Fundamentals of Mathematics-I

SETS

A set is a collection of well defined objects which are distinct from each other. Sets are generally denoted by capital letters A, B, C, ... etc. and the elements of the set by small letters a, b, c .... etc.

If a is an element of a set A, then we write $a \in A$ and say a belongs to A.

If a does not belong to A then we write a A,

e.g. the collection of first five prime natural numbers is a set containing the elements 2, 3, 5, 7, 11.

METHODS TO WRITE A SET :

(i) **Roster Method or Tabular Method :** In this method a set is described by listing elements, separated by commas and enclose then by curly brackets. Note that while writing the set in roster form, an element is not generally repeated e.g. the set of letters of word SCHOOL may be written as {S, C, H, O, L}.

(ii) **Set builder form (Property Method) :** In this we write down a property or rule which gives us all the element of the set.

$A = \{x : P(x)\}$ where P(x) is the property by which $x \in A$ and colon (:) stands for 'such that'

Example # 1 : Express set $A = \{x : x \in N \text{ and } x = 2^n \text{ for } n \in N\}$ in roster form

Solution : $A = \{2, 4, 8, 16, ...\}$

Example # 2 : Express set $B = \{x^3 : x < 5, x \in W\}$ in roster form

Solution : $B = \{0, 1, 8, 27, 64\}$

Example # 3 : Express set A = {0, 7, 26, 63, 124} in set builder form

Solution : $A = \{x : x = n^3 - 1, n \in N, 1 \le n \le 5\}$

TYPES OF SETS

Null set or empty set : A set having no element in it is called an empty set or a null set or void set, it is denoted by ϕ or { }. A set consisting of at least one element is called a non-empty set or a non-void set.

Singleton set : A set consisting of a single element is called a singleton set.

Finite set : A set which has only finite number of elements is called a finite set.

Order of a finite set : The number of distinct elements in a finite set A is called the order of this set and denoted by O(A) or n(A). It is also called cardinal number of the set.

e.g. A = {a, b, c, d} $\Rightarrow$ n(A) = 4

Infinite set : A set which has an infinite number of elements is called an infinite set.

Equal sets : Two sets A and B are said to be equal if every element of A is member of B, and every element of B is a member of A. If sets A and B are equal, we write A = B and if A and B are not equal then

$A \neq B$

Equivalent sets : Two finite sets A and B are equivalent if their cardinal number is same i.e. n(A) = n(B)

e.g. A = {1, 3, 5, 7}, B = {a, b, c, d} $\Rightarrow$ n(A) = 4 and n(B) = 4

$\Rightarrow$ A and B are equivalent sets

Note - Equal sets are always equivalent but equivalent sets may not be equal

Example # 4 : Identify the type of set :

(i) $A = \{x \in W : 3 \le x < 10\}$ (ii) $A = \{\alpha, \beta, \gamma, \delta\}$
(iii) $A = \{1, 0, -1, -2, -3, ...\}$ (iv) $A = \{1, 8, -2, 6, 5\}$ and $B = \{1, 8, -2, 6, 5\}$
(v) A = {x : x is number of students in a class room}

Solution : (i) finite set (ii) finite set
(iii) infinite set (iv) equal sets
(v) singleton set

Self Practice Problem :

(1) Write the set of all integers 'x' such that $-2 < x - 4 < 5$.
(2) Write the set {1, 2, 5, 10} in set builder form.
(3) If $A = \{x : x^2 < 9, x \in Z\}$ and $B = \{-2, -1, 1, 2\}$ then find whether sets A and B are equal or not.

Answers (1) {3, 4, 5, 6, 7, 8}
(2) {x : x is a natural number and a divisor of 10}
(3) Not equal sets

SUBSET AND SUPERSET :

Let A and B be two sets. If every element of A is an element of B then A is called a subset of B and B is called superset of A. We write it as $A \subset B$.

e.g. $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\}$ $\Rightarrow$ $A \subset B$

If A is not a subset of B then we write $A \not\subset B$

PROPER SUBSET :

If A is a subset of B but $A \neq B$ then A is a proper subset of B. Set A is not proper subset of A so this is improper subset of A

Note : (i) Every set is a subset of itself
(ii) Empty set ϕ is a subset of every set
(iii) $A \subset B$ and $B \subset A \Leftrightarrow A = B$
(iv) The total number of subsets of a finite set containing n elements is 2^n.
(v) Number of proper subsets of a set having n elements is $2^n - 1$.
(vi) Empty set ϕ is proper subset of every set except itself.

