of velocity ($\vec{v}$) gives total acceleration.

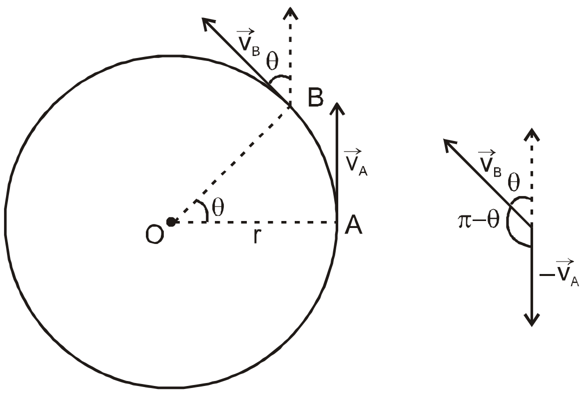
(iii) $\left|\frac{d\vec{v}}{dt}\right|$ & $\frac{d|\vec{v}|}{dt}$ are not same physical quantity. $\left|\frac{d\vec{v}}{dt}\right|$ is the magnitude of rate of change of velocity, i.e. magnitude of total acceleration and $\frac{d|\vec{v}|}{dt}$ is a rate of change of speed, i.e. tangential acceleration.

4.1 Calculation of centripetal acceleration :

Consider a particle which moves in a circle with constant speed v as shown in figure.



∴ change in velocity between the point A and B is ;



$\Delta \vec{v} = \vec{v}_B - \vec{v}_A$

Magnitude of change in velocity.

$|\Delta\vec{v}| = |\vec{v}_B - \vec{v}_A| = \sqrt{v_B^2 + v_A^2 + 2v_A v_B \cos\ (\pi - \theta)}$

($v_A = v_B = v$, since speed is same)

∴ $|\Delta \vec{v}| = 2v \sin\frac{\theta}{2}$

Distance travelled by particle between A and B = rθ

Hence time taken, $\Delta t = \frac{r\theta}{v}$

Net acceleration, $|\vec{a}_{net}| = \left|\frac{\overrightarrow{\Delta v}}{\Delta t}\right| = \frac{2v\sin\theta/2}{r\theta/v} = \frac{v^2}{r}\frac{2\sin\theta/2}{\theta}$

If $\Delta t \to 0$, then θ is small, sin (θ/2) = θ/2

$$\lim_{\Delta t\to 0}\left|\frac{\Delta\vec{v}}{\Delta t}\right| = \left|\frac{d\vec{v}}{dt}\right| = \frac{v^2}{r}$$

i.e. net acceleration is $\frac{v^2}{r}$ but speed is constant so that tangential acceleration, $a_t = \frac{dv}{dt} = 0$.

∴ $a_{net} = a_r = \frac{v^2}{r}$

** Through we have derived the formula of centripetal acceleration under condition of constant speed, the same formula is applicable even when speed is variable.

☞ **IMPORTANT POINT**

In vector form $\vec{a}_c = \vec{\omega} \times \vec{v}$

Example 11. The speed of a particle traveling in a circle of radius 20 cm increases uniformly from 6.0 m/s to 8.0 m/s in 4.0 s, find the angular acceleration.

Solution : Since speed increases uniformly, average tangential acceleration is equal to instantaneous tangential acceleration

$\therefore$ The instantaneous tangential acceleration is given by

$$a_t = \frac{dv}{dt} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{8.0 - 6.0}{4.0} \text{ m/s}^2 = 0.5 \text{ m/s}^2.$$

The angular acceleration is $\alpha = a_t / r = \frac{0.5 m/s^2}{20cm} = 2.5 \text{ rad/s}^2$.

Example 12. A particle is moving in a circle of radius 10 cm with uniform speed completing the circle in 4s, find the magnitude of its acceleration.

Solution : The distance covered in completing the circle is $2\pi r = 2\pi \times 10$ cm. The linear speed is

$$v = 2\pi r/t = \frac{2\pi \times 10cm}{4s} = 5\pi \text{ cm/s}.$$

The acceleration is $a = \frac{v^2}{r} = \frac{(5\pi cm/s)^2}{10 cm} = 2.5\pi^2 \text{ cm/s}^2$.

Example 13. A particle moves in a circle of radius 2.0 cm at a speed given by v = 4t, where v is in cm/s and t is in seconds.

(a) Find the tangential acceleration at t = 1s.

(b) Find total acceleration at t = 1s.

Solution : (a) Tangential acceleration

$$a_t = \frac{dv}{dt} \quad \text{or} \quad a_t = \frac{d}{dt}(4t) = 4 \text{ cm/s}^2$$

$$a_C = \frac{v^2}{R} = \frac{(4)^2}{2} = 8 \text{ cm/s}^2 \quad \Rightarrow \quad a = \sqrt{a_t^2 + a_C^2} = \sqrt{(4)^2 + (8)^2} = 4\sqrt{5} \text{ cm/s}^2$$

Example 14. A particle begins to move with a tangential acceleration of constant magnitude 0.6 m/s^2 in a circular path. If it slips when its total acceleration becomes 1 m/s^2, Find the angle through which it would have turned before it starts to slip.

Solution : $a_{Net} = \sqrt{a_t^2 + a_c^2} \quad \Rightarrow \quad \omega^2 = \omega_0^2 + 2\alpha\theta$

$\because \omega_0 = 0 \quad$ so $\quad \omega^2 = 2\alpha\theta$

$\omega^2 R = 2(\alpha R\theta)$

$a_c = \omega^2 R = 2a_t\theta$

$1 = \sqrt{0.36 + (1.2 \times \theta)^2} \quad \Rightarrow \quad 1 - 0.36 = (1.2\,\theta)^2$

$\Rightarrow \frac{0.8}{1.2} = \theta \quad \Rightarrow \quad \theta = \frac{2}{3}$ radian **Ans.**

5. DYNAMICS OF CIRCULAR MOTION :

If there is no force acting on a body it will move in a straight line (with constant speed). Hence if a body is moving in a circular path or any curved path, there must be some force acting on the body.

If speed of body is constant, the net force acting on the body is along the inside normal to the path of the body and it is called centripetal force.

Centripetal force $(F_c) = ma_c = \frac{mv^2}{r} = m\,\omega^2 r$

However if speed of the body varies then, in addition to above centripetal force which acts along inside normal, there is also a force acting along the tangent of the path of the body which is called tangential force.

Tangential force $(F_t) = Ma_t = M\frac{dv}{dt} = M\,\alpha\, r$; where α is the angular acceleration

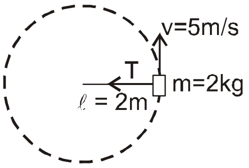
☞ **IMPORTANT POINT**

Remember $\frac{mv^2}{r}$ is not a force itself. It is just the value of the net force acting along the inside normal which is responsible for circular motion. This force may be friction, normal, tension, spring force, gravitational force or a combination of them.
So to solve any problem in uniform circular motion we identify all the forces acting along the normal (towards center), calculate their resultant and equate it to $\frac{mv^2}{r}$.
If circular motion is non uniform then in addition to above step we also identify all the forces acting along the tangent to the circular path, calculate their resultant and equate it to $\frac{mdv}{dt}$ or $\frac{md|\vec{v}|}{dt}$.

6. CIRCULAR MOTION IN HORIZONTAL PLANE :

Example 15. A block of mass 2kg is tied to a string of length 2m, the other end of which is fixed. The block is moved on a smooth horizontal table with constant speed 5 m/s. Find the tension in the string.

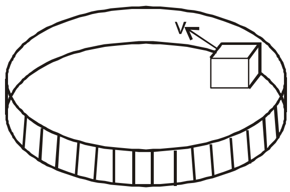
Solution :



here centripetal force is provided by tension.

$T = \frac{mv^2}{r} = \frac{2\times 5^2}{2} = 25$ N

Example 16. A block of mass m moves with speed v against a smooth, fixed vertical circular groove of radius r kept on smooth horizontal surface.



