CHAPTER-5
RELATIVE MOTION

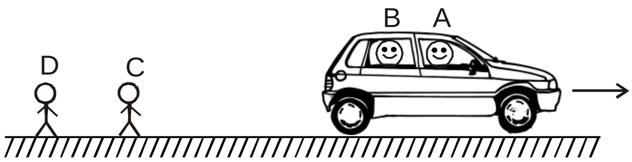
1 RELATIVE MOTION

Motion is a combined property of the object under study as well as the observer. It is always relative ; there is no such thing like absolute motion or absolute rest. Motion is always defined with respect to an observer or reference frame.

Reference frame :

Reference frame is an axis system from which motion is observed along with a clock attached to the axis, to measure time. Reference frame can be stationary or moving.

☞ Suppose there are two persons A and B sitting in a car moving at constant speed. Two stationary persons C and D observe them from the ground.



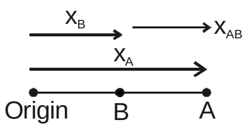
Here B appears to be moving for C and D, but at rest for A. Similarly C appears to be at rest for D but moving backward for A and B.

2 RELATIVE MOTION IN ONE DIMENSION :

2.1 Relative Position :

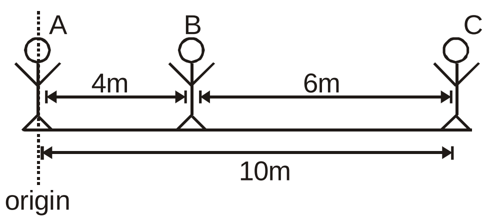
It is the position of a particle w.r.t. observer.

In general if position of A w.r.t. to origin is x_A and that of B w.r.t. origin is x_B then "Position of A w.r.t. B" x_{AB} is



$$\boxed{x_{AB} = x_A - x_B}$$

Example 1. See the figure (take +ve direction towards right and –ve towards left). Find x_{BA}, x_{CA}, x_{CB}, x_{AB} and x_{AC}.



Here, Position of B w.r.t. A is 4 m towards right. (x_{BA} = +4m)

Position of C w.r.t. A is 10 m towards right. (x_{CA} = +10m)

Position of C w.r.t. B is 6 m towards right (x_{CB} = +6m)

Position of A w.r.t. B is 4 m towards left. (x_{AB} = –4 m)

Position of A w.r.t. C is 10 m towards left. (x_{AC} = –10m)

2.2 Relative Velocity

Definition : Relative velocity of a particle A with respect to B is defined as the velocity with which A appears to move if B is considered to be at rest. In other words, it is the velocity with which A appears to move as seen by B considering itself to be at rest.

NOTE 1 : All velocities are relative & have no significance unless observer is specified. However, when we say "velocity of A", what we mean is , velocity of A w.r.t. ground which is assumed to be at rest.

Relative velocity in one dimension -

If x_A is the position of A w.r.t. ground, x_B is position of B w.r.t. ground and x_{AB} is position of A w.r.t. B then we can say v_A = velocity of A w.r.t. ground = $\frac{dx_A}{dt}$

v_B = velocity of B w.r.t. ground = $\frac{dx_B}{dt}$

and v_{AB} = velocity of A w.r.t. B = $\frac{dx_{AB}}{dt} = \frac{d}{dt}(x_A - x_B) = \frac{dx_A}{dt} - \frac{dx_B}{dt}$

Thus

$$\boxed{v_{AB} = v_A - v_B}$$

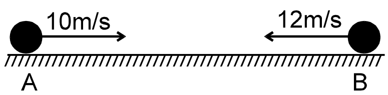
NOTE 2. : Velocity of an object w.r.t. itself is always zero.

Example 2. An object A is moving with 5 m/s and B is moving with 20 m/s in the same direction. (Positive x-axis)
(i) Find velocity of B with respect to A.
(ii) Find velocity of A with respect to B

Solution :
(i) v_B = +20 m/s, v_A = +5 m/s,
$v_{BA} = v_B - v_A$ = +15 m/s
(ii) v_B = +20 m/s, v_A = +15 m/s
$v_{AB} = v_A - v_B$ = – 15 m/s

Note : $v_{BA} = - v_{AB}$

Example 3. Two objects A and B are moving towards each other with velocities 10 m/s and 12 m/s respectively as shown.
(i) Find the velocity of A with respect to B.
(ii) Find the velocity of B with respect to A



Solution : v_A = +10, v_B = – 12
(i) $v_{AB} = v_A - v_B$ = (10) – (–12) = 22 m/s.
(ii) $v_{BA} = v_B - v_A$ = (–12) – (10) = –22 m/s.

2.3 Relative Acceleration

It is the rate at which relative velocity is changing.

$$a_{AB} = \frac{dv_{AB}}{dt} = \frac{dv_A}{dt} - \frac{dv_B}{dt} = a_A - a_B$$

Equations of motion when relative acceleration is constant.

$v_{rel} = u_{rel} + a_{rel}\, t$

$s_{rel} = u_{rel}\, t + \frac{1}{2} a_{rel}\, t^2$

$v^2_{rel} = u^2_{rel} + 2a_{rel}\, s_{rel}$

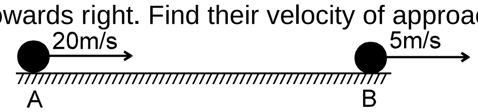
2.4 Velocity of Approach / Separation

It is the component of relative velocity of one particle w.r.t. another, along the line joining them. If the separation is decreasing, we say it is velocity of approach and if separation is increasing, then we say it is velocity of separation. In one dimension, since relative velocity is along the line joining A and B, hence velocity of approach / separation is simply equal to magnitude of relative velocity of A w.r.t. B.

Example 4. A particle A is moving with a speed of 10 m/s towards right and another particle B is moving at speed of 12 m/s towards left. Find their velocity of approach.

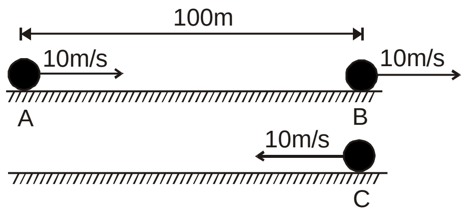
Solution : $V_A = +10$, $V_B = -12$ $\Rightarrow$ $V_{AB} = V_A - V_B$ $\Rightarrow$ $10 - (-12) = 22$ m/s
since separation is decreasing hence $V_{app} = |V_{AB}| = 22$ m/s
Ans. : 22 m/s

Example 5 A particle A is moving with a speed of 20 m/s towards right and another particle B is moving at a speed of 5 m/s towards right. Find their velocity of approach.



Solution : $V_A = +20$, $V_B = +5$
$V_{AB} = V_A - V_B$
$20 - (+5) = 15$ m/s
since separation is decreasing hence $V_{app} = |V_{AB}| = 15$ m/s
Ans. : 15 m/s

Example 6. A particle A is moving with a speed of 10 m/s towards right, particle B is moving at a speed of 10 m/s towards right and another particle C is moving at speed of 10 m/s towards left. The separation between A and B is 100 m. Find the time interval between C meeting B and C meeting A.



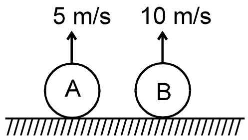
Solution : $t = \dfrac{\text{seperation between A and C}}{V_{app} \text{ of A and C}} = \dfrac{100}{10-(-10)} = 5$ sec.

Ans. : 5 sec.