POWER SET :

Let A be any set. The set of all subsets of A is called power set of A and is denoted by P(A)

Example # 5 : Examine whether the following statements are true or false :
(i) $\{a\} \not\subset \{b, c, a\}$
(ii) $\{x, p\} \not\subset$ {x : x is a consonant in the English alphabet}
(iii) $\{\alpha, \beta, \gamma, \delta\} \subset \{\alpha, \beta, \phi, \psi\}$
(iv) $\{a, b\} \in \{a, \{a\}, b, c\}$

Solution : (i) False as {a} is subset of {b, c, a}
(ii) False as x, p are consonant
(iii) False as element γ, δ is not in the set $\{\alpha, \beta, \phi, \psi\}$
(iv) False as $a, b \in \{a, \{a\}, b, c\}$ and $\{a, b\} \subset \{a, \{a\}, b, c\}$

Example # 6 : Find power set of set A = {1, 2, 3}

Solution : $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Example # 7 : If ϕ denotes null set then find

	(a)	$P(\phi)$	(b)	$P(P(\phi))$
	(c)	$n(P(P(P(\phi))))$	(d)	$n(P(P(P(P(\phi)))))$
Solution :	(a)	$P(\phi) = \{\phi\}$	(b)	$P(P(\phi)) = \{\phi, \{\phi\}\}$
	(c)	$n(P(P(P(\phi)))) = 2^2 = 4$	(d)	$n(P(P(P(P(\phi))))) = 2^4 = 16$

Self Practice Problem :

(4) State true/false : $A = \{p, q, r, s\}$, $B = \{p, q, r, p, t\}$ then $A \subset B$.

(5) State true/false : $A = \{p, q, r, s\}$, $B = \{s, r, q, p\}$ then $A \subset B$.

(6) State true/false : $[4, 15) \subset [-15, 15]$

Answers (4) False (5) True (6) True

UNIVERSAL SET :

A set consisting of all possible elements which occur in the discussion is called a universal set and is denoted by U.

e.g. if $A = \{1, 2, 3\}$, $B = \{2, 4, 5, 6\}$, $C = \{1, 3, 5, 7\}$ then $U = \{1, 2, 3, 4, 5, 6, 7\}$ can be taken as the universal set.

SOME OPERATION ON SETS :

(i) **Union of two sets :** $A \cup B = \{x : x \in A \text{ or } x \in B\}$
e.g. $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ then $A \cup B = \{1, 2, 3, 4\}$

(ii) **Intersection of two sets :** $A \cap B = \{x : x \in A \text{ and } x \in B\}$
e.g. $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ then $A \cap B = \{2, 3\}$

(iii) **Difference of two sets :** $A - B = \{x : x \in A \text{ and } x \notin B\}$. It is also written as $A \cap B'$.
Similarly $B - A = B \cap A'$ e.g. $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$; $A - B = \{1\}$

(iv) **Symmetric difference of sets :** It is denoted by $A \Delta B$ and $A \Delta B = (A - B) \cup (B - A)$

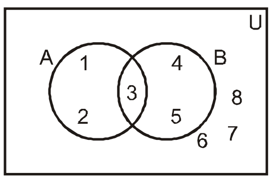
(v) **Complement of a set :** $A' = \{x : x \notin A \text{ but } x \in U\} = U - A$
e.g. $U = \{1, 2, ..., 10\}$, $A = \{1, 2, 3, 4, 5\}$ then $A' = \{6, 7, 8, 9, 10\}$

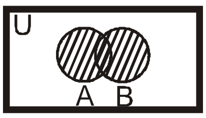
(vi) **Disjoint sets :** If $A \cap B = \phi$, then A, B are disjoint sets.
e.g. If $A = \{1, 2, 3\}$, $B = \{7, 8, 9\}$ then $A \cap B = \phi$

VENN DIAGRAM :

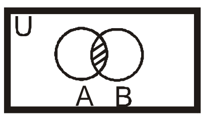
Most of the relationships between sets can be represented by means of diagrams which are known as venn diagrams. These diagrams consist of a rectangle for universal set and circles in the rectangle for subsets of universal set. The elements of the sets are written in respective circles.