Find :
(i) normal reaction of the floor on the block.
(ii) normal reaction of the vertical wall on the block.

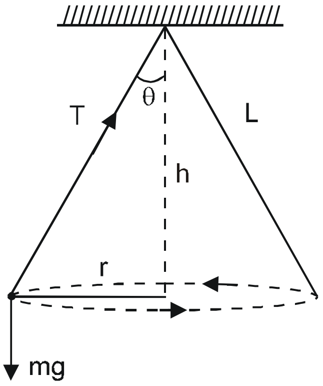
Solution : Here centripetal force is provided by normal reaction of vertical wall.
(i) normal reaction of floor $N_F = mg$
(ii) normal reaction of vertical wall $N_W = \frac{mv^2}{r}$.

Example 17. A block of mass m is kept on the edge of a horizontal turn table of radius R, which is rotating with constant angular velocity ω (along with the block) about its axis. If coefficient of friction is μ, find the friction force between block and table

Solution : Here centripetal force is provided by friction force.
Friction force = centripetal force = $m\omega^2R$

Example 18. Consider a conical pendulum having bob of mass m is suspended from a ceiling through a string of length L. The bob moves in a horizontal circle of radius r. Find (a) the angular speed of the bob and (b) the tension in the string.

Solution : The situation is shown in figure. The angle θ made by the string with the vertical is given by



$\sin\theta = r / L$, $\cos\theta = h/L = \dfrac{\sqrt{L^2 - r^2}}{L}$ ...(i)

The forces on the particle are

(a) the tension T along the string and
(b) the weight mg vertically downward.

The particle is moving in a circle with a constant speed v. Thus , the radial acceleration towards the centre has magnitude v^2 / r. Resolving the forces along the radial direction and applying Newton's second law,

$T\sin\theta = m(v^2 / r)$ ....(ii)

As there is no acceleration in vertical direction, we have from Newton's law,

$T\cos\theta = mg$ ....(iii)

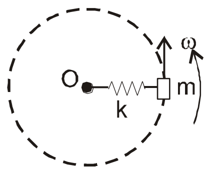
Dividing (ii) by (iii),

$\tan\theta = \dfrac{v^2}{rg}$ or, $v = \sqrt{rg\ \tan\theta}$

$\Rightarrow\ \omega = \dfrac{v}{r} = \sqrt{\dfrac{g\tan\theta}{r}} = \sqrt{\dfrac{g}{h}} = \sqrt{\dfrac{g}{L\cos\theta}} = \sqrt{\dfrac{g}{(L^2 - r^2)^{\frac{1}{2}}}}$ Ans.

And from (iii), $T = \dfrac{mg}{\cos\theta} = \dfrac{mgL}{(L^2 - r^2)^{\frac{1}{2}}}$ **Ans.**

Example 19. A block of mass m is tied to a spring of spring constant k, natural length ℓ, and the other end of spring is fixed at O. If the block moves in a circular path on a smooth horizontal surface with constant angular velocity ω, find tension in the spring.



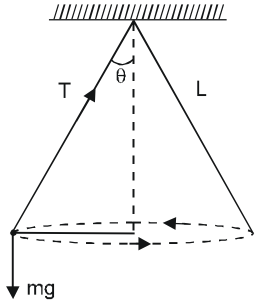
Solution : Assume extension in the spring is x.
Here centripetal force is provided by spring force.

Centripetal force, $kx = m\omega^2(\ell + x)$ $\Rightarrow$ $x = \dfrac{m\omega^2\ell}{k - m\omega^2}$

therefore, tension $= kx = \dfrac{km\omega^2\ell}{k - m\omega^2}$ **Ans.**

Example 20. A string breaks under a load of 50 kg. A mass of 1 kg is attached to one end of the string 10 m long and is rotated in horizontal circle. Calculate the greatest number of revolutions that the mass can make in one second without breaking the string.

Solution : $\omega = 2\pi n$,



$T_{max} = 500$ N, $r = L\sin\theta$

$T\sin\theta = m\omega^2 r$

$\Rightarrow\ T = m\omega^2 L$

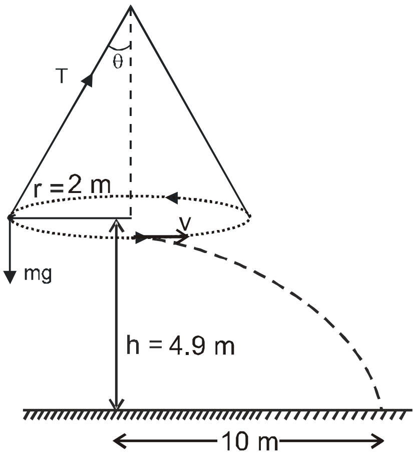
$\Rightarrow\ T_{max} = m\,\omega_{max}^2 L$

$\Rightarrow\ T_{max} = m(2\pi n_{max})^2 L$

$n_{max} = \dfrac{1}{2\pi}\sqrt{\dfrac{T_{max}}{mL}} = \dfrac{1}{2\pi}\sqrt{\dfrac{500}{1\times 10}} = \dfrac{\sqrt{50}}{2\pi}$ revolution per second. **Ans.**

Example 21. A boy whirls a stone in a horizontal circle of radius 2 m and at height 4.9 m above level ground. The string breaks, and the stone files off horizontally and strikes the ground at a point which is 10 m away from the point on the ground directly below the point where the string had broken. What is the magnitude of the centripetal acceleration of the stone while in circular motion? ($g = 9.8$ m/s^2)

Solution :

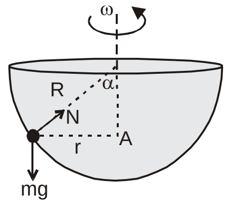


$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 4.9}{9.8}} = 1s$

$v = \frac{10}{t} = 10$ m/s ; $a = \frac{v^2}{R} = 50$ m/s^2

Example 22. A hemispherical bowl of radius R is rotating about its axis of symmetry which is kept vertical. A small ball kept in the bowl rotates with the bowl without slipping on its smooth surface and the angle made by the radius through the ball with the vertical is α. Find the angular speed at which the bowl is rotating.

Solution : Let ω be the angular speed of rotation of the bowl. Two force are acting on the ball.



1. Normal reaction N
2. weight mg

The ball is rotating in a circle of radius r (= R sin α) with centre at A at an angular speed ω. Thus,

$N \sin \alpha = mr\omega^2 = mR\omega^2 \sin \alpha$

$N = mR\omega^2$ .....(i)

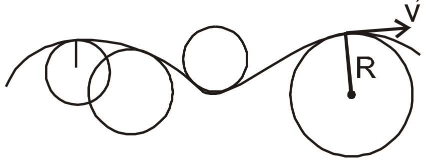
and $N \cos \alpha = mg$ .....(ii)

Dividing Eqs. (i) by (ii), we get $\frac{1}{\cos\alpha} = \frac{\omega^2 R}{g}$

$\therefore \quad \omega = \sqrt{\frac{g}{R \cos\alpha}}$

7. RADIUS OF CURVATURE

Any curved path can be assumed to be made of infinite circular arcs. Radius of curvature at a point is the radius of the circular arc at a particular point which fits the curve at that point.



If R is radius of the circular arc at a given point P, where velocity is $\vec{v}$, then centripetal force at that point is,

$F_c = \frac{mv^2}{R} \Rightarrow R = \frac{mv^2}{F_C}$

Now centripetal force F_c is simply the component of force perpendicular to velocity (let us say $F_\perp$).

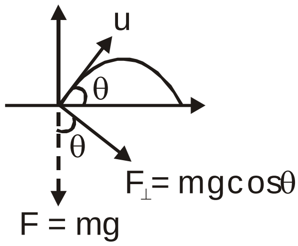
$\therefore \quad R = \frac{mv^2}{F_\perp} \quad \Rightarrow \quad R = \frac{v^2}{a_\perp}$

Where, $a_\perp$ = Component of acceleration perpendicular to velocity.