Note : $a_{app} = \left(\dfrac{d}{dt}\right) v_{app}$, $a_{sep} = \dfrac{d}{dt} v_{sep}$

$v_{app} = \int a_{app} dt$, $v_{sep} = \int a_{sep} dt$

Example 7. A and B are thrown vertically upward with velocity, 5 m/s and 10 m/s respectively (g = 10 m/s^2. Find separation between them after one second



Solution : $S_A = ut - \dfrac{1}{2} gt^2$

$= 5t - \dfrac{1}{2} \times 10 \times t^2 = 5 \times 1 - 5 \times 1^2 = 5 - 5 = 0$

$S_B = ut - \dfrac{1}{2} gt^2 = 10 \times 1 - \dfrac{1}{2} \times 10 \times 1^2 = 10 - 5 = 5$

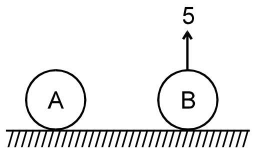
$\therefore$ $S_B - S_A$ = separation = 5m.

Aliter : $\vec{a}_{BA} = \vec{a}_B - \vec{a}_A = (-10) - (-10) = 0$

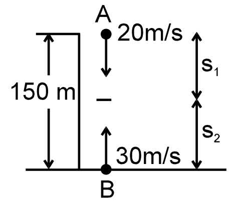
Also $\vec{v}_{BA} = \vec{v}_B - \vec{v}_A = 10 - 5 = 5$ m/s

$\therefore$ $\vec{s}_{BA}$ (in 1 sec) $= \vec{v}_{BA} \times t = 5 \times 1 = 5$ m

$\therefore$ Distance between A and B after 1 sec = 5 m.



Example 8. A ball is thrown downwards with a speed of 20 m/s from the top of a building 150 m high and simultaneously another ball is thrown vertically upwards with a speed of 30 m/s from the foot of the building. Find the time after which both the balls will meet. ($g = 10\ m/s^2$)



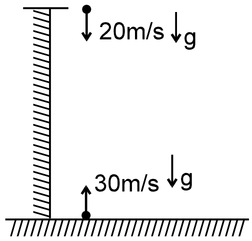
Solution : $S_1 = 20\,t + 5\,t^2$

$S_2 = 30\,t - 5\,t^2$

$S_1 + S_2 = 150$

$\Rightarrow\ 150 = 50\,t \ \Rightarrow\ t = 3\ s$

Aliter : Relative acceleration of both is zero since both have same acceleration in downward direction



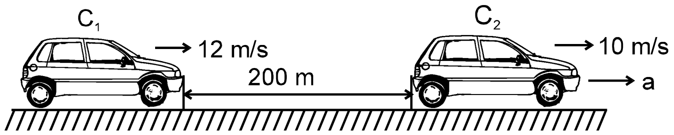
$\vec{a}_{AB} = \vec{a}_A - \vec{a}_B = g - g = 0$

$\vec{v}_{BA} = 30 - (-20) = 50$

$s_{BA} = v_{BA} \times t$

$t = \dfrac{s_{BA}}{v_{BA}} = \dfrac{150}{50} = 3\ s$

Example 9. Two cars C_1 and C_2 moving in the same direction on a straight single lane road with velocities 12 m/s and 10 m/s respectively. When the separation between the two was 200 m C_2 started accelerating to avoid collision. What is the minimum acceleration of car C_2 so that they don't collide.



Solution : Acceleration of car 1 w.r.t. car 2

$\vec{a}_{12} = \vec{a}_1 - \vec{a}_2 = \vec{a}_{C_1} - \vec{a}_{C_2} = 0 - a = (-a)$

$\vec{u}_{12} = \vec{u}_1 - \vec{u}_2 = 12 - 10 = 2\ m/s.$

The collision is just avoided if relative velocity becomes zero just at the moment the two cars meet each other.

i.e. $v_{12} = 0$ When $s_{12} = 200$

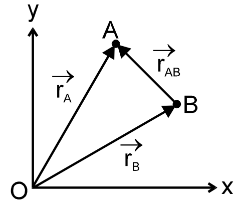
Now $v_{12} = 0$, $\vec{u}_{12} = 2$, $\vec{a}_{12} = -a$ and $s_{12} = 200$

$\therefore\ v_{12}^2 - u_{12}^2 = 2a_{12}s_{12}$

$\Rightarrow\ 0 - 2^2 = -2 \times a \times 200 \qquad \Rightarrow \qquad a = \dfrac{1}{100}\ m/s^2 = 0.1\ m/s^2 = 1\ cm/s^2.$

$\therefore$ Minimum acceleration needed by car $C_2 = 1\ cm/s^2$

3. RELATIVE MOTION IN TWO DIMENSION



$\vec{r}_A$ = position of A with respect to O

$\vec{r}_B$ = position of B with respect to O

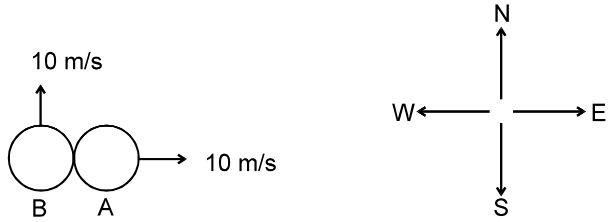
$\vec{r}_{AB}$ = position of A with respect to B.

$\vec{r}_{AB} = \vec{r}_A - \vec{r}_B$ (The vector sum $\vec{r}_A - \vec{r}_B$ can be done by Δ law of addition or resolution method)

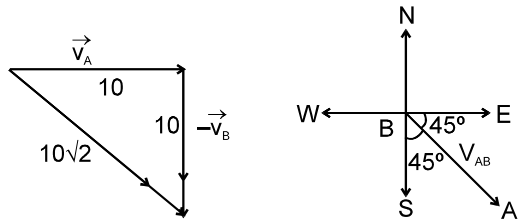
$\therefore\ \dfrac{d(\vec{r}_{AB})}{dt} = \dfrac{d(\vec{r}_A)}{dt} - \dfrac{d(\vec{r}_B)}{dt}.$

$\Rightarrow\ \vec{v}_{AB} = \vec{v}_A - \vec{v}_B\ ;\ \dfrac{d(\vec{v}_{AB})}{dt} = \dfrac{d(\vec{v}_A)}{dt} - \dfrac{d(\vec{v}_B)}{dt} \ \Rightarrow\ \vec{a}_{AB} = \vec{a}_A - \vec{a}_B$

Example 10. Object A and B both have speed of 10 m/s. A is moving towards East while B is moving towards North starting from the same point as shown. Find velocity of A relative to B.



Solution : **Method 1 :** $\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$ $\therefore$ $|\vec{v}_{AB}| = 10\sqrt{2}$

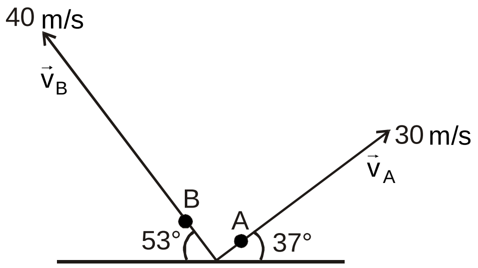


Method 2 : $\vec{v}_A = 10\hat{i}$, $\vec{v}_B = 10\hat{j}$

$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B = 10\hat{i} - 10\hat{j}$ $\therefore$ $|\vec{v}_{AB}| = 10\sqrt{2}$

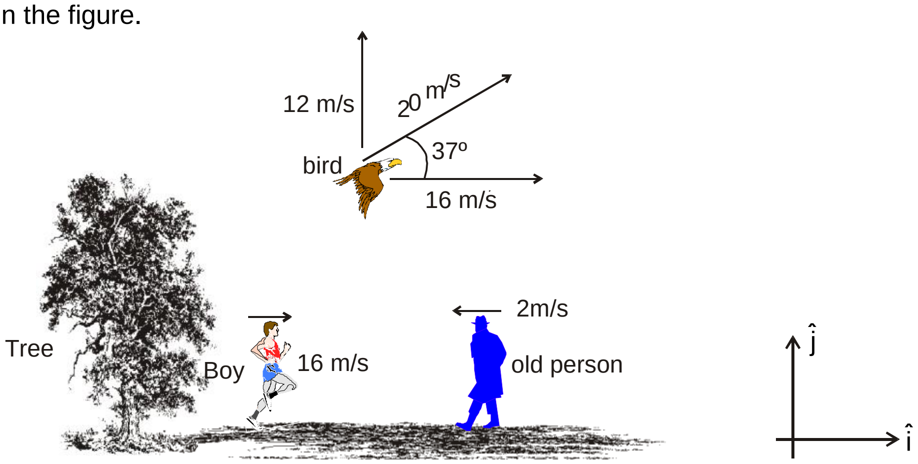
Note : $|\vec{v}_A - \vec{v}_B| = \sqrt{v_A^2 + v_B^2 - 2v_A v_B \cos\theta}$, where θ is angle between $\vec{v}_A$ and $\vec{v}_B$.