For example If $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ then their venn diagram is

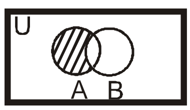




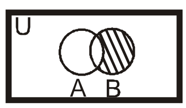
$A \cup B$



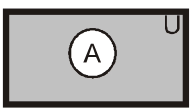
$A \cap B$



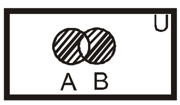
$A - B$



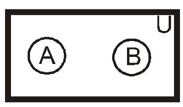
$B - A$



A'



$(A \Delta B) = (A - B) \cup (B - A)$



Disjoint

LAWS OF ALGEBRA OF SETS (PROPERTIES OF SETS):

(i) **Commutative law :** $(A \cup B) = B \cup A$; $A \cap B = B \cap A$

(ii) **Associative law :** $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap C = A \cap (B \cap C)$

(iii) **Distributive law :** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(iv) **De-morgan law :** $(A \cup B)' = A' \cap B'$; $(A \cap B)' = A' \cup B'$

(v) **Identity law :** $A \cap U = A$; $A \cup \phi = A$

(vi) **Complement law :** $A \cup A' = U$, $A \cap A' = \phi$, $(A')' = A$

(vii) **Idempotent law :** $A \cap A = A$, $A \cup A = A$

NOTE :

(i) $A - (B \cup C) = (A - B) \cap (A - C)$; $A - (B \cap C) = (A - B) \cup (A - C)$

(ii) $A \cap \phi = \phi$, $A \cup U = U$

Example # 8 : Let A = {1, 2, 3, 4, 5, 6} and B = {4, 5, 6, 7, 8, 9} then find $A \cup B$

Solution : $A \cup B$ = {1, 2, 3, 4, 5, 6, 7, 8, 9}

Example # 9 : Let A = {1, 2, 3, 4, 5, 6}, B = {4, 5, 6, 7, 8, 9}. Find A – B and B – A.

Solution : A – B = {x : x $\in$ A and x B} = {1, 2, 3}

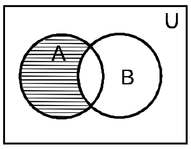
similarly B – A = {7, 8, 9}

Example # 10 : State true or false :

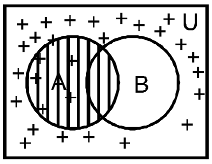
(i) $A \cup A' = A$ (ii) $U \cap A = A$

Solution : (i) false because $A \cup A' = U$ (ii) true as $U \cap A = A$

Example # 11 : Use Venn diagram to prove that $A - B = A \cap B'$.



A – B



= A, = B'

= A∩ B'

Solution :

From venn diagram we can conclude that $A - B = A \cap B'$.

Self Practice Problem :

(7) Find $A \cup B$ if A = {x : x = 2n + 1, n ≤ 5, n $\in$ N} and B = {x : x = 3n – 2, n ≤ 4, n $\in$ N}.

(8) Find A – (A – B) if A = {5, 9, 13, 17, 21} and B = {3, 6, 9, 12, 15, 18, 21, 24}

Answers (7) {1, 3, 4, 5, 7, 9, 10, 11} (8) {9, 21}

SOME IMPORTANT RESULTS ON NUMBER OF ELEMENTS IN SETS :

If A, B, C are finite sets and U be the finite universal set then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A - B) = n(A) - n(A \cap B)$

(iii) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(iv) Number of elements in exactly two of the sets A, B, C

$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

(v) Number of elements in exactly one of the sets A, B, C

$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

Example # 12 : In a group of 60 students, 36 read English newspaper, 22 read Hindi newspaper and 12 read neither of the two. How many read both English & Hindi news papers ?

Solution : $n(U) = 60$, $n(E) = 36$, $n(H) = 22$

$n(E' \cap H') = 12 \Rightarrow n(E \cup H)' = 12$

$\Rightarrow$ $n(U) - n(E \cup H) = 12$

$\Rightarrow$ $n(E \cup H) = 48$

$\Rightarrow$ $n(E) + n(H) - n(E \cap H) = 48$

$\Rightarrow$ $n(E \cap H) = 58 - 48 = 10$

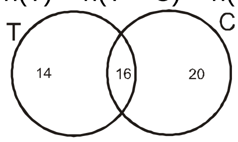
Example#13 : In a group of 50 persons, 14 drink tea but not coffee and 30 drink tea. Find

(i) How many drink tea and coffee both ? (ii) How many drink coffee but not tea ?