☞ If a particle moves in a trajectory given by y = f(x) then radius of curvature at any point (x, y) of the trajectory is given by $\Rightarrow$ $R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

Example 23. A particle of mass m is projected with speed u at an angle θ with the horizontal. Find the radius of curvature of the path traced out by the particle at the point of projection and also at the highest point of trajectory.

Solution : at point of projection



$R = \frac{mv^2}{F_\perp} = \frac{mu^2}{mg\ \cos\theta}$

$R = \frac{u^2}{g\cos\theta}$ **Ans.**

at highest point

$a_\perp = g,\ v = u\cos\theta \ : \quad R = \frac{v^2}{a_\perp} = \frac{u^2\cos^2\theta}{g}$ **Ans.**

Example 24. A particle moves along the plane trajectory y(x) with constant speed v. Find the radius of curvature of the trajectory at the point x = 0 if the trajectory has the form of a parabola $y = ax^2$ where 'a' is a positive constant.

Solution : If the equation of the trajectory of a particle is given we can find the radius of trajectory of the instantaneous circle by using the formula

$$R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

As ; $y = ax^2 \Rightarrow \frac{dy}{dx} = 2ax = 0$ (at x = 0) and $\frac{d^2y}{dx^2} = 2a$

Now radius of trajectory is given by

$R = \frac{[1+0]^{3/2}}{2a} = \frac{1}{2a}$

Aliter : This problem can also be solved by using the formula : $R = \frac{v^2}{a_\perp}$. $y = ax^2$, differentiate with respect to time $\frac{dy}{dt} = 2ax\frac{dx}{dt}$ ....(1)

at x = 0, $v_y = \frac{dy}{dt} = 0$ hence $v_x = v$

since v_x is constant, $a_x = 0$

Now, differentiate (1) with respect to time $\frac{d^2y}{dt^2} = 2ax\frac{d^2x}{dt^2} + 2a\left(\frac{dx}{dt}\right)^2$

at x = 0, $v_x = v$

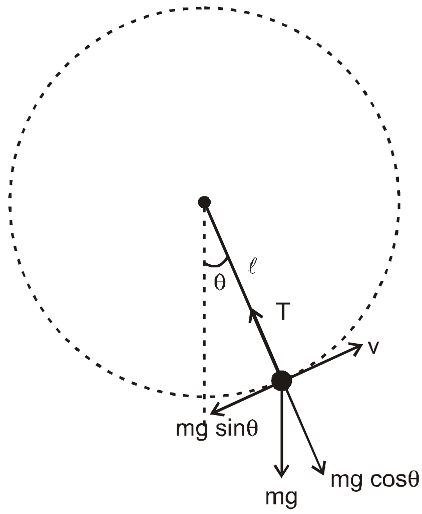
$\therefore$ net acceleration, $a = a_y = 2av^2$ (since $a_x = 0$)

this acceleration is perpendicular to velocity (v_x). Hence it is equal to centripetal acceleration

$R = \frac{v^2}{a_\perp} = \frac{v^2}{2av^2} = \frac{1}{2a}$ **Ans.**

8. MOTION IN A VERTICAL CIRCLE :

Let us consider the motion of a point mass tied to a string of length ℓ and whirled in a vertical circle. If at any time the body is at angular position θ, as shown in the figure, the forces acting on it are tension T in the string along the radius towards the center and the weight of the body mg acting vertically down wards.



Applying Newton's law along radial direction

$$T - mg\cos\theta = m.a_c = \frac{mv^2}{\ell}$$

or $T = \frac{mv^2}{\ell} + mg\cos\theta$ .....(1)

The point mass will complete the circle only and only if tension is never zero (except momentarily, if at all) if tension becomes zero at any point, string will go slack and subsequently, the only force acting on the body is gravity. Hence its subsequent motion will be similar to that of a projectile.

From equation ...(1), it is evident that tension decreases with increase in θ because cos θ is a decreasing function and v decreases with height. Hence tension is minimum at the top most point. i.e. $T_{min} = T_{topmost}$.

T > 0 at all points. $\Rightarrow$ $T_{min} > 0$.

However if tension is momentarily zero at highest point the body would still be able to complete the circle.

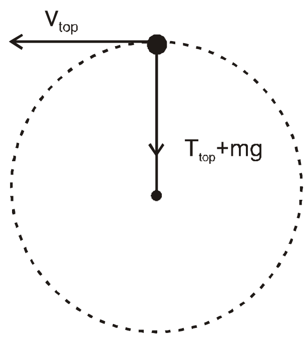
Hence condition for completing the circle (or looping the loop) is

$T_{min} \geq 0$ or $T_{top} \geq 0$.

$$T_{top} + mg = \frac{mv_{top}^2}{\ell} \quad .....(2)$$

Equation...(2) could also be obtained by putting $\theta = \pi$ in equation ..(1).

For looping the loop, $T_{top} \geq 0$.



$$\Rightarrow \frac{mv_{top}^2}{\ell} \geq mg \quad \Rightarrow \quad v_{top} \geq \sqrt{g\ell} \quad ....(3)$$

Condition for looping the loop is $v_{top} \geq \sqrt{g\ell}$.

If speed at the lowest point is u, then from conservation of mechanical energy between lowest point and top most point.

$$\frac{1}{2}mu^2 = \frac{1}{2}m v_{top}^2 + mg\,.\,2\ell$$

using equation ..(3) for v_{top} we get $u \geq \sqrt{5g\ell}$

i.e., for looping the loop, velocity at lowest point must be $\geq \sqrt{5g\ell}$.

☞ If velocity at lowest point is just enough for looping the loop, value of various quantities. (True for a point mass attached to a string or a mass moving on a smooth vertical circular track.)

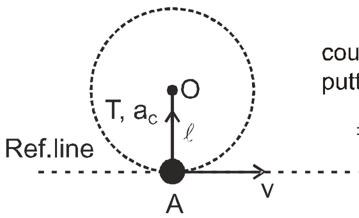


Fig. (1)

P.E. = 0

$v = \sqrt{5g\ell}$

$T - mg = \frac{mv^2}{\ell}$

could also be obtained by putting θ=0 in equation ..(1)

$\Rightarrow T = 6mg$

$a_c = 5g$

$a_t = 0$

$a_{net} = 5g$

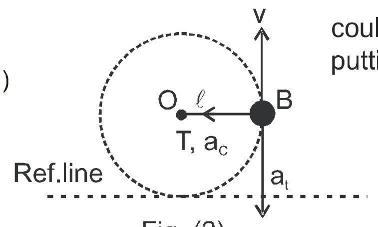


Fig. (2)

P.E.=$mg\ell$

By energy conservation, $v = \sqrt{3g\ell}$

$T = mv^2/\ell$

could also be obtained by putting θ =90º in equation ..(1)

T=3mg

$a_c = 3g$

$a_t = g$

$a_{net} = g\sqrt{10}$

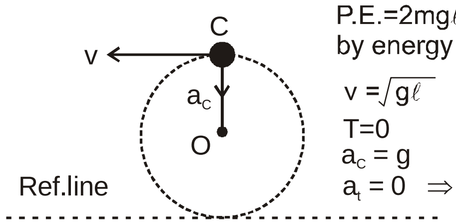


Fig. (3)

P.E.=$2mg\ell$

by energy conservation

$v = \sqrt{g\ell}$

T=0

$a_c = g$

$a_t = 0 \Rightarrow a_{net} = g$

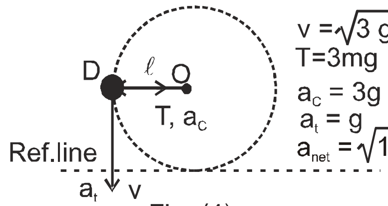


Fig. (4)