Example 11. Two particles A and B are projected in air. A is thrown with a speed of 30 m/sec and B with a speed of 40 m/sec as shown in the figure. What is the separation between them after 1 sec.



Solution : $\vec{a}_{AB} = \vec{a}_A - \vec{a}_B = \vec{g} - \vec{g} = 0$ $\therefore$ $v_{AB} = \sqrt{30^2 + 40^2} = 50$ $\therefore$ $s_{AB} = v_{AB}t = 50\ t = 50$ m

Example 12. An old man and a boy are walking towards each other and a bird is flying over them as shown in the figure.



(1) Find the velocity of tree, bird and old man as seen by boy.
(2) Find the velocity of tree, bird and boy as seen by old man
(3) Find the velocity of tree, boy and old man as seen by bird.

Solution : (1) With respect to boy :

v_{tree} = 16 m/s (←) or $-16\hat{i}$

v_{bird} = 12 m/s (↑) or $12\hat{j}$

$v_{old\ man}$ = 18 m/s (←) or $-18\hat{i}$

(2) With respect to old man :

v_{Boy} = 18 m/s (→) or $18\hat{i}$

v_{Tree} = 2 m/s (→) or $2\hat{i}$

v_{Bird} = 18 m/s (→) and 12 m/s (↑) or $18\hat{i} + 12\hat{j}$

(3) With respect to Bird : v_{Tree} = 12 m/s (↓) and 16 m/s (←) or $-12\hat{j} - 16\hat{i}$

$v_{old\ man}$ = 18 m/s (←) and 12 m/s (↓) or $-18\hat{i} - 12\hat{j}$

v_{Boy} = 12 m/s (↓) or $-12\hat{j}$

3.1 Relative Motion in Lift

Projectile motion in a lift moving with acceleration a upwards

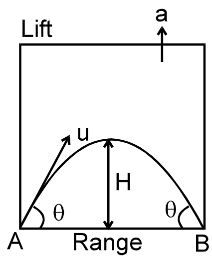
(1) In the reference frame of lift, acceleration of a freely falling object is (g + a)

(2) Velocity at maximum height = ucosθ

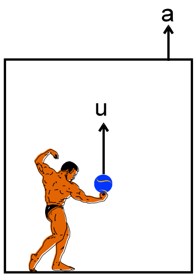
(3) $T = \dfrac{2u\sin\theta}{g+a}$

(4) Maximum height (H) = $\dfrac{u^2\sin^2\theta}{2(g+a)}$

(5) Range = $\dfrac{u^2\sin 2\theta}{g+a}$



Example 13. A lift is moving up with acceleration a. A person inside the lift throws the ball upwards with a velocity u relative to hand.



(a) What is the time of flight of the ball?

(b) What is the maximum height reached by the ball in the lift?

Solution : (a) $\vec{a}_{BL} = \vec{a}_B - \vec{a}_L = g + a$

$\vec{s} = \vec{u}\,t + \dfrac{1}{2}\vec{a}_{BL}t^2$

$0 = uT - \dfrac{1}{2}(g + a)T^2$

$\therefore \quad T = \dfrac{2u}{(g+a)}$

(b) $v^2 - u^2 = 2as$

$0 - u^2 = -2(g + a)H$

$H = \dfrac{u^2}{2(g+a)}$

4. RELATIVE MOTION IN RIVER FLOW

If a man can swim relative to water with velocity $\vec{v}_{mR}$ and water is flowing relative to ground with velocity $\vec{v}_R$, velocity of man relative to ground $\vec{v}_m$ will be given by :

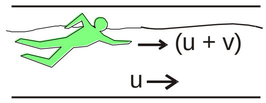
$\vec{v}_{mR} = \vec{v}_m - \vec{v}_R$ or $\vec{v}_m = \vec{v}_{mR} + \vec{v}_R$

If $\vec{v}_R = 0$, then $\vec{v}_m = \vec{v}_{mR}$ in words, velocity of man in still water = velocity of man w.r.t. river

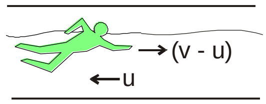
4.1 River Problem in One Dimension :

☞ Velocity of river is u & velocity of man in still water is v.

Case-1 : Man swimming downstream (along the direction of river flow).
In this case velocity of river $v_R = + u$
velocity of man w.r.t. river $v_{mR} = +v$
now $\vec{v}_m = \vec{v}_{mR} + \vec{v}_R = u + v$



Case-2 : Man swimming upstream (opposite to the direction of river flow). In this case velocity of river $\vec{v}_R = -u$
velocity of man w.r.t. river $\vec{v}_{mR} = +v$
now $\vec{v}_m = \vec{v}_{mR} + \vec{v}_R = (v - u)$



Example 14 A swimmer capable of swimming with velocity 'v' relative to water jumps in a flowing river having velocity 'u'. The man swims a distance d down stream and returns back to the original position. Find out the time taken in complete motion.

Solution : Total time = time of swimming downstream + time of swimming upstream

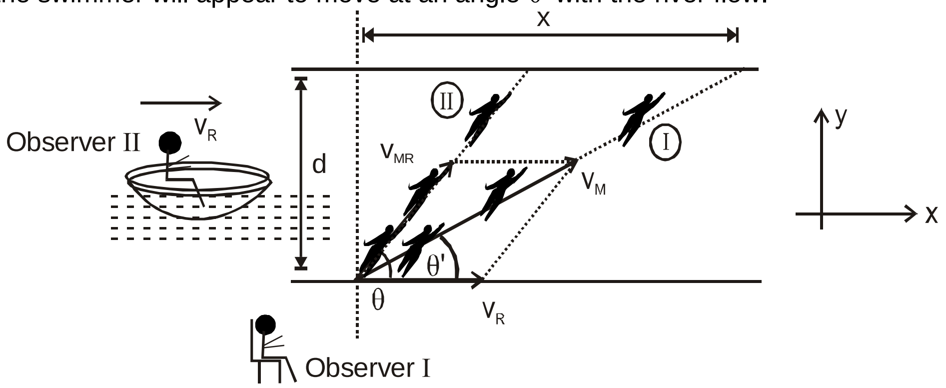
$$t = t_{down} + t_{up} = \frac{d}{v+u} + \frac{d}{v-u} = \frac{2dv}{v^2 - u^2} \quad \textbf{Ans.}$$

4.2 Motion of Man Swimming in a River

Consider a man swimming in a river with a velocity of $\vec{v}_{MR}$ relative to river at an angle of θ with the river flow. The velocity of river is $\vec{V}_R$. Let there be two observers I and II, observer I is on ground and observer II is on a raft floating along with the river and hence moving with the same velocity as that of river. Hence motion w.r.t. observer II is same as motion w.r.t. river. i.e., the man will appear to swim at an angle θ with the river flow for observer II .

