Definite Integration & its Application

Newton-Leibnitz formula.

Let $\frac{d}{dx}(F(x)) = f(x)\ \forall\ x \in (a, b)$. Then $\int_a^b f(x)\ dx = \lim_{x\to b^-} F(x) - \lim_{x\to a^+} F(x)$.

Note : 1. If $a > b$, then $\int_a^b f(x)\ dx = \lim_{x\to b^+} F(x) - \lim_{x\to a^-} F(x)$.

2. If F(x) is continuous at a and b, then = F(b) – F(a)

Example # 1 : Evaluate $\int_1^2 \frac{dx}{(x+1)(x+2)}$

Solution : $\because \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$ (by partial fractions)

$$\int_1^2 \frac{dx}{(x+1)(x+2)} = [\ell n(x+1) - \ell n(x+2)]_1^2 = \ell n3 - \ell n4 - \ell n2 + \ell n3 = \ell n\left(\frac{9}{8}\right)$$

Self practice problems :

Evaluate the following

(1) $\int_1^2 \frac{5x^2}{x^2+4x+3}\ dx$ (2) $\int_0^{\pi/4} \left(2\sec^2 x + x^3 + 2\right)\ dx$ (3) $\int_0^{\pi/3} \frac{x}{1+\sec x}\ dx$

Ans. (1) $5 - \frac{5}{2}\left(9\ell n\ \frac{5}{4} - \ell n\ \frac{3}{2}\right)$ (2) $\frac{\pi^4}{1024} + \frac{\pi}{2} + 2$ (3) $\frac{\pi^2}{18} - \frac{\pi}{3\sqrt{3}} + 2\ \ell n\left(\frac{2}{\sqrt{3}}\right)$

Property (1) $\int_a^b f(x)\ dx = \int_a^b f(t)\ dt$

i.e. definite integral is independent of variable of integration.

Property (2) $\int_a^b f(x)\ dx = -\int_b^a f(x)\, dx$

Property (3) $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$, where c may lie inside or outside the interval [a, b].

Example # 2 : If $f(x) = \begin{cases} x+3 & : \ x < 3 \\ 3x^2+1 & : \ x \ge 3 \end{cases}$, then find $\int_2^5 f(x)\ dx$.

Solution $\int_2^5 f(x)\ dx = \int_2^3 f(x)\ dx + \int_3^5 f(x)\, dx = \int_2^3 (x+3)\ dx + \int_3^5 (3x^2+1)\ dx = \left[\frac{x^2}{2} + 3x\right]_2^3 + \left[x^3 + x\right]_3^5$

$= \frac{9-4}{2} + 3\ (3 - 2) + 5^3 - 3^3 + 5 - 3 = \frac{211}{2}$

Example # 3 : Evaluate $\int_2^8 |x-5|\ dx$.

Solution : $\int_2^8 |x-5|\, dx = \int_2^5 (-x+5)\ dx + \int_5^8 (x-5)\, dx = 9$

Example # 4 : Show that $\int_0^2 (2x+1)\ dx = \int_0^5 (2x+1) + \int_5^2 (2x+1)$

Solution : L.H.S. = $x^2 + x]_0^2$ = 4 + 2 = 6 ; R.H.S. = 25 + 5 – 0 + (4 + 2) – (25 + 5) = 6

$\therefore$ L.H.S. = R.H.S

Self practice problems :

Evaluate the following

(4) $\int_0^4 (|x-1|+|x-3|)dx$ (5) $\int_{-2}^4 [x]dx$, where [x] is integral part of x.

(6) $\int_0^9 \left[\sqrt{t}\right]$ dt.

Ans. (4) 10 (5) 3 (6) 13

Property (4) $\int_{-a}^{a} f(x)\ dx = \int_0^a (f(x)+f(-x))\ dx = \begin{cases} 2\int_0^a f(x)\ dx, & \text{if } f(-x)=f(x) \text{ i.e. } f(x) \text{ is even} \\ 0, & \text{if } f(-x)=-f(x) \text{ i.e. } f(x) \text{ is odd} \end{cases}$

Example # 5 : Evaluate $\int_{-1}^{1} \frac{3^x+3^{-x}}{1+3^x}\ dx$

Solution : $\int_{-1}^{1} \frac{3^x+3^{-x}}{1+3^x}\ dx = \int_0^1 \left(\frac{3^x+3^{-x}}{1+3^x}+\frac{3^{-x}+3^x}{1+3^{-x}}\right) dx = \int_0^1 \left(\frac{3^x+3^{-x}}{1+3^x}+\frac{3^x(3^{-x}+3^x)}{1+3^x}\right)$