P.E.=$mg\ell$

by energy conservation

$v = \sqrt{3g\ell}$

T=3mg

$a_c = 3g$

$a_t = g$

$a_{net} = \sqrt{10}\ g$

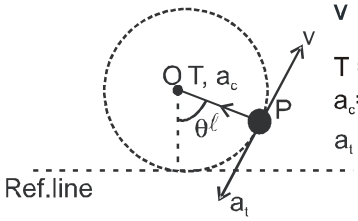


Fig. (5)

P.E.=$mg\ell(1 - \cos\theta)$

by energy conservation

$v = \sqrt{g\ell(3 + 2\cos\theta)}$

$T = 3mg(1 + \cos\theta)$

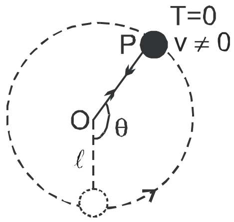
$a_c = g(3 + 2\cos\theta)$

$a_t = g\sin\theta$

		A	B,D	C	P(generalpoint)
1	Velocity	$\sqrt{5g\ell}$	$\sqrt{3g\ell}$	$\sqrt{g\ell}$	$\sqrt{g\ell(3+2\cos\theta)}$
2	Tension	6mg	3mg	0	$3mg(1+\cos\theta)$
3	PotentialEnergy	0	$mg\ell$	$2mg\ell$	$mg\ell(1-\cos\theta)$
4	Radial acceleration	5g	3g	g	$g(3+2\cos\theta)$
5	Tangential acceleration	0	g	0	$g\sin\theta$

Note : From above table we can see, $T_{bottom} - T_{top} = T_C - T_A = 6$ mg , this difference in tension remain same even if V > $\sqrt{5g\ell}$

Example 25. Find minimum speed at A so that the ball can reach at point B as shown in figure. Also discuss the motion of particle when T = 0, v = 0 simultaneously at θ = 90º.



(B)

For Leaving the circular path after which motion converts into projectile motion.

$\sqrt{2g\ell} < v_L < \sqrt{5g\ell}$

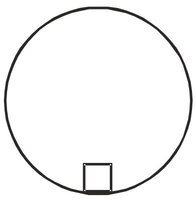
$90° < \theta < 180°$

In this case a component of gravity will always act towards center, hence centripetal acceleration or speed will remain nonzero. Hence tension becomes zero first.

As soon as, Tension becomes zero at any point, string will go slack and subsequently, the only force acting on the body is gravity. Hence its subsequent motion will be similar to that of a projectile. In this case motion is a combination of circular and projectile motion.

8.2 CONDITION FOR LOOPING THE LOOP IN SOME OTHER CASES

Case 1 : A mass moving on a smooth vertical circular track.



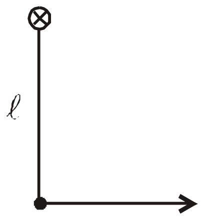
(B)

Mass moving along a smooth vertical circular loop.
condition for just looping the loop, normal at highest point = 0

By calculation similar to article (motion in vertical circle)

Minimum horizontal velocity at lowest point = $\sqrt{5g\ell}$

Case 2 : A particle attached to a light rod rotated in vertical circle. Condition for just looping the loop, velocity v = 0 at highest point (even if tension is zero, rod won't slack, and a compressive force can appear in the rod).



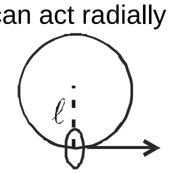
By energy conservation,

velocity at lowest point = $\sqrt{4g\ell}$

$V_{min} = \sqrt{4g\ell}$ (for completing the circle)

Case 3 : A bead attached to a ring and rotated.

Condition for just looping the loop, velocity v = 0 at highest point (even if normal is zero, the bead will not lose contact with the track, normal can act radially outward). By energy conservation,

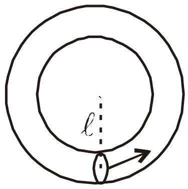


velocity at lowest point = $\sqrt{4g\ell}$

$V_{min} = \sqrt{4g\ell}$ (for completing the circle)

Case 4 : A block rotated between smooth surfaces of a pipe.

Condition for just looping the loop, velocity v = 0 at highest point (even if normal is zero, the bead will not lose contact with the track, normal can act radially outward).



By energy conservation, velocity at lowest point = $\sqrt{4g\ell}$

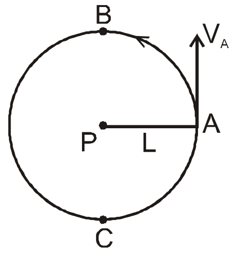
$V_{min} = \sqrt{4g\ell}$ (for completing the circle)

Example 26. If a particle of mass M is tied to a light inextensible string fixed at point P and particle is projected at A with velocity $V_A = \sqrt{4gL}$ as shown. Find :

(i) velocity at points B and C

(ii) tension in the string at B and C

Assume particle is projected in the vertical plane.



Solution : $V_B = \sqrt{2gL}$ (from energy conservation) ; $V_C = \sqrt{6gL}$

$T_B + Mg = \dfrac{Mv_B^2}{L}$

$T_B = Mg$

$T_C - Mg = \dfrac{Mv_C^2}{L}$; $T_C = 7Mg$ (where M $\Rightarrow$ Mass of the particle)

Example 27. Two point mass m are connected the light rod of length ℓ and it is free to rotate in vertical plane as shown. Calculate the minimum horizontal velocity is given to mass so that it completes the circular motion in vertical lane.

Solution : Here tension in the rod at the top most point of circle can be zero or negative for completing the loop. So velocity at the top most point is zero.

From energy conservation $\frac{1}{2}mv^2 + \frac{1}{2}m\frac{v^2}{4} = mg(2\ell) + mg(4\ell) + 0 \Rightarrow v = \sqrt{\frac{48g\ell}{5}}$ **Ans.**

Example 28. You may have seen in a circus a motorcyclist driving in vertical loops inside a 'death well' (a hollow spherical chamber with holes, so that the cyclist does not drop down when he is at the uppermost point, with no support from below. What is the minimum speed required at the uppermost position to perform a vertical loop if the radius of the chamber is 25 m?

Solution : When the motorcyclist is at the highest point of the death-well, the normal reaction R on the motorcyclist by the ceiling of the chamber acts downwards. His weight mg also act downwards.

$F_{net} = ma_c \Rightarrow \therefore R + mg = \dfrac{mv^2}{r}$

Here v is the speed of the motorcyclist and m is the mass of the motorcyclist (including the mass of the motor cycle). Because of the balancing of the forces, the motorcyclist does not fall down. The minimum speed required to perform a vertical loop is given by equation (1) when R = 0.

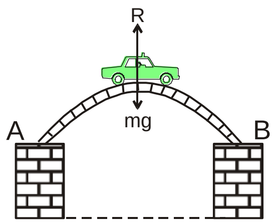
$\therefore$ $mg = \dfrac{mv_{min}^2}{r}$ or $v^2_{min} = gr$ or $v_{min} = \sqrt{gr} = \sqrt{9.8 \times 25}$ m s^{-1} = 15.65 ms^{-1}.

So, the minimum speed, at the top, required to perform a vertical loop is 15.65 m s^{-1}.

Example 29. Prove that a motor car moving over a convex bridge is lighter than the same car resting on the same bridge.

Solution : The motion of the motor car over a convex bridge AB is the motion along the segment of a circle AB (Figure) ;

The centripetal force is provided by the difference of weight mg of the car and the normal reaction R of the bridge.

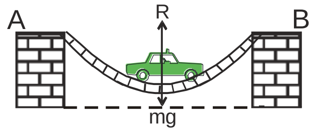


$\therefore$ $mg - R = \dfrac{mv^2}{r}$ or $R = mg - \dfrac{mv^2}{r}$

Clearly R < mg, i.e., the weight of the moving car is less than the weight of the stationary car.

Example 30. Prove that a motor car moving over a concave bridge is heavier than the same car resting on the same bridge.