Solution : T : people drinking tea

C : people drinking coffee

(i) $n(T) = n(T - C) + n(T \cap C) \Rightarrow 30 = 14 + n(T \cap C) \Rightarrow n(T \cap C) = 16$



(ii) $n(C - T) = n(T \cup C) - n(T) = 50 - 30 = 20$

Self Practice Problem :

(9) Let A and B be two finite sets such that $n(A - B) = 15$, $n(A \cup B) = 90$, $n(A \cap B) = 30$. Find $n(B)$

(10) A market research group conducted a survey of 1000 consumers and reported that 720 consumers liked product A and 450 consumers liked product B. What is the least number that must have liked both products ?

Answers (9) 75 (10) 170

Intervals :

Intervals are basically subsets of R and are commonly used in solving inequalities or in finding domains. If there are two numbers $a, b \in R$ such that $a < b$, we can define four types of intervals as follows :

Name	**Representation**	**Discription**
Open Interval	(a, b)	$\{x : a < x < b\}$ i.e. end points are not included.
Close Interval	[a, b]	$\{x : a \le x \le b\}$ i.e. end points are also included. This is possible only when both a and b are finite.
Open - Closed Interval	(a, b]	$\{x : a < x \le b\}$ i.e. a is excluded and b is included.
Close - Open Interval	[a, b)	$\{x : a \le x < b\}$ i.e. a is included and b is excluded.

Note : (1) The infinite intervals are defined as follows :

(i) $(a, \infty) = \{x : x > a\}$ (ii) $[a, \infty) = \{x : x \ge a\}$

(iii) $(-\infty, b) = \{x : x < b\}$ (iv) $(-\infty, b] = \{x : x \le b\}$

(v) $(-\infty, \infty) = \{x : x \in R\}$

(2) $x \in \{1, 2\}$ denotes some particular values of x, i.e. $x = 1, 2$

(3) If there is no value of x, then we say $x \in \phi$ (null set)

General Method to solve Inequalities :

(Method of intervals (Wavy curve method)

Let $g(x) = \left(\dfrac{(x-b_1)^{k_1}(x-b_2)^{k_2} --- (x-b_n)^{k_n}}{(x-a_1)^{r_1}(x-a_2)^{r_2} --- (x-a_n)^{r_n}}\right)$ ... (i)

Where $k_1, k_2 .....k_n$ and $r_1, r_2 ....r_n \in N$ and $b_1, b_2.....b_n$ and $a_1, a_2 ......a_n$ are real numbers.

Then to solve the inequality following steps are taken.

Steps : -

Points where numerator becomes zero are called zeros or roots of the function and where denominator becomes zero are called poles of the function.

(i) First we find the zeros and poles of the function.

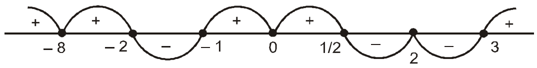
(ii) Then we mark all the zeros and poles on the real line and put a vertical bar there dividing the real line in many intervals.

(iii) Determine sign of the function in any of the interval and then alternates the sign in the neghbouring interval if the poles or zeros dividing the two interval has appeared odd number of times otherwise retain the sign.

(iv) Thus we consider all the intervals. The solution of the g(x) > 0 is the union of the intervals in which we have put the plus sign and the solution of g(x) < 0 is the union of all intervals in whichwe have put the minus sign.

Example# 14 : Solve the inequality if $f(x) = \dfrac{(x-2)^{10}(x+1)^3\left(x-\dfrac{1}{2}\right)^5(x+8)^2}{x^{24}(x-3)^3(x+2)^5}$ is > 0 or < 0.

