Differential Equation

It is not certain that everything is uncertain ....Pascal, Blaise

Introduction :

An equation involving independent and dependent variables and the derivatives of the dependent variables is called a **differential equation**. There are two kinds of differential equation:

1.1Ordinary Differential Equation : If the dependent variables depend on one independent variable x, then the differential equation is said to be ordinary.

for example $\frac{dy}{dx} + \frac{dz}{dx} = y + z,$

$$\frac{dy}{dx} + xy = \sin x, \quad \frac{d^3y}{dx^3} + 2\frac{dy}{dx} + y = e^x,$$

$$k\frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}, \quad y = x\frac{dy}{dx} + k\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}$$

1.2 Partial differential equation : If the dependent variables depend on two or more indepen dent variables, then it is known as partial differential equation

for example $y^2\frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial y^2} = ax, \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

Order and Degree of a Differential Equation:

2.1 Order : Order is the highest differential appearing in a differential equation.

2.2 Degree :

It is determined by the highest degree of the highest order derivative present in it after the differential equation is cleared of radicals and fractions so far as the derivatives are concerned.

Note : In the differential equation, all the derivatives should be expressed in the polynomial form

$$f_1(x, y)\left[\frac{d^my}{dx^m}\right]^{n_1} + f_2(x, y)\left[\frac{d^{m-1}y}{dx^{m-1}}\right]^{n_2} + \ldots\ldots\ldots f_k(x, y)\left[\frac{dy}{dx}\right]^{n_k} = 0$$

The above differential equation has the order m and degree n_1.

Example # 1: Find the order & degree of following differential equations.

(i) $\frac{d^2y}{dx^2} = \left[y+\left(\frac{dy}{dx}\right)^6\right]^{1/4}$ (ii) $y = \log_e\left(\frac{d^3y}{dx^3}+\left(\frac{dy}{dx}\right)^2\right)$

(iii) $\tan^{-1}\left(x\frac{dy}{dx}+\frac{d^2y}{dx^2}\right) = y$ (iv) $e^{y'''} - xy'' + y = 0$

Solution : (i) $\left(\frac{d^2y}{dx^2}\right)^4 = y+\left(\frac{dy}{dx}\right)^6$ $\therefore$ order = 2, degree = 4

(ii) $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 = e^y$ $\therefore$ order = 3, degree = 1

(iii) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \tan y$ $\therefore$ order = 2, degree = 1

(iv) $e^{\frac{d^3y}{dx^3}} - x\frac{d^2y}{dx^2} + y = 0$

$\because$ equation can not be expressed as a polynomial in differential coefficients, so degree is not applicable but order is 3.

Self Practice Problems :

(1) Find order and degree of the following differential equations.

(i) $\frac{dy}{dx} + y = \frac{1}{\frac{dy}{dx}}$ (ii) $\sqrt{\frac{dy}{dx}} - 4\frac{dy}{dx} - 7x = 0$

(iii) $\left[\left(\frac{dy}{dx}\right)^{1/2} + y\right]^2 = \frac{d^2y}{dx^2}$

Ans. (1) (i) order = 1, degree = 2 (ii) order = 1, degree = 2
(iii) order = 2, degree = 2

Formation of Differential Equation:

Differential equation corresponding to a family of curve will have :

(a) Order exactly same as number of essential arbitrary constants in the equation of curve.
(b) No arbitrary constant present in it.

The differential equation corresponding to a family of curve can be obtained by using the following steps:
(i) Identify the number of essential arbitrary constants in equation of curve.

NOTE : If arbitrary constants appear in addition, subtraction, multiplication or division, then we can club them to reduce into one new arbitrary constant.
(ii) Differentiate the equation of curve till the required order.
(iii) Eliminate the arbitrary constant from the equation of curve and additional equations obtained in step (ii) above.

Example # 2 : Form a differential equation of family of straight lines passing through (0,2)
Solution : Family of straight lines passing through (0,2) is y = mx + 2 where'm' is a parameter.
Differentiating w.r.t. x

$\frac{dy}{dx} = m$

Eliminating 'm' from both equations, we obtain

$y = x\frac{dy}{dx} + 2$ which is the required differential equation.

Example # 3 : Form a differential equation of family of parabolas having x axis as line of symmetry and tangent at vertex is y-axes
Solution : Let equation of parabola

$y^2 = 4ax$ ....(i)

$2y\frac{dy}{dx} = 4a$ ....(ii)

by (i) and (ii)

$\Rightarrow y^2 = 2yx\frac{dy}{dx}$ $\Rightarrow$ $y = 2x\frac{dy}{dx}$

Self Practice Problems :

(2) Obtain a differential equation of the family of curves y = a sin (bx + c) where a and c being arbitrary constant.