For observer I the velocity of swimmer will be $\vec{v}_M = \vec{v}_{MR} + \vec{v}_R$,

Hence the swimmer will appear to move at an angle θ' with the river flow.

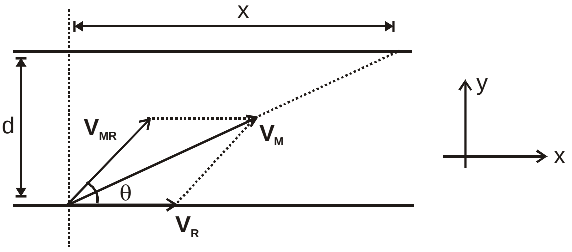


(I) : Motion of swimmer for observer I

(II) : Motion of swimmer for observer II

4.3 River problem in two dimension (crossing river) :

Consider a man swimming in a river with a velocity of $\vec{v}_{MR}$ relative to river at an angle of θ with the river flow. The velocity of river is V_R and the width of the river is d



$\vec{v}_M = \vec{v}_{MR} + \vec{v}_R \Rightarrow \vec{v}_M = (v_{MR}\cos\theta\,\hat{i} + v_{MR}\sin\theta\,\hat{j}) + v_R\,\hat{i} \Rightarrow \vec{v}_M = (v_{MR}\cos\theta + v_R)\,\hat{i} + v_{MR}\sin\theta\,\hat{j}$

Here $v_{MR}\sin\theta$ is the component of velocity of man in the direction perpendicular to the river flow. This component of velocity is responsible for the man crossing the river. Hence if the time to cross the river is t, then $t = \dfrac{d}{v_y} = \dfrac{d}{v_{MR}\sin\theta}$

DRIFT

It is defined as the displacement of man in the direction of river flow. (See the figure). It is simply the displacement along x-axis, during the period the man crosses the river. $(v_{MR}\cos\theta + v_R)$ is the component of velocity of man in the direction of river flow and this component of velocity is responsible for drift along the river flow. If drift is x then,

Drift = $v_x \times t$

$$x = (v_{MR}\cos\theta + v_R) \times \frac{d}{v_{MR}\sin\theta}$$

4.4 Crossing the river in shortest time

As we know that $t = \dfrac{d}{v_{MR}\sin\theta}$. Clearly t will be minimum when $\theta = 90°$ i.e. time to cross the river will be minimum if man swims perpendicular to the river flow. Which is equal to $\dfrac{d}{v_{MR}}$.

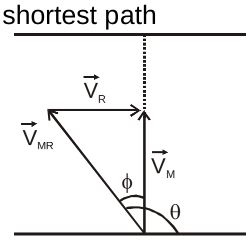
4.5 Crossing the river in shortest path, Minimum Drift

The minimum possible drift is zero. In this case the man swims in the direction perpendicular to the river flow as seen from the ground. This path is known as **shortest path**

here $x_{min} = 0 \Rightarrow (v_{MR}\cos\theta + v_R) = 0$ or $\cos\theta = -\dfrac{v_R}{v_{MR}}$

- ☞ since $\cos\theta$ is –ve, $\therefore\ \theta > 90°$, i.e. for minimum drift the man must swim at some angle ϕ with the perpendicular in backward direction. Where $\sin\phi = \dfrac{v_R}{v_{MR}}$
- ☞ $\theta = \cos^{-1}\left(\dfrac{-v_R}{v_{MR}}\right) \quad \therefore \left|\dfrac{v_R}{v_{MR}}\right| \le 1$ i.e. $v_R \le v_{MR}$

  i.e. minimum drift is zero if and only if velocity of man in still water is greater than or equal to the velocity of river.
- ☞ Time to cross the river along the shortest path



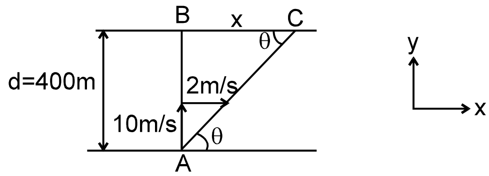
$$t = \frac{d}{v_{MR}\sin\theta} = \frac{d}{\sqrt{v_{MR}^2 - V_R^2}}$$

Note :

- **If $v_R > v_{MR}$ then it is not possible to have zero drift. In this case the minimum drift (corresponding to shortest possible path is non zero and the condition for minimum drift can be proved to be $\cos\theta = -\left(\frac{v_{MR}}{v_R}\right)$ or $\sin\phi = \left(\frac{v_{MR}}{v_R}\right)$ for minimum but non zero drift.**

Example 15. A 400 m wide river is flowing at a rate of 2.0 m/s.A boat is sailing with a velocity of 10 m/s with respect to the water, in a direction perpendicular to the river.
(a) Find the time taken by the boat to reach the opposite bank.
(b) How far from the point directly opposite to the starting point does the boat reach the opposite bank.
(c) In what direction does the boat actually move, with river flow (downstream).

Solution :



(a) time taken to cross the river $t = \frac{d}{v_y} = \frac{400m}{10m/s} = 40$ s **Ans.**

(b) drift (x) = $(v_x)(t)$ = (2m/s) (40s) = 80 m **Ans.**

(c) Actual direction of boat, $\theta = \tan^{-1}\left(\frac{10}{2}\right) = \tan^{-1} 5$, (downstream) with the river flow.

Example 16. A man can swim at the rate of 5 km/h in still water. A 1 km wide river flows at the rate of 3 km/h. The man wishes to swim across the river directly opposite to the starting point.
(a) Along what direction must the man swim?
(b) What should be his resultant velocity?
(c) How much time will he take to cross the river?

Solution : The velocity of man with respect to river v_{mR}= 5 km/hr, this is greater than the river flow velocity, therefore, he can cross the river directly (along the shortest path). The angle of swim must be

$$\theta = \frac{\pi}{2} + \sin^{-1}\left(\frac{v_r}{v_{mR}}\right) = 90^o + \sin^{-1}\left(\frac{v_r}{v_{mR}}\right) = 90^o + \sin^{-1}\left(\frac{3}{5}\right) = 90^o + 37^o$$

= 127° w.r.t. the river flow or 37° w.r.t. perpendicular in backward direction **Ans.**

(b) Resultant velocity will be

$$v_m = \sqrt{v_{mR}^2 - v_R^2} = \sqrt{5^2 - 3^2} = 4 \text{ km/hr}$$

along the direction perpendicular to the river flow.

(c) time taken to cross the $t = \frac{d}{\sqrt{v_{mR}^2 - v_R^2}} = \frac{1km}{4km/hr} = \frac{1}{4}$ h = 15 min

Example 17. A man wishing to cross a river flowing with velocity u jumps at an angle θ with the river flow.
(i) Find the net velocity of the man with respect to ground if he can swim with speed v in still water.
(ii) In what direction does the man actually move.
(iii) Find how far from the point directly opposite to the starting point does the man reach the opposite bank, if the width of the river is d. (i.e. drift)

Solution : (i) $v_{MR} = v$, $v_R = u$; $\vec{v}_M = \vec{v}_{MR} + \vec{v}_R$

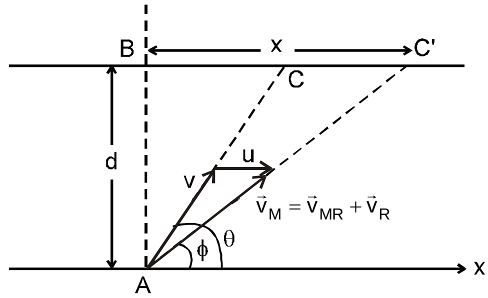
$\therefore$ Velocity of man,

$v_M = \sqrt{u^2 + v^2 + 2vu\cos\theta}$ **Ans.**

(ii) $\tan\phi = \dfrac{v\sin\theta}{u + v\cos\theta}$ **Ans.**

(iii) $(v \sin\theta)\, t = d \Rightarrow t = \dfrac{d}{v\sin\theta}$;

$x = (u + v\cos\theta)\, t = (u + v\cos\theta)\dfrac{d}{v\sin\theta}$



Example 18. A boat moves relative to water with a velocity v which is n times less than the river flow velocity u. At what angle to the stream direction must the boat move to minimize drifting ?