Solution. Let $f(x) = \dfrac{(x-2)^{10}(x+1)^3\left(x-\dfrac{1}{2}\right)^5(x+8)^2}{x^{24}(x-3)^3(x+2)^5}$ the poles and zeros are 0, 3,– 2,– 1, $\dfrac{1}{2}$,– 8, 2



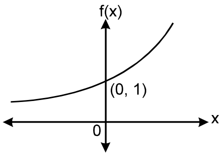
If f(x) > 0, then $x \in (-\infty, -8) \cup (-8, -2) \cup (-1, 0) \cup \left(0, \dfrac{1}{2}\right) \cup (3, \infty)$

and if f(x) < 0, then $x \in (-2, -1) \cup \left(\dfrac{1}{2}, 2\right) \cup (2, 3)$ **Ans.**

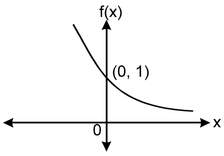
Exponential Function

A function $f(x) = a^x = e^{x \ln a}$ ($a > 0$, $a \neq 1$, $x \in R$) is called an exponential function. Graph of exponential function can be as follows :

Case - I
For a > 1



Case - II
For 0 < a 1



Logarithm of A Number :

The logarithm of the number N to the base 'a' is the exponent indicating the power to which the base 'a' must be raised to obtain the number N. This number is designated as $\log_a N$. Hence:

$\log_a N = x \Leftrightarrow a^x = N$, a > 0, a ≠ 1 & N > 0

If a = 10, then we write log b rather than $\log_{10} b$.
If a = e, we write *l*n b rather than $\log_e b$. Here 'e' is called as Napier's base & has numerical value equal to 2.7182.

Remember

$\log_{10} 2 \simeq 0.3010$; $\log_{10} 3 \simeq 0.4771$
$\ell n\, 2 \simeq 0.693$; $\ell n\, 10 \simeq 2.303$

Domain of Definition :

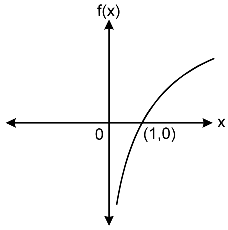
The existence and uniqueness of the number $\log_a N$ can be determined with the help of set of conditions, $a > 0$ & $a \neq 1$ & $N > 0$.

The base of the logarithm 'a' must not equal unity otherwise numbers not equal to unity will not have a logarithm and any number will be the logarithm of unity.

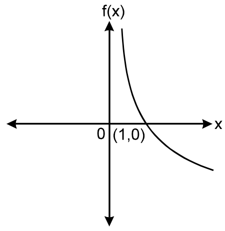
Graph of Logarithmic function :

$f(x) = \log_a x$ is called logarithmic function where $a > 0$ and $a \neq 1$ and $x > 0$. Its graph can be as follows:

Case- I
For $a > 1$



Case- II
For $0 < a < 1$



Fundamental Logarithmic Identity :

$a^{\log_a N} = N$, $a > 0$, $a \neq 1$ & $N > 0$

The Principal Properties of Logarithm:

Let M & N are arbitrary positive numbers, $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$ and α, β are any real numbers, then :

(i) $\log_a (M.N) = \log_a M + \log_a N$; in general $\log_a (x_1\ x_2\ ......x_n) = \log_a x_1 + \log_a x_2 + ...+ \log_a x_n$

(ii) $\log_a (M/N) = \log_a M - \log_a N$

(iii) $\log_a M^\alpha = \alpha . \log_a M$

(iv) $\log_{a^\beta} M = \dfrac{1}{\beta} \log_a M$

(v) $\log_b M = \dfrac{\log_a M}{\log_a b}$ (base changing theorem)

NOTE :

- $\log_a 1 = 0$
- $\log_a a = 1$
- $\log_{1/a} a = -1$
- $\log_b a = \dfrac{1}{\log_a b}$
- $a^x = e^{x \ell n\, a}$
- $a^{\log_c b} = b^{\log_c a}$

Note : (i) If the number and the base are on the same side of the unity, then the logarithm is positive.

(ii) If the number and the base are on the opposite sides of unity, then the logarithm is negative.

Example#15: Find the value of the followings :

(i) $\log_2 72 + \log_2\left(\dfrac{32}{81}\right) + \log_2\left(\dfrac{9}{64}\right)$ **Ans.** 2

(ii) $7^{\frac{1}{\log_{25} 49}}$ **Ans.** 5

Solution. (i) $\log_2 72 + ...$

$= \log_2\left\{2^3.3^2.\frac{2^5}{3^4}.\frac{3^2}{2^6}\right\} = \log_2 4 = 2$

(ii) $7^{\frac{1}{\log_{25} 49}} = 7^{\log_{49} 25} = 7^{\frac{2}{2}\log_7 5} = 5^{\log_7 7} = 5$

Self practice problem :

(11) Find the value of the followings :