(3) Show that the differential equation of the system of parabolas $y^2 = 4a(x - b)$ is given by

$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$

(4) Form a differential equation of family of parabolas with focus as origin and axis of symmetry along the x-axis.

Ans. (2) $\frac{d^2y}{dx^2} + b^2y = 0$ (4) $y^2 = y^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx}$

Solution of a Differential Equation:

Finding the dependent variable from the differential equation is called solving or integrating it. The solution or the integral of a differential equation is, therefore, a relation between dependent and independent variables (free from derivatives) such that it satisfies the given differential equation

NOTE : The solution of the differential equation is also called its primitive, because the differential equation can be regarded as a relation derived from it.

There can be three types of solution of a differential equation:

(i) **General solution (or complete integral or complete primitive) :** A relation in x and y satisfying a given differential equation and involving exactly same number of arbitrary constants as order of differential equation.

(ii) **Particular Solution :** A solution obtained by assigning values to one or more than one arbitrary constant of general solution.

(iii) **Singular Solution :** It is not obtainable from general solution. Geometrically, **Singular solution** acts as an envelope to **General solution**.

4.1. Differential Equation of First Order and First Degree :

A differential equation of first order and first degree is of the type $\frac{dy}{dx}$ + f(x, y) = 0, which can also be written as : Mdx + Ndy = 0, where M and N are functions of x and y.

Solution methods of First Order and First Degree Differential Equations :

5.1 Variables separable : If the differential equation can be put in the form, f(x) dx = ϕ(y) dywe say that variables are separable and solution can be obtained by integrating each side separately. A general solution of this will be $\int f(x)\,dx = \int \phi(y)\,dy + c$, where c is an arbitrary constant.

Example # 4 : Solve the differential equation (1 + x) y dx = (y – 1) x dy

Solution : The equation can be written as -

$$\left(\frac{1+x}{x}\right)dx = \left(\frac{y-1}{y}\right)dy \quad\Rightarrow\quad \int\left(\frac{1}{x}+1\right)dx = \int\left(1-\frac{1}{y}\right)dy$$

$$\ell n\,x + x = y - \ell ny + c \quad\Rightarrow\quad \ell ny + \ell nx = y - x + c \quad\Rightarrow\quad xy = ce^{y-x}$$

Example # 5 : Solve : $(e^x + 1)\,y\,dy = (y + 1)\,e^x\,dx$

Solution : The given differential equation is $(e^x + 1)\,y\,dy = (y + 1)\,e^x\,dx$

$$\frac{ydy}{(y+1)} = \frac{e^x}{(e^x+1)}$$

Integrating both sides

$\Rightarrow$ y – log |y + 1|= log $(e^x + 1)$ + log k $\Rightarrow$ y = log |(y + 1)$(e^x + 1)$| + log k $\Rightarrow$ (y + 1)$(e^x + 1)=e^y c$

Example # 6 : $\frac{dy}{dx} = \frac{x(2\ell nx+1)}{\sin y + y\cos y}$ Solve :

Solution : $\frac{dy}{dx} = \frac{x(2\ell nx+1)}{\sin y + y\cos y}$

(siny + ycosy)dy = x(2ℓnx + 1)dx

Integrating both sides

$$\Rightarrow -\cos y + \{(y\sin y) + \cos y\} = 2 \times \left\{\frac{x^2}{2}\ln x - \frac{1}{2}\int\frac{x}{1}dx\right\} + \frac{x^2}{2} \Rightarrow y\sin y = x^2\,\ell n\,x$$

5.1.1 Polar coordinates transformations :

Sometimes transformation to the polar co-ordinates facilitates separation of variables. In this connection it is convenient to remember the following differentials:

(a) If x = r cos θ ; y = r sin θ then,

(i) x dx + y dy = r dr (ii) $dx^2 + dy^2 = dr^2$ (iii) x dy – y dx = $r^2 d\theta$

(b) If x = r sec θ & y = r tan θ then

(i) x dx – y dy = r dr (ii) x dy – y dx = $r^2 \sec\theta\, d\theta$.