Solution : (In this problem, one thing should be carefully noted that the velocity of boat is less than the river flow velocity. Hence boat cannot reach the point directly opposite to its starting point. i.e. drift can never be zero)

Suppose boat starts at an angle θ from the normal direction up stream as shown.

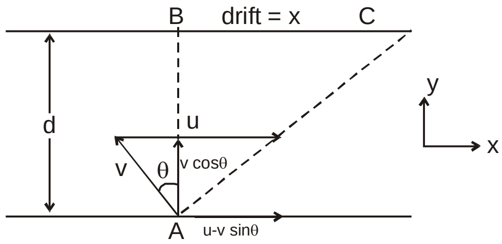
Component of velocity of boat along the river, $v_x = u - v\sin\theta$

and velocity perpendicular to the river,

$v_y = v\cos\theta$.

time taken to cross the river is

$t = \dfrac{d}{v_y} = \dfrac{d}{v\cos\theta}$.



Drift $x = (v_x)t = (u - v\sin\theta)\dfrac{d}{v\cos\theta}$

$= \dfrac{ud}{v}\sec\theta - d\tan\theta$

The drift x is minimum, when $\dfrac{dx}{d\theta} = 0$,

or $\left(\dfrac{ud}{v}\right)(\sec\theta \,.\, \tan\theta) - d\sec^2\theta = 0$

or $\dfrac{u}{v}\sin\theta = 1$ or $\boxed{\sin\theta = \dfrac{v}{u}}$

This is the result we stated without proof as a note in section 4.5

so, for minimum drift, the boat must move at an angle $\theta = \sin^{-1}\left(\dfrac{v}{u}\right) = \sin^{-1}\dfrac{1}{n}$ from normal direction.

5. WIND AIRPLANE PROBLEMS

This is very similar to boat river flow problems. The only difference is that boat is replaced by aeroplane and river is replaced by wind.

Thus, velocity of aeroplane with respect to wind

$\vec{v}_{aw} = \vec{v}_a - \vec{v}_w$ or $\vec{v}_a = \vec{v}_{aw} + \vec{v}_w$

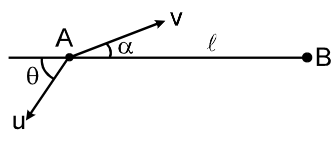
where, $\vec{v}_a$ = velocity of aeroplane w.r.t. ground and, $\vec{v}_w$ = velocity of wind.

Example 19 An aeroplane flies along a straight path A to B and returns back again. The distance between A and B is ℓ and the aeroplane maintains the constant speed v w.r.t. wind. There is a steady wind with a speed u at an angle θ with line AB. Determine the expression for the total time of the trip.

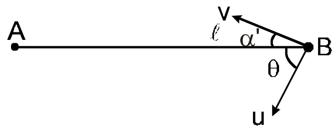
Solution : Suppose plane is oriented at an angle α w.r.t. line AB while the plane is moving from A to B :

Velocity of plane along AB = $v\cos\alpha - u\cos\theta$,

and for no-drift from line AB ; $v\sin\alpha = u\sin\theta$



$\Rightarrow \quad \sin\alpha = \dfrac{u\sin\theta}{v}$



time taken from A to B : $t_{AB} = \dfrac{\ell}{v\cos\alpha - u\cos\theta}$

Suppose plane is oriented at an angle α' w.r.t. line AB
while the plane is moving from B to A :
velocity of plane along BA = $v\cos\alpha + u\cos\theta$ and for no drift from line AB ; $v\sin\alpha = u\sin\theta$

$\Rightarrow \quad \sin\alpha = \dfrac{u\sin\theta}{v} \qquad \Rightarrow \qquad \alpha = \alpha'$

time taken from B to A : $t_{BA} = \dfrac{\ell}{v\cos\alpha + u\cos\theta}$

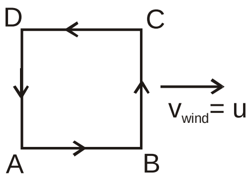
total time taken = $t_{AB} + t_{BA} = \dfrac{\ell}{v\cos\alpha - u\cos\theta} + \dfrac{\ell}{v\cos\alpha + u\cos\theta}$

$= \dfrac{2v\ell\cos\alpha}{v^2\cos^2\alpha - u^2\cos^2\theta} = \dfrac{2v\ell\sqrt{1-\dfrac{u^2\sin^2\theta}{v^2}}}{v^2 - u^2}$.

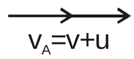
Example 20. Find the time an aeroplane having velocity v, takes to fly around a square with side a if the wind is blowing at a velocity u along one side of the square.

Answer : $\dfrac{2a}{v^2 - u^2}\left(v + \sqrt{v^2 - u^2}\right)$

Solution :

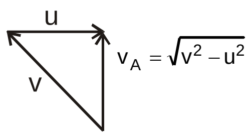


Velocity of aeroplane while flying through AB



$v_A = v + u \quad ; \quad t_{AB} = \dfrac{a}{v+u}$

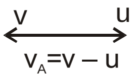
Velocity of aeroplane while flying through BC



$v_A = \sqrt{v^2 - u^2}$;

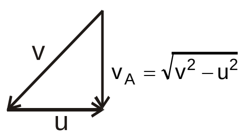
$t_{BC} = \dfrac{a}{\sqrt{v^2 - u^2}}$

Velocity of aeroplane while flying through CD



$v_A = v - u \quad ; \quad t_{CD} = \dfrac{a}{v-u}$

Velocity of aeroplane while flying through DA



$v_A = \sqrt{v^2 - u^2}$; $t_{DA} = \dfrac{a}{\sqrt{v^2 - u^2}}$

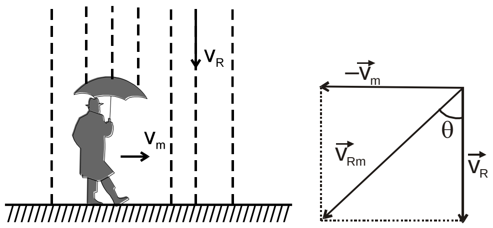
Total time = $t_{AB} + t_{BC} + t_{CD} + t_{DA} = \dfrac{a}{v+u} + \dfrac{a}{\sqrt{v^2-u^2}} + \dfrac{a}{v-u} + \dfrac{a}{\sqrt{v^2-u^2}} = \dfrac{2a}{v^2-u^2}\left(v + \sqrt{v^2 - u^2}\right)$

6. RAIN PROBLEM

If rain is falling vertically with a velocity $\vec{v}_R$ and an observer is moving horizontally with velocity $\vec{v}_m$, the velocity of rain relative to observer will be :

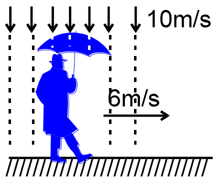
$\vec{v}_{Rm} = \vec{v}_R - \vec{v}_m$ or $v_{Rm} = \sqrt{v_R^2 + v_m^2}$

and direction $\theta = \tan^{-1}\left(\frac{v_m}{v_R}\right)$ with the vertical as shown in figure



Example 21. Rain is falling vertically at speed of 10 m/s and a man is moving with velocity 6 m/s. Find the angle at which the man should hold his umbrella to avoid getting wet.