(i) $\log_{49} 343$ (ii) $4\log_{27} 243$

(iii) $\log_{(1/100)} 1000$ (iv) $\log_{(7-4\sqrt{3})}(7+4\sqrt{3})$

(v) $\log_{125} 625$

(12) $\log_8 9.\log_9 10 .... \log_{63} 64$

(13) Find the value of $\log \cot 1° + \log \cot 2° + \log \cot 3° + ... + \log \cot 89°$

Ans. (11) (i) 3/2 (ii) 20/3 (iii) – 3/2 (iv) –1 (v) 4/3

(12) 2 (13) 0

Logarithmic Equation :

The equality $\log_a x = \log_a y$ is possible if and only if $x = y$ i.e.

$\boldsymbol{\log_a x = \log_a y \Leftrightarrow x = y}$

Always check validity of given equation, $(x > 0, y > 0, a > 0, a \neq 1)$

Example#16 : $\log_x(4x - 3) = 2$ **Ans.** $x = 3$

Solution. Domain : $x > 0, 4x - 3 > 0, x \neq 1$

Hence $4x - 3 = x^2 \Rightarrow x^2 - 4x + 3 = 0$

$x = 3$ or $x = 1$ (rejected as not in domain)

Exmaple#17 : $\log_2 (\log_3\{\log_5 (x^2 + 4)\} = 0$ **Ans.** $x = \pm 11$

Solution. $\log_3(\log_5(x^2 + 4)\} = 2° = 1$

$\Rightarrow \log_5(x^2 + 4) = 3^1 = 3$

$\Rightarrow (x^2 + 4) = 5^3 = 125 \Rightarrow x^2 = 121 \Rightarrow x = \pm 11$

Example#18 : $\log_2(x^2) + \log_2 (x + 2) = 4$ **Ans.** $x = 2$

Solution. $\log_2 (x^2(x + 2) = 4 \Rightarrow x^3 + 2x^2 - 16 = 0 \Rightarrow (x - 2)\underbrace{(x^2 + 4x + 8)}_{D<0} = 0$

$x = 2$

Self practice problem

(14) $3^{3\log_3 x} = 27$ (15) $(\log_{10} x)^2 - (\log_{10} x) - 6 = 0$

(16) $3(\log_7 x + \log_x 7) = 10$ (17) $(x+2)^{\log_2(x+2)} = 8(x+2)^2$

Ans. (14) $x = 3$ (15) $x = 10^3, \frac{1}{10^2}$ (16) $x = 343, \sqrt[3]{7}$ (17) $x = 6$ or $-3/2$

Logarithmic Inequality :

Let 'a' is a real number such that

(i) If $a > 1$, then $\log_a x > \log_a y \Rightarrow x > y$

(ii) If $a > 1$, then $\log_a x < \alpha \Rightarrow 0 < x < a^\alpha$

(iii) If $a > 1$, then $\log_a x > \alpha \Rightarrow x > a^\alpha$

(iv) If $0 < a < 1$, then $\log_a x > \log_a y \Rightarrow 0 < x < y$

(v) If $0 < a < 1$, then $\log_a x < \alpha \Rightarrow x > a^\alpha$

Form - I : $f(x) > 0$, $g(x) > 0$, $g(x) \neq 1$

	Form		**Collection of system**
(a)	$\log_{g(x)} f(x) \geq 0$	$\Leftrightarrow$	$\begin{cases} f(x) \geq 1 \quad , \quad g(x) > 1 \\ 0 < f(x) \leq 1 \quad , \quad 0 < g(x) < 1 \end{cases}$
(b)	$\log_{g(x)} f(x) \leq 0$	$\Leftrightarrow$	$\begin{cases} f(x) \geq 1 \quad , \quad 0 < g(x) < 1 \\ 0 < f(x) \leq 1 \quad , \quad g(x) > 1 \end{cases}$
(c)	$\log_{g(x)} f(x) \geq a$	$\Leftrightarrow$	$\begin{cases} f(x) \geq (g(x))^a \quad , \quad g(x) > 1 \\ 0 < f(x) \leq (g(x))^a \quad , \quad 0 < g(x) < 1 \end{cases}$
(d)	$\log_{g(x)} f(x) \leq a$	$\Leftrightarrow$	$\begin{cases} 0 < f(x) \leq (g(x))^a \quad , \quad g(x) > 1 \\ f(x) \geq (g(x))^a \quad , \quad 0 < g(x) < 1 \end{cases}$