Example # 7 : Solve the differential equation xdx + ydy = x (xdy – ydx)

Solution : Taking $x = r\cos\theta$, $y = r\sin\theta$

$x^2 + y^2 = r^2$

$2x\,dx + 2y\,dy = 2r\,dr$

$x\,dx + y\,dy = r\,dr$ ....(i)

$$\frac{y}{x} = \tan\theta \quad \Rightarrow \quad \frac{x\frac{dy}{dx} - y}{x^2} = \sec^2\theta \cdot \frac{d\theta}{dx}$$

$x\,dy - y\,dx = x^2\sec^2\theta \cdot d\theta$

$x\,dy - y\,dx = r^2\,d\theta$ ....(ii)

Using (i) & (ii) in the given differential equation then it becomes

$r\,dr = r\cos\theta \cdot r^2\,d\theta$

$$\frac{dr}{r^2} = \cos\theta\,d\theta \quad \Rightarrow -\frac{1}{r} = \sin\theta + \lambda \Rightarrow -\frac{1}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} + \lambda \Rightarrow \frac{y+1}{\sqrt{x^2+y^2}} = c$$

where $-\lambda' = c \Rightarrow (y + 1)^2 = c(x^2 + y^2)$

5.1.2 Equations Reducible to the Variables Separable form : If a differential equation can be reduced into a variables separable form by a proper substitution, then it is said to be "Reducible to the variables separable type". Its general form is $\frac{dy}{dx} = f(ax + by + c)$ a, b $\neq$ 0. To solve this, put ax + by + c = t.

Example # 8 : Solve $\frac{dy}{dx} = (4x + y + 1)^2$

Solution : Putting $4x + y + 1 = t \quad \Rightarrow \quad 4 + \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 4$

Given equation becomes

$$\frac{dt}{dx} - 4 = t^2 \qquad \Rightarrow \qquad \frac{dt}{t^2+4} = dx \qquad \text{(Variables are separated)}$$

Integrating both sides,

$$\int \frac{dt}{4+t^2} = \int dx \Rightarrow \frac{1}{2}\tan^{-1}\frac{t}{2} = x + c \quad \Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c$$

Example # 9 : Solve $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

Solution : $\frac{dy}{dx} = \sin(x + y)$

putting x + y = t

$$\frac{dy}{dx} = \frac{dt}{dx} - 1 \therefore \frac{dt}{dx} - 1 = \sin t \Rightarrow \frac{dt}{dx} = 1 + \sin t \Rightarrow \frac{dt}{1+\sin t} = dx$$

Integrating both sides,

$$\int \frac{dt}{1+\sin t} = \int dx \qquad \Rightarrow \qquad \int \frac{1-\sin t}{\cos^2 t}\,dt = x + c \qquad \Rightarrow \int (\sec^2 t - \sec t\,\tan t)\,dt = x + c$$

$$\tan t - \sec t = x + c \qquad \Rightarrow \qquad -\frac{1-\sin t}{\cos t} = x + c \qquad \Rightarrow \qquad \sin t - 1 = x\cos t + c\cos t$$

substituting the value of t

$\sin(x + y) = x\cos(x + y) + c\cos(x + y) + 1$

Self Practice Problems :

(5) Solve the differential equation $x^2 y\frac{dy}{dx} = (x + 1)(y + 1)$

(6) Solve the differential equation $\frac{xdx + ydy}{\sqrt{x^2 + y^2}} = \frac{ydx - xdy}{x^2}$

(7) Solve : $\frac{dy}{dx} = e^{x+y} + x^2e^y$

(8) Solve : $xy\frac{dy}{dx} = 1 + x + y + xy$

(9) Solve $\frac{dy}{dx} = 1 + e^{x-y}$

(10) $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$

(11) Find the solution of the differential equation $(x + y)^2 \frac{dy}{dx} = 1$, satisfying the condition $y(1) = 0$

Ans. (5) $y - \ell n(y + 1) = \ell nx - \frac{1}{x} + c$ (6) $\sqrt{x^2 + y^2} + \frac{y}{x} = c$

(7) $-\frac{1}{e^y} = e^x + \frac{x^3}{3} + c$ (8) $y = x + \ell n|x(1 + y)| + c$

(9) $e^{y-x} = x + c$ (10) $\log\left|\tan\frac{x+y}{2} + 1\right| = x + c$

(11) $y + \frac{\pi}{4} = \tan^{-1}(x + y)$

5.2 Homogeneous Differential Equations :

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where f and g are homogeneous function of x and y, and of the same degree, is called homogeneous differential equaiton and can be solved easily by putting y = vx.