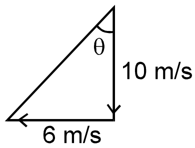
Solution :



$\vec{v}_{rain} = -10\hat{j} \Rightarrow \vec{v}_{man} = 6\hat{j}$

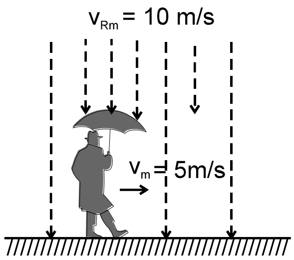
$\vec{v}_{r.w.r.t.\ man} = -10\hat{j} - 6\hat{i}$

$\tan\theta = \frac{6}{10} \Rightarrow \theta = \tan^{-1}\left(\frac{3}{5}\right)$



Where θ is angle with vertical

Example 22. A man moving with 5m/s observes rain falling vertically at the rate of 10 m/s. Find the speed and direction of the rain with respect to ground.



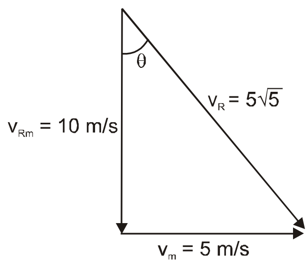
Solution : $v_{RM} = 10$ m/s, $v_M = 5$ m/s

$\vec{v}_{RM} = \vec{v}_R - \vec{v}_M$

$\Rightarrow \vec{v}_R = \vec{v}_{RM} + \vec{v}_M$

$\Rightarrow \vec{v}_R = 5\sqrt{5}$

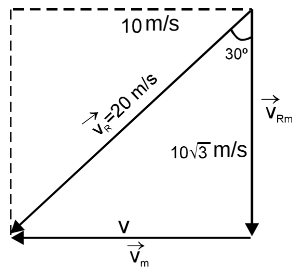
$\tan\theta = \frac{1}{2}$, $\theta = \tan^{-1}\frac{1}{2}$.



Example 23. A standing man, observes rain falling with velocity of 20 m/s at an angle of 30º with the vertical.

(i) Find the velocity with which the man should move so that rain appears to fall vertically to him.

(ii) Now if he further increases his speed, rain again appears to fall at 30º with the vertical. Find his new velocity.



Solution : (i) $\vec{v}_m = -v\hat{i}$ (let)

$\vec{v}_R = -10\hat{i} - 10\sqrt{3}\,\hat{j}$

$\vec{v}_{RM} = -(10\hat{i} - v) - 10\sqrt{3}\,\hat{j}$

$\Rightarrow\; -(10 - v) = 0$

(for vertical fall, horizontal component must be zero)

or $v = 10$ m/s **Ans.**

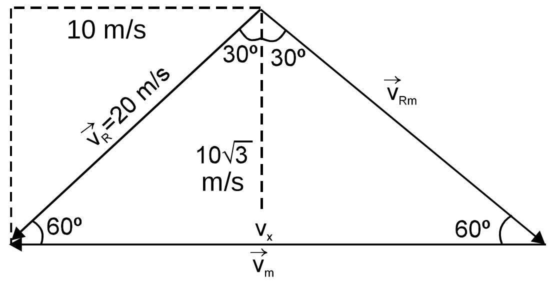
(ii) $\vec{v}_R = -10\hat{i} - 10\sqrt{3}\,\hat{j}$

$\vec{v}_m = -v_x\hat{i}$

$\vec{v}_{RM} = -(10 - v_x)\hat{i} - 10\sqrt{3}\,\hat{j}$

Angle with the vertical = 30º

$\Rightarrow\; \tan 30^\circ = \dfrac{10 - v_x}{-10\sqrt{3}} \;\Rightarrow\; v_x = 20$ m/s

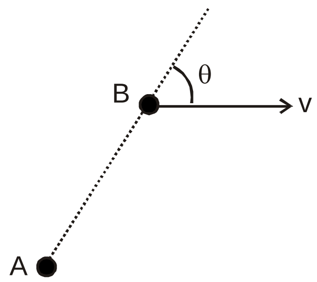


7. VELOCITY OF APPROACH / SEPARATION IN TWO DIMENSION

It is the component of relative velocity of one particle w.r.t. another, along the line joining them.

If the separation is decreasing, we say it is velocity of approach and if separation is increasing, then we say it is velocity of separation.

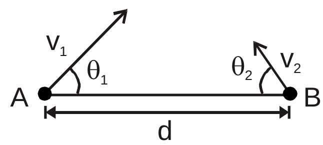
Example 24. Particle A is at rest and particle B is moving with constant velocity v as shown in the diagram at t = 0. Find their velocity of separation



Solution : $v_{BA} = v_B - v_A = v$

v_{sep} = component of v_{BA} along line AB = $v\cos\theta$

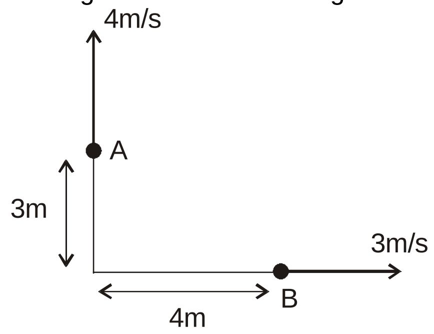
Example 25. Two particles A and B are moving with constant velocities v_1 and v_2. At t = 0, v_1 makes an angle θ_1 with the line joining A and B and v_2 makes an angle θ_2 with the line joining A and B. Find their velocity of approach.



Solution : Velocity of approach is relative velocity along line AB

$v_{APP} = v_1\cos\theta_1 + v_2\cos\theta_2$

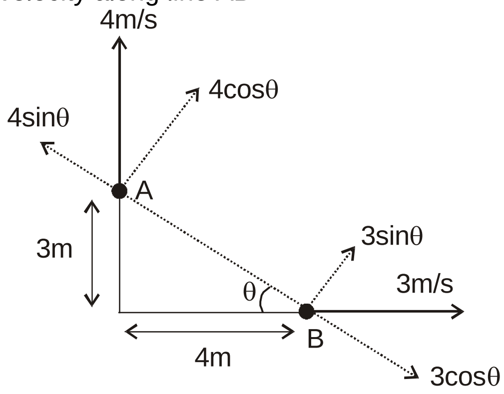
Example 26. Particles A and B are moving as shown in the diagram at t = 0. Find their velocity of separation



(i) at t = 0 (ii) at t = 1 sec.

Solution : (i) $\tan\theta = 3/4$

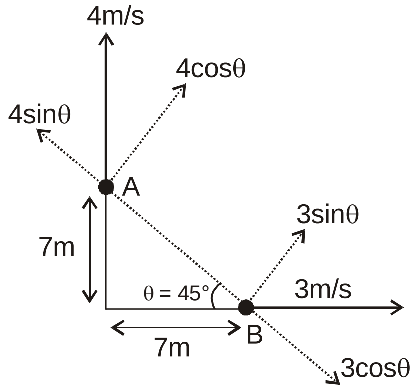
v_{sep} = relative velocity along line AB



$= 3\cos\theta + 4\sin\theta = 3.\ \frac{4}{5} + 4.\ \frac{3}{5} = \frac{24}{5} = 4.8$ m/s

(ii) $\theta = 45°$

v_{sep} = relative velocity along line AB

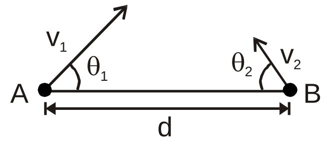


$= 3\cos\theta + 4\sin\theta = 3.\ \frac{1}{\sqrt{2}} + 4.\ \frac{1}{\sqrt{2}} = \frac{7}{\sqrt{2}}$ m/s

7.1 Condition for uniformly moving particles to collide

If two particles are moving with uniform velocities and the relative velocity of one particle w.r.t. other particle is directed towards each other then they will collide.

Example 27. Two particles A and B are moving with constant velocities v_1 and v_2. At t = 0, v_1 makes an angle θ_1 with the line joining A and B and v_2 makes an angle θ_2 with the line joining A and B.