From - II : When the inequality of the form

	Form		**Collection of system**
(a)	$\log_{\phi(x)} f(x) \geq \log_{\phi(x)} g(x)$	$\Leftrightarrow$	$\begin{cases} f(x) \geq g(x),\ \phi(x) > 1, \\ 0 < f(x) \leq g(x); 0 < \phi(x) < 1 \end{cases}$
(b)	$\log_{\phi(x)} f(x) \leq \log_{\phi(x)} g(x)$	$\Leftrightarrow$	$\begin{cases} 0 < f(x) \leq g(x), \phi(x) > 1, \\ f(x) \geq g(x) > 0,\ 0 < \phi(x) < 1 \end{cases}$

Example # 19 : Solve the logarithmic inequality $\log_{1/5} (2x^2 + 7x + 7) \geq 0$.

Solution. Since $\log_{1/5} 1 = 0$, the given inequality can be written as.

$\log_{1/5} (2x^2 + 7x + 7) \geq \log_{1/5} 1$

when the domain of the function is taken into account the inequality is equivalent to the system of inequalities $\begin{cases} 2x^2 + 7x + 7 > 0 \\ 2x^2 + 7x + 7 \leq 1 \end{cases}$

Solving the inequalities by using method of intervals $x \in \left[-2, \frac{-3}{2}\right]$

Example # 20 : Solve the inequality $\log_{1/3} (5x - 1) > 0$.

Solution. by using the basic property of logarithm.

$$\begin{cases} 5x - 1 < 1 \\ 5x - 1 > 0 \end{cases} \Rightarrow \begin{cases} 5x < 2 \\ 5x > 1 \end{cases} \Rightarrow \begin{matrix} x < \frac{2}{5} \\ x > \frac{1}{5} \end{matrix}$$

$\Rightarrow$ The solution of the inequality is given by $\left(\frac{1}{5}, \frac{2}{5}\right)$ Ans.

Example # 21 : Solve the inequality $\log_{(2x + 3)} x^2 < \log_{(2x + 3)} (2x + 3)$.

Solution. The given inequality is equivalent to the collection of the systems

$$\left[\begin{array}{l}\begin{cases}0 < 2x+3 < 1 & \text{(i)}\\ x^2 > 2x+3\end{cases}\\ \begin{cases}2x+3 > 1 & \text{(ii)}\\ 0 < x^2 < 2x+3\end{cases}\end{array}\right.$$

Solving system (i) we obtain

$$\begin{cases}\frac{-3}{2} < x < -1\\ (x-3)(x+1) > 0 \quad \text{(iii)}\end{cases}$$

System (iii) is equivalent to the collection of two systems

$$\left[\begin{array}{l}\frac{-3}{2} < x < -1, \quad x > 3; \quad \text{(iv)}\\ \frac{-3}{2} < x < -1, \quad x < -1 \quad \text{(v)}\end{array}\right.$$

system (iv) has no solution. The solution of system (v) is $x \in \left(\frac{-3}{2}, -1\right)$,

solving system (ii) we obtain.

$$\begin{cases}x > -1\\ (x-3)(x+1) < 0\end{cases} \quad \text{or} \quad \begin{cases}x > -1\\ -1 < x < 3\end{cases} \quad \Rightarrow \quad x \in (-1, 3)$$

$$x \in \left(\frac{-3}{2}, -1\right) \cup (-1, 3)$$

Example # 22 : Solve the in equation $\log_{\left(\frac{x^2-12x+30}{10}\right)}\left(\log_2 \frac{2x}{5}\right) > 0$.

Solution. This in equation is equivalent to the collection of following systems.

$$\begin{cases}\frac{x^2-12x+30}{10} > 1,\\ \log_2\left(\frac{2x}{5}\right) > 1,\end{cases} \quad \text{and} \quad \begin{cases}0 < \frac{x^2-12x+30}{10} < 1,\\ 0 < \log_2\left(\frac{2x}{5}\right) < 1\end{cases}$$

Solving the first system we have.

$$\begin{cases}x^2-12x+20 > 0\\ \frac{2x}{5} > 2\end{cases} \Leftrightarrow \begin{cases}(x-10)(x-2) > 0\\ x > 5\end{cases} \Leftrightarrow \begin{cases}x < 2 \text{ or } x > 10\\ x > 5\end{cases}$$