Example # 10 : Solve $x\frac{dy}{dx} = y + x\tan\left(\frac{y}{x}\right)$

Solution : $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ (Homogeneous differential equation)

put $y = vx$ $\Rightarrow$ $v + x\frac{dv}{dx} = v + \tan v$ $\Rightarrow \cot v.dv = \frac{dx}{x}$

Integrating both sides we have

$\ell n \sin v = \ell nx + \ell nc$ $\Rightarrow$ $\sin v = cx$ $\Rightarrow$ $\sin\left(\frac{y}{x}\right) = cx$

Example # 11 : Solve : $x^2dy + y(x + y)dx = 0$ given that y = 1 when x = 1

Solution : $\frac{dy}{dx} = -\frac{y}{x} - \frac{y^2}{x^2}$ put $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ $\Rightarrow$ $v + x\frac{dv}{dx} = -v - v^2 \Rightarrow \frac{dv}{(2+v)v} = -\frac{dx}{x}$

$\Rightarrow$ $\frac{1}{2}\int\left(\frac{1}{v} - \frac{1}{v+2}\right)dv = \int -\frac{dx}{x}$ $\Rightarrow$ $\frac{1}{2}\ell n\left(\frac{v}{v+2}\right) + \ell nx = \ell nC$

$\Rightarrow$ $x\sqrt{\frac{v}{v+2}} = C$ $\Rightarrow$ $x\sqrt{\frac{y}{y+2x}} = C$

When x = 1 then y = 1 $\Rightarrow C = \frac{1}{\sqrt{3}}$ $\Rightarrow 3x^2y = (y + 2x)$

5.2.1 Equations Reducible to the Homogeneous form

Equations of the form $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$ ....(1)

can be made homogeneous (in new variables X and Y) by substituting x = X + h and y = Y + k, where h and k are constants to obtain, $\frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{AX+BY+(Ah+Bk+C)}$ ....(2)

These constants are chosen such that ah + bk + c = 0, and Ah + Bk + C = 0. Thus we obtain the following differential equation $\frac{dY}{dX} = \frac{aX+bY}{AX+BY}$

The differential equation can now be solved by substituting Y = vX.

Example # 12 : Solve the differential equation $\frac{dy}{dx} = \frac{x+2y-5}{2x+y-4}$

Solution : Let x = X + h, y = Y + k

$\frac{dy}{dX} = \frac{d}{dX}(Y + k)$

$\frac{dy}{dX} = \frac{dY}{dX}$ ....(i) $\frac{dx}{dX} = 1 + 0$ ....(ii)

on dividing (i) by (ii) $\frac{dy}{dx} = \frac{dY}{dX}$

$$\frac{dY}{dX} = \frac{X+h+2(Y+k)-5}{2X+2h+Y+k-4} = \frac{X+2Y+(h+2k-5)}{2X+Y+(2h+k-4)}$$

h & k are such that h + 2k – 5 = 0 & 2h + k – 4 = 0

h = 1, k = 2

$\frac{dY}{dX} = \frac{X+2Y}{2X+Y}$ which is homogeneous differential equation.

Now, substituting Y = vX

$\frac{dY}{dX} = v + X\frac{dv}{dX}$ $\therefore X\frac{dv}{dX} = \frac{1+2v}{2+v} - v$ $\Rightarrow \int \frac{2+v}{1-v^2}\,dv = \int \frac{dX}{X}$

$\int\left(\frac{1}{2(v+1)} + \frac{3}{2(1-v)}\right)dv = \ell n\, X + c$ $\Rightarrow \frac{1}{2}\ell n\,(v+1) - \frac{3}{2}\ell n\,(1-v) = \ell n\, X + c$

$\ell n\left|\frac{v+1}{(1-v)^3}\right| = \ell n X^2 + 2c \Rightarrow \frac{(Y+X)}{(X-Y)^3}\frac{X^2}{X^2} = e^{2c}$

$X + Y = c'(X - Y)^3$ where $e^{2c} = c'$

$x - 1 + y - 2 = c'(x - 1 - y + 2)^3$

$x + y - 3 = c'(x - y + 1)^3$

Special case :

Case - 1 In equation (1) if $\frac{a}{A} = \frac{b}{B}$, then the substitution ax + by = v will reduce it to the form in which variables are separable.