$= \int_0^1 (3^x+3^{-x})\ dx = \left(\frac{3^x}{\ell n3}-\frac{3^{-x}}{\ell n3}\right)_0^1 = \left(\frac{3}{\ell n3}-\frac{3^{-1}}{\ell n3}\right) - \left(\frac{1}{\ell n3}-\frac{1}{\ell n3}\right) = \frac{1}{\ell n3}\left[3-\frac{1}{3}\right] = \frac{8}{3\ell n3}$

Example # 6 : Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\ dx.$

Solution : $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\ dx = 2\int_0^{\frac{\pi}{2}} \cos x\ dx = 2$ ($\because$ cos x is even function)

Example # 7 : Evaluate $\int_{-1}^{1} \log_e\left(\frac{2-x}{2+x}\right) dx.$

Solution : Let $f(x) = \log_e\left(\frac{2-x}{2+x}\right)$ $\Rightarrow$ $f(-x) = \log_e\left(\frac{2+x}{2-x}\right) = -\log_e\left(\frac{2-x}{2+x}\right) = -f(x)$

i.e. f(x) is odd function $\therefore$ $\int_{-1}^{1} \log_e\left(\frac{2-x}{2+x}\right) dx = 0$

Self practice problems :

Evaluate the following

(7) $\int_{-1}^{1} |x|\ dx.$ (8) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\ dx.$ (9) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec x dx}{1+2^x}\ dx.$

Ans. (7) 1 (8) 0 (9) $\ell n\left(\sqrt{2}+1\right)$

Property (5) $\int_a^b f(x)\ dx = \int_a^b f(a+b-x)\ dx.$ Further $\int_0^a f(x) dx = \int_0^a f(a-x)\ dx$

Example # 8 : Prove that $\int_0^{\frac{\pi}{2}} \frac{g(\sin x)}{g(\sin x)+g(\cos x)}\, dx = \int_0^{\frac{\pi}{2}} \frac{g(\cos x)}{g(\sin x)+g(\cos x)}\, dx = \frac{\pi}{4}$.

Solution : Let $I = \int_0^{\frac{\pi}{2}} \frac{g(\sin x)}{g(\sin x)+g(\cos x)}\, dx \Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{g\left(\sin\left(\frac{\pi}{2}-x\right)\right)}{g\left(\sin\left(\frac{\pi}{2}-x\right)\right)+g\left(\cos\left(\frac{\pi}{2}-x\right)\right)} dx$

$$= \int_0^{\frac{\pi}{2}} \frac{g(\cos x)}{g(\cos x)+g(\sin x)} dx$$

on adding, we obtain

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{g(\sin x)}{g(\sin x)+g(\cos x)} + \frac{g(\cos x)}{g(\cos x)+g(\sin x)}\right) dx = \int_0^{\frac{\pi}{2}} dx \quad \Rightarrow I = \frac{\pi}{4}$$

Self practice problems:

Evaluate the following

(10) $\int_0^{\pi} \frac{x}{1+\sin x}\, dx.$ (11) $\int_0^{\frac{\pi}{2}} \frac{x}{\sin x+\cos x}\, dx.$

(12) $\int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sin^4 x+\cos^4 x}\, dx.$ (13) $\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{dx}{1+\sqrt{\cot x}}$

Ans. (10) π (11) $\frac{\pi}{2\sqrt{2}} \log_e\left(1+\sqrt{2}\right)$ (12) $\frac{\pi^2}{16}$ (13) $\frac{\pi}{6}$

Property (6) $\int_0^{2a} f(x)\, dx = \int_0^{a} (f(x)+f(2a-x))\, dx = \begin{cases} 2\int_0^{a} f(x)\, dx, & \text{if } f(2a-x)=f(x) \\ 0, & \text{if } f(2a-x)=-f(x) \end{cases}$

Example # 9 : Evaluate $\int_0^{\pi} \cot x . \cos 2x\, dx$

Solution : Let f(x) = cotx cos2x

$\Rightarrow f(\pi - x) = \cot(\pi-x)\cos 2(\pi-x) = -\cot x \cos 2x = -f(x)$

$\therefore \int_0^{\pi} \cot x \cos 2x\, dx = 0$

Example # 10 : Evaluate $\int_0^{\pi} \frac{dx}{1+3\cos^2 x}\, dx.$

Solution : Let $f(x) = \frac{1}{1+3\cos^2 x} \Rightarrow f(\pi - x) = f(x) \Rightarrow \int_0^{\pi} \frac{dx}{1+3\cos^2 x}$

$$= 2\int_0^{\frac{\pi}{2}} \frac{dx}{1+3\cos^2 x} = 2\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\, dx}{1+\tan^2 x+3} = 2\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\, dx}{4+\tan^2 x} = \left[\tan^{-1}\left(\frac{\tan x}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$\because \tan\frac{\pi}{2}$ is undefined, we take limit $= \lim_{x\to\pi/2-} \tan^{-1}\left(\frac{\tan x}{2}\right) - \tan^{-1}\left(\frac{\tan 0}{2}\right) = \pi/2 - 0 = \pi/2$