(i) Find the condition for A and B to collide.

(ii) Find the time after which A and B will collide if separation between them is d at t = 0

Solution : (i) For A and B to collide, their relative velocity must be directed along the line joining them. Therefore their relative velocity along the perpendicular to this line must be zero.
Thus $v_1 \sin\theta_1 = v_2 \sin\theta_2$.

(ii) $v_{APP} = v_1\cos\theta_1 + v_2\cos\theta_2$; $t = \frac{d}{v_{app}} = \frac{d}{v_1\cos\theta_1 + v_2\cos\theta_2}$

7.2 Minimum / Maximum distance between two particles

If the separation between two particles decreases and after some time it starts increasing then the separation between them will be minimum at the instant, velocity of approach changes to velocity of separation. (at this instant $v_{app} = 0$)

Mathematically S_{AB} is minimum when $\frac{dS_{AB}}{dt} = 0$

Similarly for maximum separation $v_{sep} = 0$.

Note :

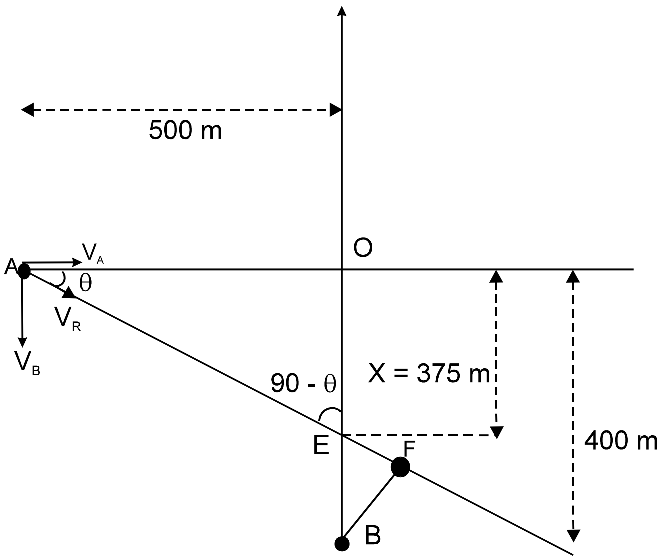
- If the initial position of two particles are $\vec{r}_1$ and $\vec{r}_2$ and their velocities are $\vec{v}_1$ and $\vec{v}_2$ then shortest distance between the particles, $d_{shortest} = \frac{|\vec{r}_{12} \times \vec{v}_{12}|}{|\vec{v}_{12}|}$ and time after which this situation will occur, $t = -\frac{\vec{r}_{12}.\vec{v}_{12}}{|\vec{v}_{12}|^2}$

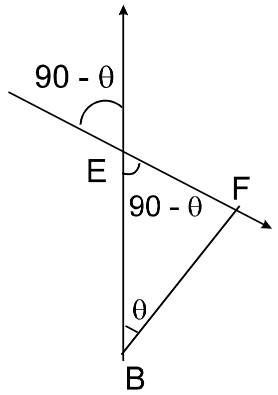
Example 28. Two cars A and B are moving west to east and south to north respectively along crossroads. A moves with a speed of 72 kmh^{-1} and is 500 m away from point of intersection of cross roads and B moves with a speed of 54 kmh^{-1} and is 400 m away from point of intersection of cross roads. Find the shortest distance between them?

Solution : **Method – I (Using the concept of relative velocity)**

In this method we watch the velocity of A w.r.t. B. To do this we plot the resultant velocity V_r. Since the accelerations of both the bodies is zero, so the relative acceleration between them is also zero. Hence the relative velocity will remain constant. So the path of A with respect to B will be straight line and along the direction of relative velocity of A with respect to B. The shortest distance between A & B is when A is at point F (i.e. when we drop a perpendicular from B on the line of motion of A with respect to B).

From figure





$\tan\theta = \dfrac{V_B}{V_A} = \dfrac{15}{20} = \dfrac{3}{4}$ ....(i)

This θ is the angle made by the resultant velocity vector with the x-axis.
Also we know that from figure

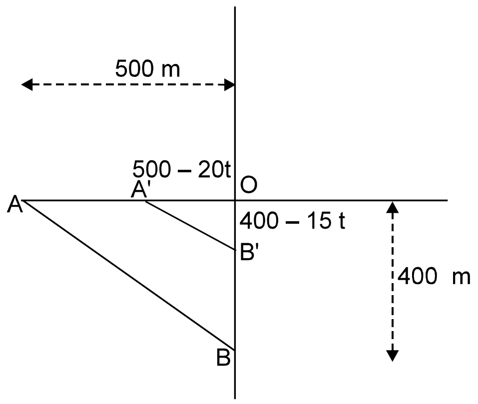
$OE = \dfrac{x}{500} = \dfrac{3}{4}$ ....(ii)

From equation (i) & (ii) we get
x = 375 m
∴ EB = OB – OE = 400 – 375 = 25 m
But the shortest distance is BF.

From magnified figure we see that $BF = EB\cos\theta = 25 \times \dfrac{4}{5}$

∴ BF = 20 m

Method II (Using the concept of maxima – minima)
A & B be are the initial positions and A', B' be the final positions after time t.
B is moving with a speed of 15 m/sec so it will travel a distance of BB' = 15t during time t.
A is moving with a speed of 20 m/sec so it will travel a distance of AA' = 20t during time t.
So
OA' =500 – 20 t
OB' = 400 – 15 t
∴ $A'B'^2 = OA'^2 + OB'^2$
$= (500 - 20t)^2 + (400 - 15t)^2$ .....(i)
For A'B' to be minimum $A'B'^2$ should also be minimum



$\therefore \dfrac{d(A'B'^2)}{dt} = \dfrac{d(400-15t)^2 + (500-20t)^2}{dt} = 0$

= 2(400 – 15t) (–15) + 2(500 – 20t) (–20) = 0
= – 1200 + 45t = 2000 – 80 t
∴ 125 t = 3200

$\therefore \quad t = \dfrac{128}{5}$ s.

Hence A and B will be closest after $\dfrac{128}{5}$ s.

Now $\dfrac{d^2A'B'}{dt^2}$ comes out to be positive hence it is a minima.

On substituting the value of t in equation (i) we get

$\therefore$ $A'B'^2 = \left(400 - 15 \times \frac{128}{5}\right)^2 + \left(500 - 20.\frac{128}{5}\right)^2 = \sqrt{16^2 + (-12)^2}$ = 20 m

$\therefore$ Minimum distance A'B' = 20 m.

Method III (Using the concept of relative velocity of approach)

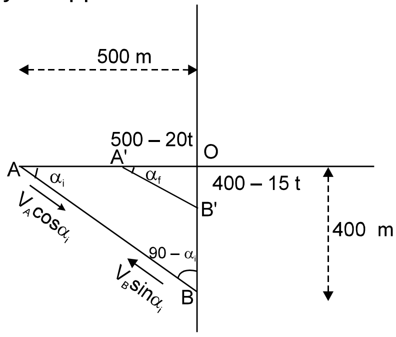
After time t let us plot the components of velocity of A and B in the direction along AB. When the distance between the two is minimum, the relative velocity of approach is zero.

$\therefore$ $V_A \cos\alpha_f + V_B \sin\alpha_f = 0$

(where α_f is the angle made by the line A'B' with the x-axis)

$20 \cos\alpha_f = -15 \sin\alpha_f$

$\therefore$ $\tan\alpha_f = -\frac{20}{15} = -\frac{4}{3}$



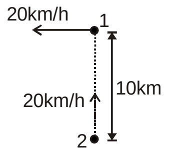
(Here do not confuse this angle with the angle θ in method (I) because that θ is the angle made by the net relative velocity with x-axis, but α_f is the angle made by the line joining the two particles with x-axis when velocity of approach in zero.)