Example # 13 : Solve $\frac{dy}{dx} = \frac{2x+3y-1}{4x+6y-5}$

Solution : Putting u = 2x + 3y

$\frac{du}{dx} = 2 + 3 . \frac{dy}{dx}$ $\Rightarrow$ $\frac{1}{3}\left(\frac{du}{dx} - 2\right) = \frac{u-1}{2u-5} \Rightarrow$ $\frac{du}{dx} = \frac{3u-3+4u-10}{2u-5}$

$\int \frac{2u-5}{7u-13}\,dx = \int dx$ $\Rightarrow$ $\frac{2}{7}\int 1.du - \frac{9}{7}\int \frac{1}{7u-13}.\,du = x + c$

$\Rightarrow \frac{2}{7}u - \frac{9}{7}.\frac{1}{7}\ell n\,(7u - 13) = x + c \Rightarrow 4x + 6y - \frac{9}{7}\ell n\,(14x + 21y - 13) = 7x + 7c$

$\Rightarrow -3x + 6y - \frac{9}{7}\ell n\,(14x + 21y - 13) = c'$

Case - 2 In equation (1), if b + A = 0, then by a simple cross multiplication equation (1) becomes an **exact differential equation**.

Example # 14 : Solve $\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1}$

Solution : Cross multiplying,

$2xdy + y\,dy - dy = xdx - 2ydx + 5dx$

$2\,(xdy + y\,dx) + ydy - dy = xdx + 5\,dx$

$2\,d(xy) + y\,dy - dy = xdx + 5dx$

On integrating,

$2xy + \frac{y^2}{2} - y = \frac{x^2}{2} + 5x + c \Rightarrow x^2 - 4xy - y^2 + 10x + 2y = c'$ where $c' = -2c$

Case - 3 If the homogeneous equation is of the form :
$yf(xy)\,dx + xg(xy)dy = 0$, the variables can be separated by the substitution $xy = v$.

Self Practice Problems :

Solve the following differential equations

(12) $\left(x\frac{dy}{dx} - y\right)\tan^{-1}\frac{y}{x} = x$ given that $y = 0$ at $x = 1$

(13) $x\frac{dy}{dx} = y - x\tan\frac{y}{x}$

(14) $\frac{dy}{dx} = \frac{2x-y+3}{x+2y+4}$

(15) $(3x - 2y + 1)\,dy + (4y - 6x + 3)dx = 0$

(16) $\frac{dy}{dx} = \frac{3x+2y-5}{3y-2x+5}$

Ans. (12) $\sqrt{x^2+y^2} = e^{\frac{y}{x}\tan^{-1}\frac{y}{x}}$

(13) $x\sin\frac{y}{x} = C$

(14) $y^2 - x^2 + xy + 4y - 3x + C = 0$

(15) $10\,\ell n\,|3x - 2y - 9| = 2y - 4x + C$

(16) $3x^2 + 4xy - 3y^2 - 10x - 10y = C$

5.3 Exact Differential Equation :

The differential equation $M + N\frac{dy}{dx} = 0$ ....(1)

Where M and N are functions of x and y is said to be exact if it can be derived by direct differentiation (without any subsequent multiplication, elimination etc.) of an equation of the form $f(x, y) = c$

e.g. $y^2\,dy + x\,dx + \frac{dx}{x} = 0$ is an exact differential equation.

NOTE : (i) The necessary condition for (1) to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

(ii) For finding the solution of exact differential equation, following results on exact differentials should be remembered:

(a) $xdy + y\,dx = d(xy)$

(b) $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$

(c) $2(x\,dx + y\,dy) = d\,(x^2 + y^2)$

(d) $\frac{xdy - ydx}{xy} = d\left(\ln\frac{y}{x}\right)$

(e) $\frac{xdy - ydx}{x^2+y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$

(f) $\frac{xdy + ydx}{xy} = d(\ln xy)$

(g) $\frac{xdy + ydx}{x^2\,y^2} = d\left(-\frac{1}{xy}\right)$

Example # 15 : Solve : $y^2x\,dx + ydx - xdy = 0$

Solution : $\frac{y^2xdx + ydx - xdy}{y^2} = 0 \Rightarrow xdx + d\left(\frac{x}{y}\right) = 0$

on integrating $\frac{x^2}{2} + \frac{x}{y} + c = 0$

Example # 16 : Solve : (x – y)dy + (x + y)dx = dx + dy

Solution : The given equation can be written as

(xdy + ydx) – ydy + xdx = dx + dy $\Rightarrow$ d(x.y) – ydy + xdx = dx + dy

Also integrating each term we get $xy - \frac{y^2}{2} + \frac{x^2}{2} = x + y + C$

Self Practice Problems :

(17) Solve : $xdy + ydx + xy\, e^y\, dy = 0$

(18) Solve : $ye^{-x/y}\, dx - (xe^{-x/y} + y^3)\, dy = 0$

Ans. (17) $\ell n\,(xy) + e^y = c$ (18) $2e^{-x/y} + y^2 = c$

Linear Differential Equation :

A linear differential equation has the following characteristics :

(i) Dependent variable and its derivative in first degree only and are not multiplied together

(ii) All the derivatives should be in a polynomial form

(iii) The order may be more than one

The m^{th} order linear differential equation is of the form.

$$P_0(x)\frac{d^m y}{dx^m} + P_1(x)\frac{d^{m-1}y}{dx^{m-1}} + \ldots\ldots\ldots\ldots + P_{m-1}(x)\frac{dy}{dx} + P_m(x)\, y = \phi(x),$$

where $P_0(x), P_1(x)$...............$P_m(x)$ are called the coefficients of the differential equation.