Solution : The motion of the motor car over a concave bridge AB is the motion along the segment of a circle AB (Figure) ;
The centripetal force is provided by the difference of normal reaction R of the bridge and weight mg of the car.

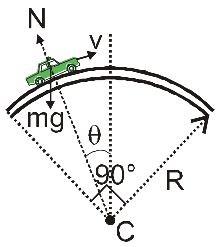


$\therefore \quad R - mg = \frac{mv^2}{r}$ or $R = mg + \frac{mv^2}{r}$

Clearly R > mg, i.e., the weight of the moving car is greater than the weight of the stationary car.

Example 31. A car is moving with uniform speed over a circular bridge of radius R which subtends an angle. of 90° at its centre. Find the minimum possible speed so that the car can cross the bridge without losing the contact any where.

Solution : Let the car losses the contact at angle θ with the vertical



$mg\cos\theta - N = \frac{mv^2}{R}$

$N = mg\cos\theta - \frac{mv^2}{R}$ ....(1)

for losing the contact N = 0 ,

$\Rightarrow$ $v = \sqrt{Rg\cos\theta}$ (from (1))

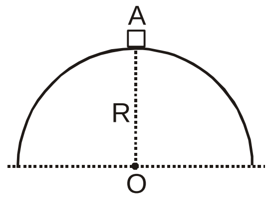
for minimum speed, $\cos\theta$ should be minimum so that θ should be maximum.

$\theta_{max} = 45° \Rightarrow \cos 45° = \frac{1}{\sqrt{2}}$

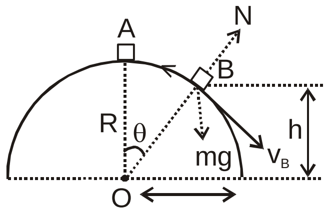
$v_{min} = \left(\frac{Rg}{\sqrt{2}}\right)^{1/2}$ **Ans.**

So that if car cannot lose the contact at initial or final point, car cannot be lose the contact anywhere.

Example 32. A block of mass m is released from the top of a frictionless fixed hemisphere as shown. Find (i) the angle with the vertical where it breaks off. (ii) the velocity at the instant when it breaks off. (iii) the height where it breaks off.



Solution : At B ; N = 0



$\therefore \quad mg\cos\theta = \frac{mv_B^2}{R}$

$\therefore v_B = \sqrt{gR\cos\theta}$ .... (1)

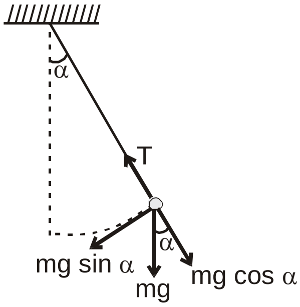
Now by equation of energy between A and B we have ; $0 + mgR = \frac{1}{2}mv_B^2 + mgh$

put v_B from (1) and $h = R\cos\theta$

$\therefore$ $v_B = \sqrt{\frac{2}{3}gR}$ and $h = \frac{2R}{3}$ from the bottom

Example 33. Consider a simple pendulum having a bob of mass m suspended by string of length L fixed at its upper end. The bob is oscillating in a vertical circle. It is found that the speed of the bob is v when the string makes an angle α with the vertical. Find (i) tension in the string and (ii) magnitude of net force on the bob at the instant.

Solution : (i) The forces acting on the bob are :
(a) the tension T (b) the weight mg

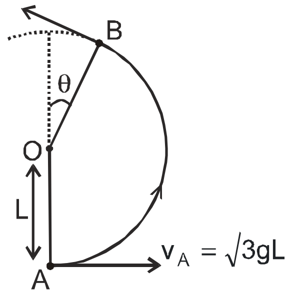


As the bob moves in a circle of radius L with centre at O. A centripetal force of magnitude $\frac{mv^2}{L}$ is required towards O. This force will be provided by the resultant of T and mg cos α. Thus,

or $T - mg\cos\alpha = \frac{mv^2}{L}$ $T = m\left(g\cos\alpha + \frac{v^2}{L}\right)$

(ii) $a_{net} = \sqrt{a_t^2 + a_r^2} = \sqrt{(g\ \sin\alpha)^2 + \left(\frac{v^2}{\ell}\right)^2}$ $\Rightarrow$ $|\vec{F}_{net}| = ma_{net} = m\sqrt{g^2\ \sin^2\alpha + \frac{v^4}{L^2}}$ **Ans.**

Example 34. A particle is projected with velocity $\sqrt{3gL}$ at point A (lowest point of the circle) in the vertical plane. Find the maximum height above horizontal level of point A if the string slacks at the point B as shown.



Solution : As tension at B ; T = 0

$\therefore$ $mg\cos\theta = \frac{mv_B^2}{L}$

$\therefore$ $v_B = \sqrt{gL\ \cos\theta}$ .....(1)

Now by equation of energy between A and B.

$0 + \frac{1}{2}m\ 3gL = \frac{1}{2}mv_B^2 + mgL(1+\cos\theta)$

put v_B

$\therefore$ $\cos\theta = \frac{1}{3}$

$\therefore$ height attend by particle after the point B where the string slacks is ;

$h' = \frac{v_B^2\ \sin^2\theta}{2g} = \frac{gL\cos\theta(1-\cos^2\theta)}{2g} = \frac{4L}{27}$

$\therefore$ Maximum height about point A is given by ; $H_{max} = L + L\cos\theta + h' = L + \frac{L}{3} + \frac{4L}{27} = \frac{40L}{27}$

9. CIRCULAR TURNING ON ROADS :

When vehicles go through turnings, they travel along a nearly circular arc. There must be some force which will produce the required centripetal acceleration. If the vehicles travel in a horizontal circular path, this resultant force is also horizontal. The necessary centripetal force is being provided to the vehicles by following three ways.

1. **By friction only**
2. **By banking of roads only.**
3. **By friction and banking of roads both.**
   In real life the necessary centripetal force is provided by friction and banking of roads both. Now let us write equations of motion in each of the three cases separately and see what are the constant in each case.

9.1 By Friction Only

Suppose a car of mass m is moving at a speed v in a horizontal circular arc of radius r. In this case, the necessary centripetal force to the car will be provided by force of friction f acting towards center

Thus, $f = \frac{mv^2}{r}$

Further, limiting value of f is μN or $f_L = \mu N = \mu mg$ (N = mg)

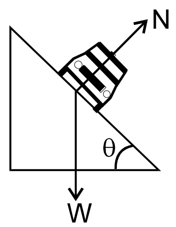
Therefore, for a safe turn without sliding $\frac{mv^2}{r} \le f_L$ or $\frac{mv^2}{r} \le \mu mg$ or $\mu \ge \frac{v^2}{rg}$ or $v \le \sqrt{\mu rg}$

Here, two situations may arise. If μ and r are known to us, the speed of the vehicle should not exceed $\sqrt{\mu rg}$ and if v and r are known to us, the coefficient of friction should be greater than $\frac{v^2}{rg}$.

Example 35. A bend in a level road has a radius of 100 m. Calculate the maximum speed which a car turning this bend may have without skidding. Given : μ = 0.8.

Solution : $V_{max} = \sqrt{\mu rg} = \sqrt{0.8 \times 100 \times 10} = \sqrt{800} = 28$ m/s

9.2. By Banking of Roads Only



Friction is not always reliable at circular turns if high speeds and sharp turns are involved to avoid dependence on friction, the roads are banked at the turn so that the outer part of the road is some what lifted compared to the inner part.

Applying Newton's second law along the radius and the first law in the vertical direction.

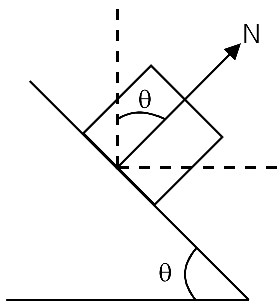
$$N\sin\theta = \frac{mv^2}{r} \quad \text{or} \quad N\cos\theta = mg$$

from these two equations, we get

$$\tan\theta = \frac{v^2}{rg} \quad \text{or} \quad v = \sqrt{rg\tan\theta}$$

Example 36. What should be the angle of banking of a circular track of radius 600 m which is designed for cars at an average speed of 180 km/hr ?