Example # 11 : Evaluate : $\int_0^{\infty} (\cot^{-1} x)^2 dx$

Solution : Let $I = \int_0^{\infty} (\cot^{-1} x)^2 dx \Rightarrow$ Let $x = \cot\theta \Rightarrow dx = -\operatorname{cosec}^2\theta\, d\theta$

$$\therefore\ I = \int_{\pi/2}^{0} \theta^2(-\operatorname{cosec}^2\theta)d\theta \Rightarrow \quad I = \int_0^{\pi/2} \theta^2(\operatorname{cosec}^2\theta)d\theta$$

$$= \left(\theta^2(-\cot\theta)\right)_0^{\pi/2} + 2\int_0^{\pi/2} \theta \ \cot\theta \ d\theta \quad \Rightarrow \quad I = 0 + 2\int_0^{\pi/2} \theta \ \cot\theta \ d\theta$$

$$= \left(2\theta \ell n \sin\theta\right)_0^{\pi/2} - 2\int_0^{\pi/2} \ell n \sin\theta \ d\theta \quad \left\{\begin{matrix}\text{Standard result} \\ \int_0^{\pi/2} \ell n \sin\theta d\theta = \frac{-\pi}{2}\ell n2\end{matrix}\right. \quad = 0 - 2 \times \left(-\frac{\pi}{2}\right)\ell n2 = \pi \ell n2.$$

Self practice problems :

Evaluate the following

(14) $\int_0^{\infty} \left(\frac{\ell n\ (1+x^2)}{1+x^2}\right)dx.$ (15) $\int_0^{\infty} \frac{\tan^{-1}x}{x(1+x^2)}dx$ (16) $\int_0^1 \ell n\ \sin\left(\frac{\pi}{2}x\right) dx.$

Ans. (14) $\pi \ell n2$ (15) $\frac{\pi}{2}\ \ell n2$ (16) $-\ell n2$

Property (7) If f(x) is a periodic function with period T, then

(i) $\int_0^{nT} f(x)\ dx = n\int_0^{T} f(x)\,dx,\ n \in z$

(ii) $\int_a^{a+nT} f(x)\ dx = n\int_0^{T} f(x)\ dx,\ n \in z,\ a \in R$

(iii) $\int_{mT}^{nT} f(x)\ dx = (n-m)\int_0^{T} f(x)\ dx,\ m, n \in z$

(iv) $\int_{nT}^{a+nT} f(x)\ dx = \int_0^{a} f(x)\ dx,\ n \in z,\ a \in R$

(v) $\int_{a+nT}^{b+nT} f(x)\ dx = \int_a^{b} f(x)\ dx,\ n \in z,\ a, b \in R$

Example # 12 : Evaluate $\int_{-3}^{5} e^{\{x\}}\ dx$, where {.} denotes the fractional part function.

Solution : $\int_{-3}^{5} e^{\{x\}}\ dx = (5-(-3))\int_0^1 e^{\{x\}}dx = 8\int_0^1 e^x dx = 8\left(e^x\right)_0^1 = 8\ (e-1)$

Example # 13 : Evaluate $\sum_{n=1}^{1000} \int_{n-1}^{n} |\cos 2\pi x|dx$

Solution : $\int_0^1 |\cos 2\pi x|\ dx + \int_1^2 |\cos 2\pi x|\ dx + \ldots\ldots + \int_{999}^{1000} |\cos 2\pi x|\ dx = \int_0^{1000} |\cos 2\pi x|\ dx$

Now $|\cos 2\pi x|$ is a periodic function of period 1/2

$I = 2000\int_0^{1/2} |\cos 2\pi x|\,dx \Rightarrow I = 2000 \times 2 = 4000$

Self practice problems :

Evaluate the following

(17) $\int_{-1}^{41/2} e^{2x-[2x]}dx$, where [•] denotes the greatest integer function.