NOTE : $\frac{dy}{dx} + y^2 \sin x = \ln x$ is not a Linear differential equation.

6.1 Linear differential equations of first order :

The differential equation $\frac{dy}{dx} + Py = Q$, is linear in y.

(Where P and Q are functions of x only).

Integrating Factor (I.F.) : It is an expression which when multiplied to a differential equation converts it into an exact form.

I.F for linear differential equation = $e^{\int P dx}$ (constant of integration will not be considered)

$\therefore$ after multiplying above equation by I.F it becomes;

$$\frac{dy}{dx}\, e^{\int P dx} + Py\,.\, e^{\int P dx} = Q.\, e^{\int P dx}$$

$$\Rightarrow \quad \frac{d}{dx}(y.e^{\int P dx}) = Q.e^{\int P dx} \quad \Rightarrow \quad y.e^{\int P dx} = \int Q.e^{\int P dx} + C.$$

NOTE : Some times differential equation becomes linear, if x is taken as the dependent variable, and y as independent variable. The differential equation has then the following form : $\frac{dx}{dy} + P_1 x = Q_1$.

where P_1 and Q_1 are functions of y. The I.F. now is $e^{\int P_1 dy}$

Example # 17 : Solve the differential equation $f(x)\frac{dy}{dx} = f^2(x) + f(x)\, y + f'(x)\, y$

Solution : $f(x)\frac{dy}{dx} = f^2(x) + f(x)\, y + f'(x)\, y$

Given DE can be written as $\frac{dy}{dx} - \left(1 + \frac{f'(x)}{f(x)}\right) y = f(x)$

Which is L.D.E.

I.F. = $e^{-x - \ell n\, f(x)} = \frac{e^{-x}}{f(x)}$

General solution $y\frac{e^{-x}}{f(x)} = \int f(x)\frac{e^{-x}}{f(x)}\, dx + c = -e^{-x} + c \Rightarrow y = -f(x) + ce^x f(x)$

Example # 18 : Solve : $x\,\ell nx \frac{dy}{dx} + y = 2\,\ell n\, x$

Solution : $\frac{dy}{dx} + \frac{1}{x\ell nx} y = \frac{2}{x} \quad \Rightarrow \quad P = \frac{1}{x\ell nx}, Q = \frac{2}{x}$

$IF = e^{\int P.dx} = e^{\int \frac{1}{x\ell nx}dx} = e^{\ell n(\ell nx)} = \ell n\, x$

$\therefore$ General solution is $y.(\ell n\, x) = \int \frac{2}{x}.\ell nx.dx + c \Rightarrow y\,(\ell n\, x) = (\ell n\, x)^2 + c$

Example # 19 : Solve the differential equation

$t\,(1 + t^2)\,dx = (x + xt^2 - t^2)\,dt$ and it given that $x = -\pi/4$ at $t = 1$

Solution : $t\,(1 + t^2)\,dx = [x\,(1 + t^2) - t^2]\,dt$

$\frac{dx}{dt} = \frac{x}{t} - \frac{t}{(1+t^2)}$

$\frac{dx}{dt} - \frac{x}{t} = -\frac{t}{1+t^2}$ which is linear in $\frac{dx}{dt}$

Here, $P = -\frac{1}{t}$, $Q = -\frac{t}{1+t^2}$ $\quad IF = e^{-\int \frac{1}{t}dt} = e^{-\ell nt} = \frac{1}{t}$

$\therefore$ General solution is -

$x : \frac{1}{t} = \int \frac{1}{t}.\left(-\frac{t}{1+t^2}\right) dt + c \quad \Rightarrow \quad \frac{x}{t} = -\tan^{-1} t + c$

putting $x = -\pi/4$, $t = 1$

$-\pi/4 = -\pi/4 + c \Rightarrow c = 0$

$\therefore \quad x = -t\tan^{-1} t$

6.2 Equations reducible to linear form

6.2.1 By change of variable.

Often differential equation can be reduced to linear form by appropriate substitution of the non-linear term

Example # 20 : Solve : $y \sin x \frac{dy}{dx} = \cos x\,(\sin x - y^2)$

Solution : The given differential equation can be reduced to linear form by change of variable by a suitable subtitution.

