FUNDAMENTALS OF MATHEMATICS

SETS

A set is a collection of well defined objects which are distinct from each other.

METHODS TO WRITE A SET :

(i) **Roster Method or Tabular Method :** In this method a set is described by listing elements, separated by commas and enclose then by curly brackets.

(ii) **Set builder form (Property Method) :** In this we write down a property or rule which gives us all the element of the set.

TYPES OF SETS

Null set or empty set : A set having no element in it is called an empty set or a null set or void set, it is denoted by ϕ or { }.

Singleton set : A set consisting of a single element is called a singleton set.

Finite set : A set which has only finite number of elements is called a finite set.

Order of a finite set : The number of distinct elements in a finite set A is called the order of this set and denoted by O(A) or n(A). It is also called cardinal number of the set.
e.g. $A = \{a, b, c, d\} \Rightarrow n(A) = 4$
Infinite set : A set which has an infinite number of elements is called an infinite set.
Equal sets : Two sets A and B are said to be equal if every element of A is member of B, and every element of B is a member of A. If sets A and B are equal, we write A = B and if A and B are not equal then $A \neq B$
Equivalent sets : Two finite sets A and B are equivalent if their cardinal number is same i.e. n(A) = n(B)
e.g. $A = \{1, 3, 5, 7\}, B = \{a, b, c, d\} \Rightarrow n(A) = 4$ and $n(B) = 4$
$\Rightarrow$ A and B are equivalent sets

Note - Equal sets are always equivalent but equivalent sets may not be equal

SUBSET AND SUPERSET :

Let A and B be two sets. If every element of A is an element of B then A is called a subset of B and B is called superset of A. We write it as $A \subset B$.
e.g. $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\} \Rightarrow A \subset B$
If A is not a subset of B then we write $A \not\subset B$

PROPER SUBSET :

If A is a subset of B but $A \neq B$ then A is a proper subset of B. Set A is not proper subset of A so this is improper subset of A

Note : (i) The total number of subsets of a finite set containing n elements is 2^n.
(ii) Number of proper subsets of a set having n elements is $2^n - 1$.

POWER SET :

Let A be any set. The set of all subsets of A is called power set of A and is denoted by P(A)

UNIVERSAL SET :

A set consisting of all possible elements which occur in the discussion is called a universal set and is denoted by U.
e.g. if $A = \{1, 2, 3\}, B = \{2, 4, 5, 6\}, C = \{1, 3, 5, 7\}$ then $U = \{1, 2, 3, 4, 5, 6, 7\}$ can be taken as the universal set.

SOME OPERATION ON SETS :

(i) **Union of two sets :** $A \cup B = \{x : x \in A \text{ or } x \in B\}$
e.g. $A = \{1, 2, 3\}, B = \{2, 3, 4\}$ then $A \cup B = \{1, 2, 3, 4\}$

(ii) **Intersection of two sets :** $A \cap B = \{x : x \in A \text{ and } x \in B\}$
e.g. $A = \{1, 2, 3\}, B = \{2, 3, 4\}$ then $A \cap B = \{2, 3\}$

(iii) **Difference of two sets :** $A - B = \{x : x \in A \text{ and } x \notin B\}$. It is also written as $A \cap B'$.
Similarly $B - A = B \cap A'$ e.g. $A = \{1, 2, 3\}, B = \{2, 3, 4\}$; $A - B = \{1\}$

(iv) **Symmetric difference of sets :** It is denoted by $A \Delta B$ and $A \Delta B = (A - B) \cup (B - A)$

(v) **Complement of a set :** $A' = \{x : x \notin A \text{ but } x \in U\} = U - A$

e.g. $U = \{1, 2, ..., 10\}$, $A = \{1, 2, 3, 4, 5\}$ then $A' = \{6, 7, 8, 9, 10\}$

LAWS OF ALGEBRA OF SETS (PROPERTIES OF SETS):

(i) **Commutative law :** $(A \cup B) = B \cup A$; $A \cap B = B \cap A$

(ii) **Associative law :** $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap C = A \cap (B \cap C)$