(18) $\int_0^{\frac{14\pi}{3}} |\sin x|\,dx$ (19) $\int_{\pi}^{\frac{3\pi}{2}} (\sin^4 x + \cos^4 x)dx$

Ans. (17) $\frac{43}{2}$ (e – 1) (18) $\frac{19}{2}$ (19) $\frac{3\pi}{8}$

Leibnitz Theorem : If $F(x) = \int_{g(x)}^{h(x)} f(t)\,dt$, then $\frac{dF(x)}{dx} = h'(x)\, f(h(x)) - g'(x)\, f(g(x))$

Proof : Let $P(t) = \int f(t)\,dt \Rightarrow F(x) = \int_{g(x)}^{h(x)} f(t)\,dt = P(h(x)) - P(g(x))$

$$\Rightarrow \frac{dF(x)}{dx} = P'(h(x))\, h'(x) - P'(g(x))\, g'(x) = f(h(x))\, h'(x) - f(g(x))\, g'(x)$$

Example # 14 : If $F(x) = \int_{x}^{x^2} \sqrt{\tan t}\,dt$, then find F′(x).

Solution : $F'(x) = 2x \,.\, \sqrt{\tan x^2} - 1.\, \sqrt{\tan x}$

Example # 15 : If $F(x) = \int_{x^2}^{x^3} \frac{1}{\ell nt}dt$ then find F'(e)

Solution : $F'(x) = \frac{3x^2}{\ell nx^3} - \frac{2x}{\ell nx^2} = \frac{x^2}{\ell nx} - \frac{x}{\ell nx} = \frac{x(x-1)}{\ell nx}$ now $F'(e) = \frac{e(e-1)}{\ell ne} = e(e-1)$

Example # 16 : Evaluate : $\lim_{x\to 0^+} \frac{\int_0^{x^2} \sin\sqrt{t}\ \tan\sqrt{t}dt}{x^4}$

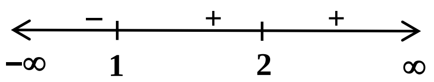
Solution : Applying L' hospital rule

$$\lim_{x\to 0^+} \frac{2x\sin x\tan x}{4x^3} \Rightarrow \lim_{x\to 0} \frac{1}{2}\left(\frac{\sin x}{x}\right)\left(\frac{\tan x}{x}\right) = \frac{1}{2}$$

Example # 17 : Let $f(x) = \int_0^x (t-1)(t-2)^2\,dt$, then find a point of minimum

Solution : $f(x) = \int_0^x (t-1)(t-2)^2\,dt$

$f'(x) = (x - 1)(x - 2)^2$



$\Rightarrow$ x = 1 is the point of minimum

$f(1) = \int_0^1 (t^3 - 5t^2 + 8t - 4)dt = \frac{1}{4} - \frac{5}{3} + 4 - 4 = -\frac{17}{12}$. Hence $(1, -\frac{17}{12})$ is a point of minimum

Example # 18 : Evaluate , $\int_0^1 \frac{x^b - 1}{\ell nx}$ 'b' being parameter.

Solution : Let $I(b) = \int_0^1 \frac{x^b-1}{\ell nx}\,dx \Rightarrow \frac{dI(b)}{db} = \int_0^1 \frac{x^b \ell nx}{\ell nx}\,dx + 0 - 0$ (using modified Leibnitz Theorem)

$= \int_0^1 x^b\,dx = \left.\frac{x^{b+1}}{b+1}\right]_0^1 = \Rightarrow I(b) = \ell n\ (b + 1) + c$

b = 0 $\Rightarrow$ I(0) = 0 $\therefore$ c = 0 $\therefore$ I(b) = ℓn (b+1)

Example # 19 : Evaluate $\int_0^1 \frac{\tan^{-1}(ax)}{x\sqrt{1-x^2}}$ dx , 'a' being parameter.

Solution : Let $I(a) = \int_0^1 \frac{\tan^{-1}(ax)}{x\sqrt{1-x^2}}\,dx \Rightarrow \frac{dI(a)}{da} = \int_0^1 \frac{x}{(1+a^2x^2)}\frac{1}{x\sqrt{1-x^2}}\,dx = \int_0^1 \frac{dx}{(1+a^2x^2)\sqrt{1-x^2}}$

Put x = sin t ⇒ dx = cos t dt

L.L. : x = 0 ⇒ t = 0

U.L. : x = 1 ⇒ $t = \frac{\pi}{2}$

$$\frac{dI(a)}{da} = \int_0^{\pi/2} \frac{1}{1+a^2\sin^2 t}\,\frac{1}{\cos t}\,\cos t\,dt = \int_0^{\pi/2}\frac{dt}{1+a^2\sin^2 t}$$

$$= \int_0^{\pi/2}\frac{\sec^2 t\,dt}{1+(1+a^2)\tan^2 t} = \frac{1}{\sqrt{1+a^2}}\tan^{-1}\left(\sqrt{1+a^2}\,\tan t\right)\Big]_0^{\pi/2} = \frac{1}{\sqrt{1+a^2}}\cdot\frac{\pi}{2}$$

$$\Rightarrow I(a) = \frac{\pi}{2}\ln\left(a+\sqrt{1+a^2}\right) + c \text{ But } I(0) = 0 \Rightarrow c = 0 \Rightarrow I(a) = \frac{\pi}{2}\ln\left(a+\sqrt{1+a^2}\right)$$

Self Practice Problems :

(20) If $f(x) = \int_0^{x^3}\sqrt{\cos t}\,dt$, find f′(x).