Substituting $y^2 = z$

$2y \frac{dy}{dx} = \frac{dz}{dx}$

differential equation becomes

$\frac{\sin x}{2}\frac{dz}{dx} + \cos x.z = \sin x \cos x$

$\frac{dz}{dx} + 2 \cot x \,.\, z = 2 \cos x$ which is linear in $\frac{dz}{dx}$

$IF = e^{\int 2\cot x\,dx} = e^{2\ell n \sin x} = \sin^2 x$

$\therefore$ General solution is -

$z.\sin^2 x = \int 2\cos x.\sin^2 x.dx + c \Rightarrow y^2 \sin^2 x = \frac{2}{3} \sin^3 x + c$

6.2.2 Bernoulli's equation :

Equations of the form $\frac{dy}{dx} + Py = Q.y^n$, $n \neq 0$ and $n \neq 1$

where P and Q are functions of x, is called Bernoulli's equation and can be made linear in v by dividing by y^n and putting $y^{-n+1} = v$. Now its solution can be obtained as in (v).

e.g. : $2 \sin x \frac{dy}{dx} - y \cos x = xy^3 e^x$.

Example # 21 : Solve : $\frac{dy}{dx} = x^3y^3 - xy$ (Bernoulli's equation)

Solution : Dividing both sides by y^3

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = x^3$$

Putting $t = 1/y^2 \Rightarrow -\frac{2}{y^3}\frac{dy}{dx} = \frac{dt}{dx} \qquad \Rightarrow \frac{dt}{dx} - 2x\,t = -2x^3$

I.F. = $e^{-\int 2x dx} = e^{-x^2}$

General solution is

$$t\,e^{-x^2} = \int -2x^3 e^{-x^2} dx + C \Rightarrow \frac{e^{-x^2}}{y^2} = -e^{-x^2}(-x^2 - 1) + C \Rightarrow \frac{1}{y^2} = (x^2 + 1) + C\,e^{x^2}$$

Self Practice Problems :

(19) Solve : $x(x^2 + 1)\frac{dy}{dx} = y(1 - x^2) + x^2 \ell nx$

(20) Solve : $(x + 2y^3)\frac{dy}{dx} = y$

(21) Solve : $x\frac{dy}{dx} + y = y^2 \log x$

(22) Solve the differential equation $xy^2\left(\frac{dy}{dx}\right) - 2y^3 = 2x^3$ given y = 1 at x = 1

Ans. (19) $\left(\frac{x^2+1}{x}\right) y = x\,\ell n\,x - x + c$ (20) $x = y(c + y^2)$

(21) $y(1 + cx + \log x) = 1$ (22) $y^3 + 2x^3 = 3x^6$

Higher Degree Equation :

The differential equation $y = mx + f(m)$ (where $m = \frac{dy}{dx}$) ....(1),

is known as Clairaut's Equation.

To solve (1), differentiate it w.r.t. x, which gives $\frac{dy}{dx} = m + x\frac{dm}{dx} + \frac{d\,f(m)}{dm}\frac{dm}{dx}$

$$x\frac{dm}{dx} + \frac{d\,f(m)}{dm}\frac{dm}{dx} = 0$$

either $\frac{dm}{dx} = 0 \Rightarrow m = c$ ....(2) or $x + f'(m) = 0$ ....(3)

NOTE : (i)If m is eliminated between (1)and(2),thesolutionobtained is a general solution of(1)
(ii) If m is eliminated between (1) and (3), then solution obtained does not contain any arbitrary constant and is not particular solution of (1). This solution is called singular solution of (1).

Example # 22 : Solve : $y = mx + m - m^3$ where, $m = \frac{dy}{dx}$

Solution : $y = mx + m - m^3$ ..... (i)

The given equation is in clairaut's form.

Now, differentiating wrt. x -

$$\frac{dy}{dx} = m + x\frac{dm}{dx} + \frac{dm}{dx} - 3m^2\frac{dm}{dx} \Rightarrow \qquad m = m + x\frac{dm}{dx} + \frac{dm}{dx} - 3m^2\frac{dm}{dx}$$

$\frac{dm}{dx}(x + 1 - 3m^2) = 0 \Rightarrow \frac{dm}{dx} = 0 \Rightarrow \qquad m = c$ ..... (ii)

or $x + 1 - 3m^2 = 0 \qquad \Rightarrow \qquad m^2 = \frac{x+1}{3}$ ..... (iii)

Eliminating 'm' between (i) & (ii) is called the general solution of the given equation.