(iii) **Distributive law :** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(iv) **De-morgan law :** $(A \cup B)' = A' \cap B'$; $(A \cap B)' = A' \cup B'$

(v) **Identity law :** $A \cap U = A$; $A \cup \phi = A$

(vi) **Complement law :** $A \cup A' = U$, $A \cap A' = \phi$, $(A')' = A$

(vii) **Idempotent law :** $A \cap A = A$, $A \cup A = A$

SOME IMPORTANT RESULTS ON NUMBER OF ELEMENTS IN SETS :

If A, B, C are finite sets and U be the finite universal set then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A - B) = n(A) - n(A \cap B)$

(iii) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(iv) Number of elements in exactly two of the sets A, B, C

$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

(v) Number of elements in exactly one of the sets A, B, C

$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

Intervals :

Intervals are basically subsets of R and are commonly used in solving inequalities or in finding domains. If there are two numbers a, $b \in R$ such that $a < b$, we can define four types of intervals as follows :

Name	Representation	Discription
Open Interval	(a, b)	$\{x : a < x < b\}$ i.e. end points are not included.
Close Interval	[a, b]	$\{x : a \le x \le b\}$ i.e. end points are also included. This is possible only when both a and b are finite.
Open - Closed Interval	(a, b]	$\{x : a < x \le b\}$ i.e. a is excluded and b is included.
Close - Open Interval	[a, b)	$\{x : a \le x < b\}$ i.e. a is included and b is excluded.

Note : (i) $(a, \infty) = \{x : x > a\}$ (ii) $[a, \infty) = \{x : x \ge a\}$ (iii) $(-\infty, b) = \{x : x < b\}$

(iv) $(-\infty, b] = \{x : x \le b\}$ (v) $(-\infty, \infty) = \{x : x \in R\}$

Graph of polynomial

To plot a graph of polynomial, several sets of Points (x, y) are required.

The key points are (i) stationary points (where $\frac{dy}{dx} = 0$)

(ii) y-intercept (where x is zero)

(iii) x-intercept (where y is zero)

and (iv) behaviour of polynomial at x tends to $\pm \infty$

Logarithm of A Number :

The logarithm of the number N to the base 'a' is the exponent indicating the power to which the base 'a' must be raised to obtain the number N. This number is designated as $\log_a N$. Hence:

$\mathbf{\log_a N = x \Leftrightarrow a^x = N\ ,\ a > 0,\ a \ne 1\ \&\ N > 0}$

Domain of Definition :

The existence and uniqueness of the number $\log_a N$ can be determined with the help of set of conditions, $a > 0$ & $a \neq 1$ & $N > 0$.

The Principal Properties of Logarithm:

Let M & N are arbitrary positive numbers, $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$ and α, β are any real numbers, then :

(i) $\log_a (M.N) = \log_a M + \log_a N$; in general $\log_a (x_1 x_2 ......x_n) = \log_a x_1 + \log_a x_2 + ...+ \log_a x_n$

(ii) $\log_a (M/N) = \log_a M - \log_a N$

(iii) $\log_a M^\alpha = \alpha. \log_a M$

(iv) $\log_{a^\beta} M = \dfrac{1}{\beta} \log_a M$

(v) $\log_b M = \dfrac{\log_a M}{\log_a b}$ (base changing theorem)

(vi) $\log_b a = \dfrac{1}{\log_a b}$

(vii) $a^x = e^{x \ell n a}$

(viii) $a^{\log_c b} = b^{\log_c a}$

(ix) $a^{\log_a N} = N$, $a > 0$, $a \neq 1$ & $N > 0$

Note : (i) If the number and the base are on the same side of the unity, then the logarithm is positive.

(ii) If the number and the base are on the opposite sides of unity, then the logarithm is negative.

Logarithmic Equation :

The equality $\log_a x = \log_a y$ is possible if and only if $x = y$ i.e.