Solution : Let the angle of banking be θ. The forces on the car are (figure)



(a) weight of the car Mg downward and
(b) normal force N.

For proper banking , static frictional force is not needed.

For vertical direction the acceleration is zero. So,

$N\cos\theta = Mg$ .....(i)

For horizontal direction , the acceleration is v^2 / r towards the centre , so that

$N\sin\theta = Mv^2/r$ .....(ii)

From (i) and (ii), $\tan\theta = v^2 / rg$

Putting the values, $\tan\theta = \dfrac{180(km/h)^2}{(600m)(10m/s^2)} = 0.4167 \Rightarrow \theta = 22.6°$.

9.3 By Friction and Banking of Road Both

If a vehicle is moving on a circular road which is rough and banked also, then three forces may act on the vehicle, of these the first force, i.e., weight (mg) is fixed both in magnitude and direction.

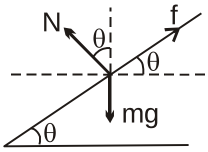
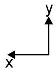


Figure (i)



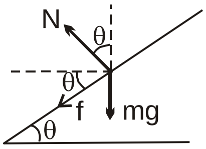


Figure (ii)

The direction of second force, i.e., normal reaction N is also fixed (perpendicular to road) while the direction of the third force i.e., friction f can be either inwards or outwards while its magnitude can be varied upto a maximum limit ($f_L = \mu N$). So the magnitude of normal reaction N and directions plus magnitude of friction f are so adjusted that the resultant of the three forces mentioned above is $\dfrac{mv^2}{r}$ towards the center. Of these m and r are also constant. Therefore, magnitude of N and directions plus magnitude of friction mainly depends on the speed of the vehicle v. Thus, situation varies from problem to problem. Even though we can see that :

(i) Friction f will be outwards if the vehicle is at rest v = 0. Because in that case the component of weight $mg\sin\theta$ is balanced by f.

(ii) Friction f will be inwards if $v > \sqrt{rg\tan\theta}$

(iii) Friction f will be outwards if $v < \sqrt{rg\tan\theta}$ and

(iv) Friction f will be zero if $v = \sqrt{rg\tan\theta}$

(v) For maximum safe speed (figure (ii)

$$N\sin\theta + f\cos\theta = \frac{mv^2}{r} \qquad \text{........(i)}$$

$$N\cos\theta - f\sin\theta = mg \qquad \text{.......(ii)}$$

As maximum value of friction

$f = \mu N$

$$\therefore \quad \frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} = \frac{v^2}{rg} \qquad \therefore \qquad v_{max} = \sqrt{\frac{rg(\tan\theta + \mu)}{(1 - \mu\tan\theta)}}$$

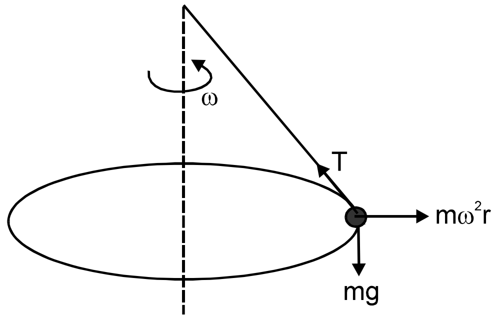
Similarly ; $v_{min} = \sqrt{\dfrac{rg(\tan\theta - \mu)}{(1 + \mu\tan\theta)}}$

Note :

- The expression $\tan\theta = \dfrac{v^2}{rg}$ also gives the angle of banking for an aircraft, i.e., the angle through which it should tilt while negotiating a curve, to avoid deviation from the circular path.
- The expression $\tan\theta = \dfrac{v^2}{rg}$ also gives the angle at which a cyclist should lean inward, when rounding a corner. In this case, θ is the angle which the cyclist must make with the vertical which will be discussed in chapter rotation.

10. CENTRIFUGAL FORCE :

When a body is rotating in a circular path and the centripetal force vanishes, the body would leave the circular path. To an observer A who is not sharing the motion along the circular path, the body appears to fly off tangentially at the point of release. To another observer B, who is sharing the motion along the circular path (i.e., the observer B is also rotating with the body which is released, it appears to B, as if it has been thrown off along the radius away from the centre by some force. This inertial force is called centrifugal force.)

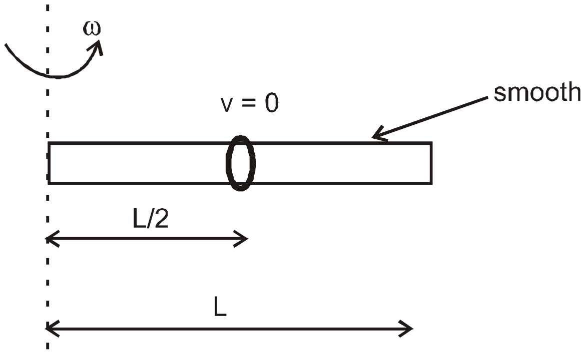


Its magnitude is equal to that of the centripetal force. = $\frac{mv^2}{r}$ = $m\omega^2 r$. Direction of centrifugal force, it is always directed radially outward.

Centrifugal force is a fictitious force which has to be applied as a concept only in a rotating frame of reference to apply Newton's law of motion in that frame. FBD of ball w.r.t. non inertial frame rotating with the ball.

Suppose we are working from a frame of reference that is rotating at a constant, angular velocity ω with respect to an inertial frame. If we analyses the dynamics of a particle of mass m kept at a distance r from the axis of rotation, we have to assume that a force $mr\omega^2$ react radially outward on the particle. Only then we can apply Newton's laws of motion in the rotating frame. This radially outward pseudo force is called the centrifugal force.

Example 37. A ring which can slide along the rod are kept at mid point of a smooth rod of length L. The rod is rotated with constant angular velocity ω about vertical axis passing through its one end. Ring is released from mid point. Find the velocity of the ring when it just leave the rod.



Solution : Centrifugal force $m\omega^2 x = ma$

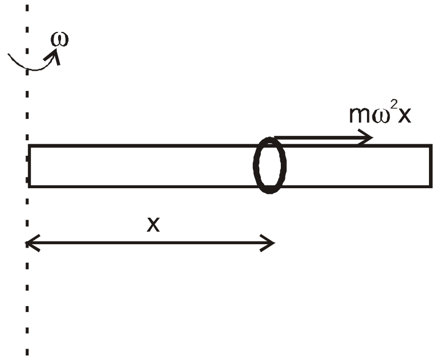
$\omega^2 x = \frac{vdv}{dx}$

$\int_{L/2}^{L} \omega^2 x \; dx = \int_0^v v \; dv$ (integrate both side.)

$\omega^2 \left(\frac{x^2}{2}\right)_{L/2}^{L} = \left(\frac{v^2}{2}\right)_0^v$

$\omega^2 \left(\frac{L^2}{2} - \frac{L^2}{8}\right) = \frac{v^2}{2}$

$v = \frac{\sqrt{3}}{2}\,\omega L.$



Velocity at time of leaving the rod

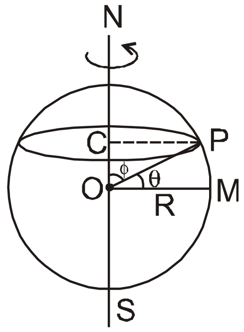
$$v' = \sqrt{(\omega L)^2 + \left(\frac{\sqrt{3}}{2}\omega L\right)^2} = \frac{\sqrt{7}}{2}\omega L$$ **Ans.**

11. EFFECT OF EARTHS ROTATION ON APPARENT WEIGHT :

The earth rotates about its axis at an angular speed of one revolution per 24 hours. The line joining the north and the south poles is the axis of rotation.