Therefore the system has solution $x > 10$
Solving the second system we have.

$$\Rightarrow \begin{cases}0 < x^2-12x+30 < 10\\ 1 < \frac{2x}{5} < 2\end{cases} \quad \Rightarrow \begin{cases}x^2-12x+30 > 0 \text{ and } x^2-12x+20 < 0\\ \frac{5}{2} < x < 5\end{cases}$$

$$\Rightarrow \begin{cases}x < 6-\sqrt{6} \text{ or } x > 6+\sqrt{6} \text{ and } 2 < x < 10\\ \frac{5}{2} < x < 5\end{cases}$$

$\therefore$ The system has solutions $\frac{5}{2} < x < 6-\sqrt{6}$ combining both systems, then solution of the original inequation is.

$x \in \left(\frac{5}{2}, 6-\sqrt{6}\right) \cup (10, \infty)$ **Ans.**

Self practice problems :

(18) Solve the following inequalities

(i) $\log_{3x+5}(9x^2 + 8x + 8) > 2$

(ii) $\log_{0.2}(x^2 - x - 2) > \log_{0.2}(-x^2 + 2x + 3)$

(iii) $\log_x (x^3 - x^2 - 2x) < 3$

Answers : (18) (i) $\left(-\frac{4}{3}, -\frac{17}{22}\right)$ (ii) $\left(2, \frac{5}{2}\right)$ (iii) $(2, \infty)$

Characteristic & Mantissa

$[\log_a N]$ is called characteristic of log of N with base 'a'. It is always an integer.

$\{\log_a N\}$ is called mantissa of log of N with base 'a'. Mantissa $\in [0, 1)$

Characteristic of log of 1 with base 10 = 0

characteristic of log of 10 with base 10 = 1

characteristic of log of 100 with base 10 = 2

characteristic of log of 1000 with base 10 = 3

characteristic of log of 83.5609 with base 10 = 1

characteristic of log of 613.0965 with base 10 = 2

Interval,	**Cha.(Base 10)**	**number of digits in no**	**No. of integers in the interval**
[1, 10)	0	1	$9 = 9 \times 10^0$
[10, 100)	1	2	$90 = 9 \times 10^1$
[100, 1000)	2	3	$900 = 9 \times 10^2$
[100, 10000)	3	4	$9000 = 9 \times 10^3$
	|		
	|		
	|		
	n	(n + 1)	9×10^n

Note :

If characteristic of a number (base of log is 10) is found to be n, then there would be (n + 1) digits in that number.

* Characteristic of log of $\frac{1}{10} = 0.1$ with base 10 = – 1

Characteristic of log of $\frac{1}{100} = 0.01$ with base 10 = – 2

Characteristic of log of $\frac{1}{1000} = 0.001$ with base 10 = – 3

Characteristic of log of $\frac{3}{100}$ with base 10 = – 2

Characteristic of log of $\frac{3}{1000}$ with base 10 = – 3

Interval	**Characteristic (base 10)**	**No. of zeros immediately after decimal**	**No.ofinteger resiprocal of which lies in interval.**
[1/10, 1)	–1	0	$9 = 9\times10^{1-1}$
[1/100, 1/10) ≡ [0.01, 0.1)	–2	1	$90 = 9\times10^{2-1}$
$[1/10^3, 1/10^2)$ ≡ [0.0001, 0.01)	–3	2	$900 = 9\times10^{3-1}$
[0.0001, 0.001)	–4	3	$9000 = 9\times10^{4-1}$
	|		
	|		
	|		
	|		
	– n	(n – 1)	$= 9 \times 10^{n-1}$

Note :

If characteristic of a number (base of log is 10) is found to be –n, then there would be (n –1) zeros immediately after decimal before first significant digit.

Example # 23 Find the total number of digits in the number 18^{50}.

(Given $\log_{10}2 = 0.3010$; $\log_{10}3 = 0.4771$) **Ans.** 63

Solution. $N = 18^{50}$

$\log_{10}N = 50 \log_{10}18 = 50\ (0.3010 + 0.9542) = 50(1.2552) = 62.76$

Characterstic = $[\log_{10}N] = 62$

No. of digits = 62 + 1 = 63

Self practice problem

(19) Find the total number of zeros immediately after the decimal in 6^{-200} .

Ans. (19) 155