$\mathbf{\log_a x = \log_a y \Leftrightarrow x = y}$

Always check validity of given equation, ($x > 0$, $y > 0$, $a > 0$, $a \neq 1$)

Logarithmic Inequality :

Let 'a' is a real number such that

(i) If $a > 1$, then $\log_a x > \log_a y \Rightarrow x > y$

(ii) If $a > 1$, then $\log_a x < \alpha \Rightarrow 0 < x < a^\alpha$

(iii) If $a > 1$, then $\log_a x > \alpha \Rightarrow x > a^\alpha$

(iv) If $0 < a < 1$, then $\log_a x > \log_a y \Rightarrow 0 < x < y$

(v) If $0 < a < 1$, then $\log_a x < \alpha \Rightarrow x > a^\alpha$

Form - I : $f(x) > 0$, $g(x) > 0$, $g(x) \neq 1$

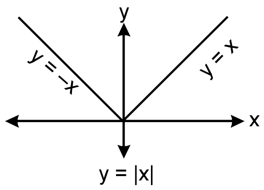
	Form		**Collection of system**
(a)	$\log_{g(x)} f(x) \geq 0$	$\Leftrightarrow$	$\begin{cases} f(x) \geq 1 \quad , \quad g(x) > 1 \\ 0 < f(x) \leq 1 \quad , \quad 0 < g(x) < 1 \end{cases}$
(b)	$\log_{g(x)} f(x) \leq 0$	$\Leftrightarrow$	$\begin{cases} f(x) \geq 1 \quad , \quad 0 < g(x) < 1 \\ 0 < f(x) \leq 1 \quad , \quad g(x) > 1 \end{cases}$
(c)	$\log_{g(x)} f(x) \geq a$	$\Leftrightarrow$	$\begin{cases} f(x) \geq (g(x))^a \quad , \quad g(x) > 1 \\ 0 < f(x) \leq (g(x))^a \quad , \quad 0 < g(x) < 1 \end{cases}$
(d)	$\log_{g(x)} f(x) \leq a$	$\Leftrightarrow$	$\begin{cases} 0 < f(x) \leq (g(x))^a \quad , \quad g(x) > 1 \\ f(x) \geq (g(x))^a \quad , \quad 0 < g(x) < 1 \end{cases}$

Form - II : When the inequality of the form

	Form		**Collection of system**
(a)	$\log_{\phi(x)} f(x) \geq \log_{\phi(x)} g(x)$	$\Leftrightarrow$	$\begin{cases} f(x) \geq g(x), \phi(x) > 1, \\ 0 < f(x) \leq g(x); 0 < \phi(x) < 1 \end{cases}$
(b)	$\log_{\phi(x)} f(x) \leq \log_{\phi(x)} g(x)$	$\Leftrightarrow$	$\begin{cases} 0 < f(x) \leq g(x), \phi(x) > 1, \\ f(x) \geq g(x) > 0, 0 < \phi(x) < 1 \end{cases}$

Absolute value function / modulus function :

The symbol of modulus function is f (x) = $|x|$ and is defined as: y = $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.



Properties of modulus : For any a, b $\in$ R

(i) $|a| \geq 0$ (ii) $|a| = |-a|$

(iii) $|a| \geq a$, $|a| \geq -a$ (iv) $|ab| = |a|\,|b|$

(v) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ (vi) $|a + b| \leq |a| + |b|$; Equality holds when $ab \geq 0$

(vii) $|a - b| \geq ||a| - |b||$; Equality holds when $ab \geq 0$

Irrational function :

An irrational function is a function y = f(x) in which the operations of addition, substraction, multiplication, division and raising to a fractional power are used.