$\therefore$ $\frac{400 - 15t}{500 - 20t} = -\frac{4}{3}$

$\therefore$ $t = \frac{128}{5}$ So, OB' = 16 m and OA' = – 12m

$A'B' = \sqrt{16^2 + (-12)^2}$ = 20 m

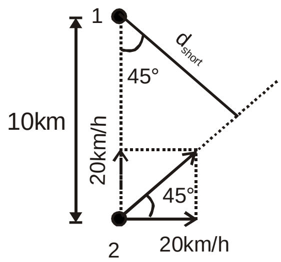
Example 29. Two ships are 10 km apart on a line joining south to north. The one farther north is steaming west at 20 km h^{-1}. The other is steaming north at 20 km h^{-1}. What is their distance of closest approach ? How long do they take to reach it ?



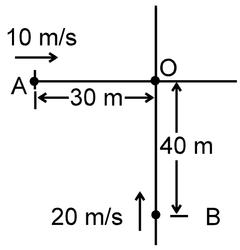
Solution : Solving from the frame of particle -1

we get $d_{short} = 10 \cos 45° = \frac{10}{\sqrt{2}} = 5\sqrt{2}$ km

$t = \frac{10 \sin 45}{|\vec{V}_{21}|} = \frac{10 \times 1/\sqrt{2}}{20\sqrt{2}} = \frac{1}{4}$ h = 15 min.

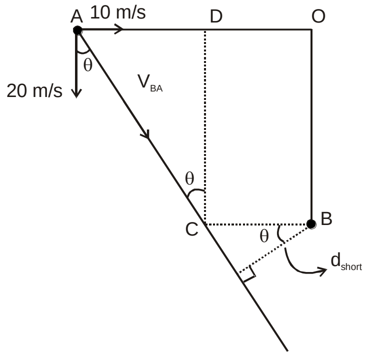


Example 30. Two particles A and B are moving with uniform velocity as shown in the figure given below at t = 0.



(i) Will the two particle collide

(ii) Find out shortest distance between two particles



Solution : Solving from the frame of B

we get $\tan\theta = \frac{10}{20} = \frac{1}{2}$

again $\tan\theta = \frac{AD}{CD} = \frac{AD}{40} = \frac{1}{2}$

$\Rightarrow$ AD = 20 $\Rightarrow$ DO = 10 $\Rightarrow$ BC = 10

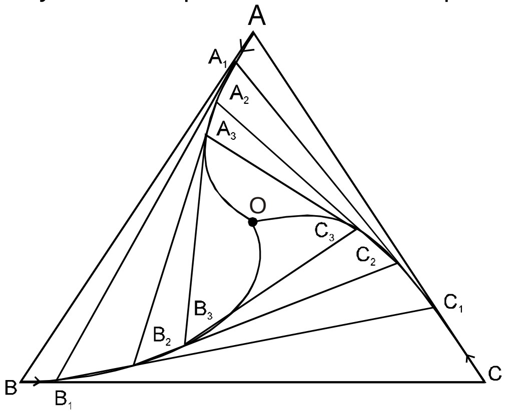
d_{short} = BC cosθ = 10 cosθ = $\frac{10 \times 2}{\sqrt{5}} = 4\sqrt{5}$ m

Since closest distance is non zero therefore they will not collide

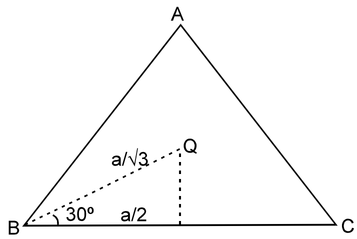
7.3 Miscellaneous Problems on collision

Example 31. There are particles A, B and C are situated at the vertices of an equilateral triangle ABC of side a at t = 0. Each of the particles moves with constant speed v. A always has its velocity along AB, B along BC and C along CA. At what time will the particle meet each other?

Solution : The motion of the particles is roughly sketched in figure. By symmetry they will meet at the centroid O of the triangle. At any instant the particles will from an equilateral triangle ABC with the same



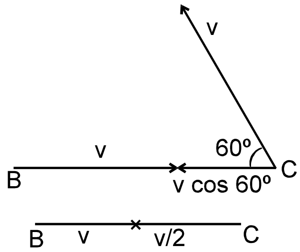
Centroid O. All the particles will meet at the centre. Concentrate on the motion of any one particle, say B. At any instant its velocity makes angle 30º with BO. The component of this velocity along BO is v cos 30º. This component is the rate of decrease of the distance BO. Initially



BO = $\frac{a/2}{\cos 30^o} = \frac{a}{\sqrt{3}}$ = displacement of each particle. Therefore, the time taken for BO to become zero

$= \frac{a/\sqrt{3}}{v\cos 30^o} = \frac{2a}{\sqrt{3}v \times \sqrt{3}} = \frac{2a}{3v}$.

Aliter : Velocity of B is v along BC. The velocity of C is along CA. Its component along BC is v cos 60º = v/2. Thus, the separation BC decreases at the rate of approach velocity.

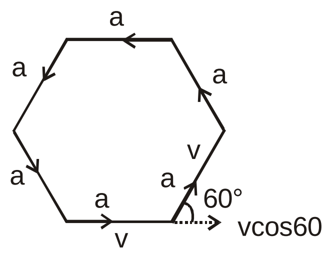


$\therefore$ approach velocity = $v + \frac{v}{2} = \frac{3v}{2}$

Since, the rate of approach is constant, the time taken in reducing the separation BC from a to zero is t = $\frac{a}{\frac{3v}{2}} = \frac{2a}{3v}$

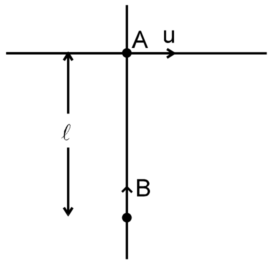
Example 32. Six particles situated at the corners of a regular hexagon of side a move at a constant speed v. Each particle maintains a direction towards the particle at the next corner. Calculate the time the particles will take to meet each other.

Solution : $V_{app} = V - V\cos 60° = V - V/2 = V/2$



$t = \frac{a}{V_{app}} = \frac{a}{V/2} = \frac{2a}{V}$

Example 33. 'A' moves with constant velocity u along the 'x' axis. B always has velocity towards A. After how much time will B meet A if B moves with constant speed v. What distance will be travelled by A and B.



Solution : Let at any instant the velocity of B makes an angle α with that of x axis and the time to collide is T.

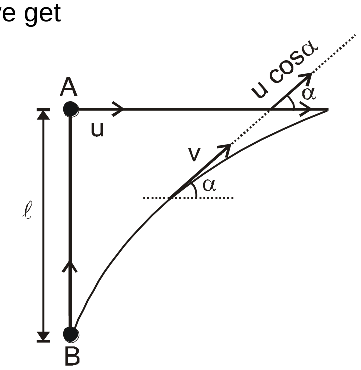
$v_{app} = v - u\cos\alpha$

$\ell = \int_0^T v_{app}\,dt = \int_0^T (v - u\cos\alpha)dt$ .... (1)

Now equating the displacement of A and B along x direction we get

$uT = \int v\cos\alpha\,dt$ .... (2)

Now from (1) and (2) we get



$\ell = vT - \int_0^T u\cos\alpha\,dt = vT - \frac{u}{v}\int_0^T v\cos\alpha\,dt = vT - \frac{u}{v} . uT$

$\Rightarrow T = \frac{\ell v}{v^2 - u^2}$

Now distance travelled by A and B

$= u \times \frac{\ell v}{v^2 - u^2}$ and $v \times \frac{\ell v}{v^2 - u^2} = \frac{uv\ell}{v^2 - u^2}$ and $\frac{v^2\ \ell}{v^2 - u^2}$