Every point on the earth moves in a circle. A point at equator moves in a circle of radius equal to the radius of the earth and the centre of the circle is same as the centre of the earth. For any other point on the earth, the circle of rotation is smaller than this. Consider a place P on the earth (figure).

Draw a perpendicular PC from P to the axis SN. The place P rotates in a circle with the centre at C. The radius of this circle is CP. The angle between the line OM and the radius OP through P is called the latitude of the place P. We have

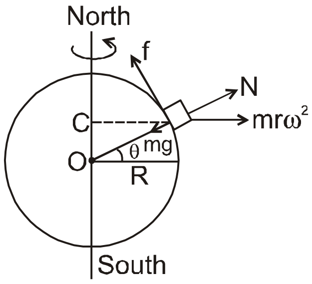


CP = OP $\cos\theta$ or, $r = R\cos\theta$

where R is the radius of the earth and ϕ is colatitude angle.

If we work from the frame of reference of the earth, we shall have to assume the existence of pseudo force. In particular, a centrifugal force mw^2r has to be assumed on any particle of mass m placed at P.

If we consider a block of mass m at point P then this block is at rest with respect to earth. If resolve the forces along and perpendicular the centre of earth then



$N + mr\omega^2 \cos\theta = mg$

$\Rightarrow$ $N = mg - mr\omega^2 \cos\theta$

$\Rightarrow$ $N = mg - mR\omega^2 \cos^2\theta$

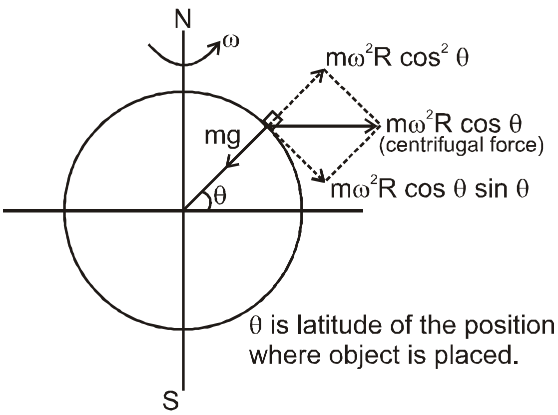


Figure (1) Earth's gravity & centrifugal force due rotation of Earth.

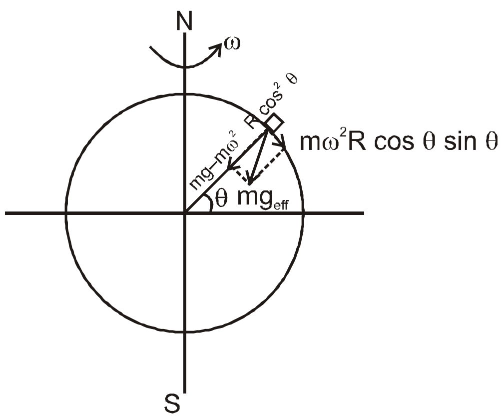


Figure (2) Resultant of Earth's gravity & centrifugal force is shown.

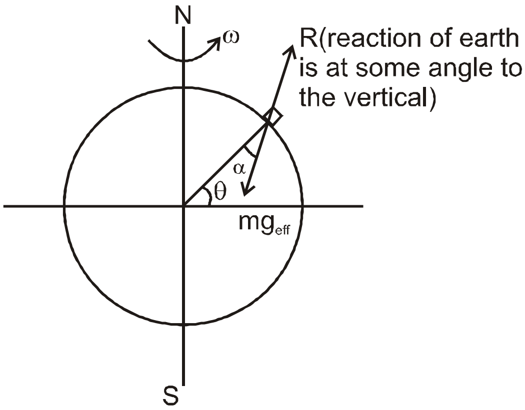


Figure (3)
The reaction on an object kept at rest w.r.t.
Earth is called a apparent weight
$W_{app.} = R = mg_{eff} \approx m(g - \omega^2 R\cos^2\theta)$

Note : At equator ($\theta = 0$) $W_{app.}$ is minimum and at pole ($\theta = \pi/2$) $W_{app.}$ is maximum.
This apparent weight is not along normal but at some angle α w.r.t. it. At all point except poles and equator ($\alpha = 0$ at poles and equator)

Example 38. A body weighs 98N on a spring balance at the north pole. What will be the reading on the same scale if it is shifted to the equator? Use $g = GM/R^2 = 9.8$ m/s^2 and $R_{earth} = 6400$ km.

Solution : At poles, the apparent weight is same as the true weight.
Thus, $98N = mg = m(9.8 \text{ m/s}^2)$
At the equator , the apparent weight is

$mg' = mg - m\omega^2 R$

The radius of the earth is 6400 km and the angular speed is

$$\omega = \frac{2\pi \text{rad}}{24 \times 60 \times 60 \text{s}} = 7.27 \times 10^{-6} \text{ rad/s}$$

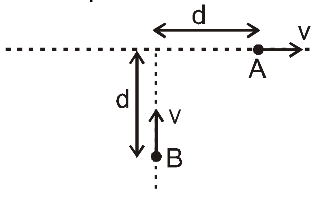
$mg' = 98N - (10 \text{ kg})(7.27 \times 10^{-5} \text{ s}^{-1})^2 (6400 \text{ km}) = 97.66N$ **Ans.**

Problem 1. A fan rotating with $\omega = 100$ rad/s, is switched off. After 2n rotation its angular velocity becomes 50 rad/s. Find the angular velocity of the fan after n rotations.

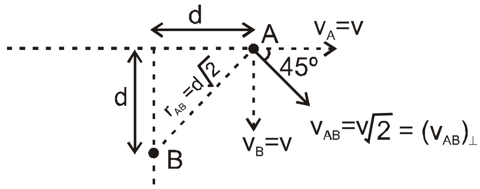
Solution : $\omega^2 = \omega_0^2 + 2\alpha\,\theta$
$50^2 = (100)^2 + 2\alpha\,(2\pi \,.\, 2n)$ .....(1)
If angular velocity after n rotation is ω_n
$\omega_n^2 = (100)^2 + 2\alpha\,(2\pi \,.\, n)$ .....(2)
from equation (1) and (2)

$$\frac{50^2 - 100^2}{\omega_n^2 - 100^2} = \frac{2\alpha(2\pi.2n)}{2\alpha 2\pi n} = 2 \Rightarrow \omega_n^2 = \frac{50^2 + 100^2}{2} \Rightarrow \omega = 25\sqrt{10} \text{ rad/s}$$ **Ans.**

Problem 2. Find angular velocity of A with respect to B at the instant shown in the figure.



Solution : Angular velocity of A with respect to B is; $\omega_{AB} = \dfrac{(v_{AB})_\perp}{r_{AB}}$



$v_{AB} = \sqrt{2}\, v = (v_{AB})_\perp \;\Rightarrow\; r_{AB} = \sqrt{2}\, d$

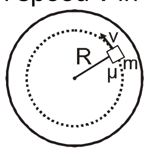
$\omega_{AB} = \dfrac{(v_{AB})_\perp}{r_{AB}} = \dfrac{v\sqrt{2}}{d\sqrt{2}} = \dfrac{v}{d}$

Problem 3. A particle is moving with a constant angular acceleration of 4 rad./sec^2 in a circular path. At time t = 0 particle was at rest. Find the time at which the magnitudes of centripetal acceleration and tangential acceleration are equal.

Solution : $a_t = \alpha R \;\Rightarrow\; v = 0 + \alpha R t \qquad \Rightarrow\; a_c = \dfrac{v^2}{R} = \dfrac{\alpha^2 R^2 t^2}{R}$

$\because\; |a_t| = |a_c| \;\Rightarrow\; \alpha R = \dfrac{\alpha^2 R^2 t^2}{R} \;\Rightarrow\; t^2 = \dfrac{1}{\alpha} = \dfrac{1}{4} \;\Rightarrow\; t = \dfrac{1}{2}$ sec. **Ans.**

Problem 4. The coefficient of friction between block and table is μ. Find the tension in the string if the block moves on the horizontal table with speed v in circle of radius R.