For example $y = \frac{x^3 + x^{1/3}}{2x + \sqrt{x}}$ is an irrational function

(a) The equation $\sqrt{f(x)} = g(x)$, is equivalent to the following system

$f(x) = g^2(x)$ & $g(x) \geq 0$

(b) The inequation $\sqrt{f(x)} < g(x)$, is equivalent to the following system

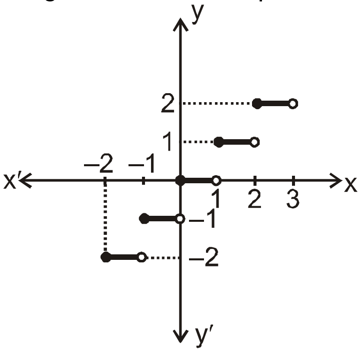
$f(x) < g^2(x)$ & $f(x) \geq 0$ & $g(x) \geq 0$

(c) The inequation $\sqrt{f(x)} > g(x)$, is equivalent to the following system

$g(x) \leq 0$ & $f(x) \geq 0$ or $g(x) \geq 0$ & $f(x) > g^2(x)$

Greatest integer function or step up function :

The function y = f (x) = [x] is called the greatest integer function where [x] equals to the greatest integer less than or equal to x. Graph of greatest integer function is



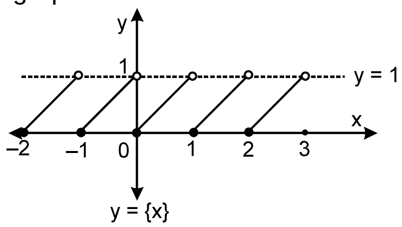
Properties of greatest integer function :

(a) $x - 1 < [x] \leq x$ (b) $[x \pm m] = [x] \pm m$ iff m is an integer.

(c) $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$ (d) $[x] + [-x] = \begin{cases} 0; & \text{if } x \text{ is an integer} \\ -1 & \text{otherwise} \end{cases}$

Fractional part function:

It is defined as $y = \{x\} = x - [x]$. It is always non-negative and varies from [0, 1). The period of this function is 1 and graph of this function is as shown.



Properties of fractional part function :

(a) $\{x \pm m\} = \{x\}$ iff m is an integer (b) $\{x\} + \{-x\} = \begin{cases} 0 & , \text{ if } x \text{ is an integer} \\ 1 & , \quad \text{otherwise} \end{cases}$

Signum function : $f(x) = \text{sgn}(x) = \begin{cases} 1 & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x = 0 \\ -1 & \text{for} \quad x < 0 \end{cases}$

Trigonometric functions of sum or difference of two angles:

(a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
(b) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
(c) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B).\sin(A-B)$
(d) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B).\cos(A-B)$
(e) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ (f) $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$
(g) $\sin(A + B + C) = \sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B - \sin A \sin B \sin C$
(h) $\cos(A + B + C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$
(i) $\tan(A + B + C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$.
(j) $\tan(\theta_1 + \theta_2 + \theta_3 + ... + \theta_n) = \dfrac{S_1 - S_3 + S_5 - ......}{1 - S_2 + S_4 - ....}$

where S_i denotes sum of product of tangent of angles taken i at a time

Transformation formulae :

(i) $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$ (a) $\sin C + \sin D = 2 \sin \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$

(ii) $\sin(A+B) - \sin(A-B) = 2 \cos A \sin B$ (b) $\sin C - \sin D = 2 \cos \dfrac{C+D}{2} \sin \dfrac{C-D}{2}$

(iii) $\cos(A+B) + \cos(A-B) = 2 \cos A \cos B$ (c) $\cos C + \cos D = 2 \cos \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$

(iv) $\cos(A-B) - \cos(A+B) = 2 \sin A \sin B$ (d) $\cos C - \cos D = 2 \sin \dfrac{C+D}{2} \sin \dfrac{D-C}{2}$

Multiple and sub-multiple angles :

(a) $\sin 2A = 2 \sin A \cos A$ Note : $\sin\theta = 2 \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2}$ etc.

(b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2 \sin^2 A$

Note : $2 \cos^2 \dfrac{\theta}{2} = 1 + \cos\theta$, $2 \sin^2 \dfrac{\theta}{2} = 1 - \cos\theta$.