$y = cx + c - c^3$ where, 'c' is an arbitrary constant.

Again, eliminating 'm' between (i) & (iii) is called singular solution of the given equation.

$y = m(x + 1 - m^2)$

$$y = \pm\left(\frac{x+1}{3}\right)^{1/2}\left(x+1-\frac{x+1}{3}\right) \Rightarrow y = \pm\left(\frac{x+1}{3}\right)^{1/2}\frac{2}{3}\ (x+1)$$

$$y = \pm\ 2\left(\frac{x+1}{3}\right)^{3/2} \Rightarrow y^2 = \frac{4}{27}\ (x+1)^3 \Rightarrow 27y^2 = 4\ (x+1)^3$$

Self Practice Problems :

(23) Solve the differential equation $y = mx + 2/m$ where, $m = \frac{dy}{dx}$

(24) Solve : $\sin px \cos y = \cos px \sin y + p$ where $p = \frac{dy}{dx}$

Ans. (23) General solution : $y = cx + 2/c$ where c is an arbitrary constant
Singular solution : $y^2 = 8x$

(24) General solution : $y = cx - \sin^{-1}(c)$ where c is an arbitrary constant.
Singular solution : $y = \sqrt{x^2-1} - \sin^{-1}\sqrt{\frac{x^2-1}{x^2}}$

Example # 23 : The normal to a given curve at each point (x,y) on the curve passes through the point (3,0). If the curve contains, the point (3,4) find its equation.

Solution : Equation of normal at any point (x,y) is

$$\frac{dy}{dx}\ (Y - y) + (X - x) = 0$$

Passes through (3,0)

$$\Rightarrow \quad (3-x)\ \frac{dy}{dx} - y = 0 \Rightarrow ydy = (3-x)dx \Rightarrow \frac{y^2}{2} = 3x - \frac{x^2}{2} + C \qquad \text{.......... (i)}$$

The curve contains the point (3,4)

$$\Rightarrow \quad 8 = 9 - \frac{9}{2} + C \Rightarrow C = 7/2$$

By equation (i)

$$\frac{y^2}{2} = 3x - \frac{x^2}{2} + \frac{7}{2} \Rightarrow x^2 + y^2 - 6x - 7 = 0$$

Example # 24 : The slope of the tangent to a curve at any point (x,y) on it given by $\frac{y}{x} - \cot\frac{y}{x}\ .\ \cos\frac{y}{x}$, (x > 0, y > 0) and curve passes through the point(1,π/4). Find the equation of the curve.

Solution : Let y = f(x) be the curve

given that $\frac{dy}{dx} = \frac{y}{x} - \cot\frac{y}{x}\ .\ \cos\frac{y}{x}$ (homogeneous differential equation)

put y = vx

$$\Rightarrow \quad v + x\frac{dv}{dx} = v - \cot v.\ \cos v \Rightarrow \tan v.\ \sec v\ dv = \frac{-dx}{x}$$

Integrating both sides, we have

$$\Rightarrow \quad \sec v = -\ \ell n|x| + C \Rightarrow \sec\frac{y}{x} + \ell n|x| = C$$

Passes through $\left(1, \frac{\pi}{4}\right) \Rightarrow C = \sqrt{2}$

The curve is $\Rightarrow \sec\frac{y}{x} + \ell n|x| = \sqrt{2}$

Example # 25 : Assume that a spherical rain drop evaporates at a rate proportional to its surface area. if its radius originally is 3mm and 1 hr later has been reduced to 2mm, find an expression for radius of the rain drop at any time.

Solution : Let r be radius , V be volume and S be surface area of rain drop at any time t.

Then $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$

given $\frac{dV}{dt} \propto S \Rightarrow \frac{dV}{dt} = kS$, k is constant of proportionality

$\Rightarrow \frac{4}{3}.3\pi r^2 \frac{dr}{dt} = k4\pi r^2 \Rightarrow \frac{dr}{dt} = k \Rightarrow dr = kdt$

Integrating both sides we have r = kt + C

when t = 0, r = 3 $\Rightarrow$ C = 3

when t = 1 hr, r = 2 $\Rightarrow$ k = – 1

Hence r = 3 – t **Ans.**

Self Practice Problems :

(25) The decay rate of radium at any time t is proportional to its mass at that time. Find the time when the mass will be halved of its initial mass.

Ans. (25) k log 2 , where k is constant of proportionality