Solution : The magnitude of centripetal force is $\dfrac{mv^2}{R}$.

(i) If limiting friction is greater than or equal to $\dfrac{mv^2}{R}$, then static friction alone provides centripetal force, so tension is equal to zero.
T = 0 **Ans.**

(ii) If limiting friction is less than $\dfrac{mv^2}{R}$, then friction as well as tension both combine to provide the necessary centripetal force.

$T + f_r = \dfrac{mv^2}{R}$. In this case friction is equal to limiting friction, $f_r = \mu mg$

$\therefore$ Tension $= T = \dfrac{mv^2}{R} - \mu mg$ **Ans.**

Problem 5. A block of mass m is kept on rough horizontal turn table at a distance r from centre of table. Coefficient of friction between turn table and block is μ. Now turn table starts rotating with uniform angular acceleration α.
(i) Find the time after which slipping occurs between block and turn table.
(ii) Find angle made by friction force with velocity at the point of slipping.

Solution : (i) $a_t = \alpha r$

speed after t time $\dfrac{dv}{dt} = \alpha r \qquad \Rightarrow \qquad v = 0 + \alpha r t$

Centripetal acceleration $a_c = \dfrac{v^2}{r} = \alpha^2 r t^2$

Net acceleration $a_{net} = \sqrt{a_t^2 + a_c^2} = \sqrt{\alpha^2 r^2 + \alpha^4 r^2 t^4}$

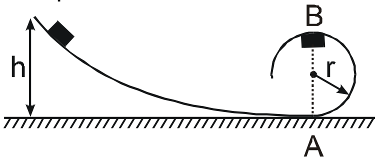
block just start slipping

$\mu mg = m a_{net} = m\sqrt{\alpha^2 r^2 + \alpha^4 r^2 t^4}$

$t = \left(\dfrac{\mu^2 g^2 - \alpha^2 r^2}{\alpha^4 r^2}\right)^{1/4} \qquad \Rightarrow\; t = \left[\left(\dfrac{\mu g}{\alpha^2 r}\right)^2 - \left(\dfrac{1}{\alpha}\right)^2\right]^{1/4}$ **Ans.**

(ii) $\tan\theta = \dfrac{a_c}{a_t} \;\Rightarrow\; \tan\theta = \dfrac{\alpha^2 r t^2}{\alpha r} \qquad \Rightarrow\; \theta = \tan^{-1}(\alpha t^2)$ **Ans.**

Problem 6. A block is released from the top of a smooth vertical track, which ends in a circle of radius r as shown.



(i) Find the minimum value of h so that the block completes the circle.
(ii) If h = 3r, find normal reaction when the block is at the points A and B.
(iii) If h = 2r, find the velocity of the block when it loses the contact with the track.

Solution : (i) For completing the circle, velocity at lowest point of circle (say A) is $\sqrt{5gr}$

from energy conservation $mgh = \frac{1}{2}m\left(\sqrt{5gr}\right)^2 \Rightarrow h = \frac{5r}{2}$ **Ans.**

(ii) h = 3r
From energy conservation velocity at point A and B are

$mg.3r = \frac{1}{2}mv_A^2 \Rightarrow v_A = \sqrt{6gr}$

$mg.3r = mg2r + \frac{1}{2}mv_B^2 \Rightarrow v_B = \sqrt{2gr}$

Therefore normal reaction at A and B is -

$N_A - mg = \frac{mv_A^2}{r} \Rightarrow N_A = 7mg$

$N_B + mg = \frac{mv_B^2}{r} \Rightarrow N_B = mg$

(iii) h = 2r
It loses contact with the track when normal reaction is zero

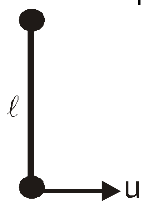
$\frac{mv^2}{r} = mg\ \cos\theta$ ......(1)

from energy conservation

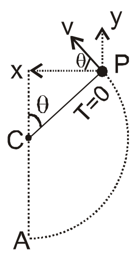
$mgh = mgr(1 + \cos\theta) + \frac{1}{2}mv^2$ ....(2)

from (1) and (2) ; $v = \sqrt{\frac{2g(h-r)}{3}} = \sqrt{\frac{2gr}{3}}$ **Ans.**

Problem 7. A point mass m connected to one end of inextensible string of length ℓ and other end of string is fixed at peg. String is free to rotate in vertical plane. Find the minimum velocity give to the mass in horizontal direction so that it hits the peg in its subsequent motion.



Solution : Tension in string is zero at point P in its subsequent motion, after this point its motion is projectile.



Velocity at point P, T = 0 $\Rightarrow mg\cos\theta = \frac{mv^2}{\ell} \Rightarrow v = \sqrt{g\ell\cos\theta}$

Assume its projectile motion start at point P and it passes through point C. So that equation of trajectory satisfy the co-ordinate of C $(\ell\sin\theta, -\ell\cos\theta)$

Equation of trajectory

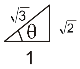
$y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta}$

$-\ell\cos\theta = \ell\sin\theta\tan\theta - \frac{g(\ell\sin\theta)^2}{2(g\ell\cos\theta)\cos^2\theta}$

$\Rightarrow \; -\cos\theta = \dfrac{\sin^2\theta}{\cos\theta} - \dfrac{1}{2}\dfrac{\sin^2\theta}{\cos^3\theta}$

$\Rightarrow \; -2\cos^4\theta = 2\sin^2\theta\cos^2\theta - \sin^2\theta$ $\quad\Rightarrow \; \sin^2\theta = 2\sin^2\theta\cos^2\theta + 2\cos^4\theta$

$\Rightarrow \; \sin^2\theta = 2\cos^2\theta(\sin^2\theta + \cos^2\theta)$ $\quad\Rightarrow \; \tan^2\theta = 2$

$\Rightarrow \; \tan\theta = \sqrt{2}$ $\quad\Rightarrow$ 

$\quad\Rightarrow \; \cos\theta = \dfrac{1}{\sqrt{3}}, \; \sin\theta = \sqrt{\dfrac{2}{3}}$

From energy conservation between point P and A.

$\dfrac{1}{2}mu^2 = \dfrac{1}{2}mv^2 + mg\ell(1+\cos\theta)$ $\quad\Rightarrow \; u^2 = v^2 + 2g\ell(1+\cos\theta)$

$\Rightarrow \; u^2 = 2g\ell + 3g\ell\cos\theta$ $\quad\Rightarrow \; u^2 = 2g\ell + 3g\ell\dfrac{1}{\sqrt{3}}$ $\quad\Rightarrow u = \left[\left(2+\sqrt{3}\right)g\ell\right]^{1/2}$ **Ans.**

Problem 8. A simple pendulum of length ℓ and mass m free to oscillate in vertical plane. A nail is located at a distance 'd = ℓ – a' vertically below the point of suspension of a simple pendulum. The pendulum bob is released from the position where the string makes an angle of 90° from vertical. Discuss the motion of the bob if (a) ℓ = 2a, (b) ℓ = 2.5 a.

Solution : (a) ℓ = 2a, Velocity at lowest point from energy conservation

$0 + mg2a = \dfrac{1}{2}mv^2$

$v = \sqrt{4ga}$

Here radius of circle is 'a' about nail and velocity at lowest point is not sufficient to complete the loop. Therefore motion of bob is combination of circular and projectile motion. Because velocity at lowest point is lie between $\sqrt{3ga}$ and $\sqrt{5ga}$.

(b) ℓ = 2.5 a, Velocity at lowest point from energy conservation

$0 + mg(2.5a) = \dfrac{1}{2}mv^2 \;\Rightarrow\; v = \sqrt{5ga}$

here radius of circle is 'a' about nail and velocity at lowest point is just sufficient to complete the loop so that here looping the loop about nail.