(c) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$ Note : $\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

(d) $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$, $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

(e) $\sin 3A = 3\sin A - 4\sin^3 A$

(f) $\cos 3A = 4\cos^3 A - 3\cos A$

(g) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

Important trigonometric ratios of standard angles :

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$, where $n \in I$

(b) $\sin 15°$ or $\sin\dfrac{\pi}{12} = \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75°$ or $\cos\dfrac{5\pi}{12}$;

$\cos 15°$ or $\cos\dfrac{\pi}{12} = \dfrac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75°$ or $\sin\dfrac{5\pi}{12}$;

$\tan 15° = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75°$; $\tan 75° = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15°$

(c) $\sin\dfrac{\pi}{10}$ or $\sin 18° = \dfrac{\sqrt{5}-1}{4} = \cos 72°$; $\cos 36°$ or $\cos\dfrac{\pi}{5} = \dfrac{\sqrt{5}+1}{4} = \sin 54°$

Conditional identities:

If $A + B + C = \pi$ then :

(i) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

(ii) $\sin A + \sin B + \sin C = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

(iii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$

(iv) $\cos A + \cos B + \cos C = 1 + 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

(v) $\tan A + \tan B + \tan C = \tan A\tan B\tan C$

(vi) $\tan\dfrac{A}{2}\tan\dfrac{B}{2} + \tan\dfrac{B}{2}\tan\dfrac{C}{2} + \tan\dfrac{C}{2}\tan\dfrac{A}{2} = 1$

(vii) $\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2} = \cot\dfrac{A}{2}\,.\,\cot\dfrac{B}{2}\,.\,\cot\dfrac{C}{2}$

(viii) $\cot A\cot B + \cot B\cot C + \cot C\cot A = 1$

Sine and Cosine series:

(i) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots + \sin\{\alpha + (n-1)\beta\} = \dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\alpha + \dfrac{n-1}{2}\beta\right)$

(ii) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos\{\alpha + (n-1)\beta\} = \dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\alpha + \dfrac{n-1}{2}\beta\right)$

where : $\beta \neq 2m\pi$, $m \in I$

Product series of cosine angles

$\cos\theta \,.\, \cos 2\theta \,.\, \cos 2^2\theta \,.\, \cos 2^3\theta \ldots\ldots \cos 2^{n-1}\theta = \dfrac{\sin 2^n\theta}{2^n\sin\theta}$

Range of trigonometric expression:

Range of $E = a\sin\theta + b\cos\theta$ is $\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$

Trigonometric Equation :

An equation involving one or more trigonometric ratios of an unknown angle is called a trigonometric equation.

Solution of Trigonometric Equation :

A solution of trigonometric equation is the value of the unknown angle that satisfies the equation.
Thus, the trigonometric equation may have infinite number of solutions (because of their periodic nature) and can be classified as :
(i) Principal solution (ii) General solution.

Principal solutions :

The solutions of a trigonometric equation which lie in the interval $[0, 2\pi)$ are called Principal solutions.

General Solution :

The expression involving an integer 'n' which gives all solutions of a trigonometric equation is called General solution. General solution of some standard trigonometric equations are given below.

General Solution of Some Standard Trigonometric Equations :

(i) If $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$.

(ii) If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$, $n \in I$.

(iii) If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$.

(iv) If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$, $n \in I$.

(v) If $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$, $n \in I$.

(vi) If $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$, $n \in I$. [**Note:** α is called the principal angle]

Some Important deductions :

(i) $\sin\theta = 0 \Rightarrow \theta = n\pi$, $n \in I$

(ii) $\sin\theta = 1 \Rightarrow \theta = (4n + 1)\frac{\pi}{2}$, $n \in I$

(iii) $\sin\theta = -1 \Rightarrow \theta = (4n - 1)\frac{\pi}{2}$, $n \in I$

(iv) $\cos\theta = 0 \Rightarrow \theta = (2n + 1)\frac{\pi}{2}$, $n \in I$

(v) $\cos\theta = 1 \Rightarrow \theta = 2n\pi$, $n \in I$

(vi) $\cos\theta = -1 \Rightarrow \theta = (2n + 1)\pi$, $n \in I$

(vii) $\tan\theta = 0 \Rightarrow \theta = n\pi$, $n \in I$