(21) Find the equation of tangent to the y = F(x) at x = 1 , where $F(x) = \int_x^{x^3}\frac{dt}{\sqrt{1+t^4}}$

(22) If $\int_0^x f(t)dt = x^2 - \int_x^{x^3}\frac{f(t)}{t}dt$ then find f(1)

(23) If $f(x) = \int_x^{x^2} x^2 \ln t\,dt$ then find f′(e)

(24) If $y = \int_4^{4x^2} t^4 e^{4t}dt$, Find $\frac{d^2y}{dx^2}$

(25) If $y = \int_0^{x^2}\ln(1+t)dt$, then find $\frac{d^2y}{dx^2}$

(26) If $\int_0^{x^2(1+x)} f(t)dt = x$ then find f(2) (27) Evaluate $\int_0^{\pi}\ln(1+b\cos x)\,dx$, 'b' being parameter.

Ans. (20) $3x^2\sqrt{\cos x^3}$ (21) $\sqrt{2}\,x - y = \sqrt{2}$ (22) 2/3

(23) $e^2(6e-1)$ (24) $2048\,e^{16x^2}$

(25) $\frac{2}{1+x^2}\,[2x^2 + (1 + x^2)\ln(1 + x^2)]$ (26) $\frac{1}{5}$ (27) $\pi\ln\left(\frac{1+\sqrt{1-b^2}}{2}\right)$

Reduction formulae in definite Integrals:

1. If $I_n = \int_0^{\pi/2}\sin^n x\,dx$, then show that $I_n = \left(\frac{n-1}{n}\right)I_{n-2}$

Proof : $I_n = \int_0^{\pi/2}\sin^n x\,dx$

$$I_n = \left[-\sin^{n-1}x\,\cos x\right]_0^{\pi/2} + \int_0^{\pi/2}(n-1)\sin^{n-2}x\,.\,\cos^2 x\,dx = (n-1)\int_0^{\pi/2}\sin^{n-2}x\,.\,(1-\sin^2 x)\,dx$$

$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \ dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^{n} x \ dx \ \Rightarrow I_n + (n-1) I_n = (n-1) I_{n-2}$

$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

Note : 1. $\int_0^{\frac{\pi}{2}} \sin^n x \ dx = \int_0^{\frac{\pi}{2}} \cos^n x \ dx$

2. $I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)$ ..... I_0 or I_1 according as n is even or odd. $I_0 = \frac{\pi}{2}$, $I_1 = 1$

Hence $I_n = \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)........\left(\frac{1}{2}\right) . \frac{\pi}{2}, & \text{if n is even} \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)........\left(\frac{2}{3}\right) . 1, & \text{if n is odd} \end{cases}$

2. If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \ dx$, then show that $I_n + I_{n-2} = \frac{1}{n-1}$

Proof : $I_n = \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} . \tan^2 x \, dx = \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} (\sec^2 x - 1) \, dx$

$= \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} \ dx = \left[\frac{(\tan x)^{n-1}}{n-1}\right]_0^{\frac{\pi}{4}} - I_{n-2}$

$I_n = \frac{1}{n-1} - I_{n-2}$ $\quad\therefore\quad$ $I_n + I_{n-2} = \frac{1}{n-1}$

3. If $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \ . \ \cos^n x \ dx$, then show that $I_{m,n} = \frac{m-1}{m+n} \ I_{m-2,\,n}$

Proof : $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^{m-1} x \ (\sin x \ \cos^n x) \ dx = \left[-\frac{\sin^{m-1} x \ . \ \cos^{n+1} x}{n+1}\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\cos^{n+1} x}{n+1} \ (m-1) \sin^{m-2} x \cos x \, dx$

$= \left(\frac{m-1}{n+1}\right)\int_0^{\frac{\pi}{2}} \sin^{m-2} x.\cos^n x.\cos^2 x \, dx = \left(\frac{m-1}{n+1}\right)\int_0^{\frac{\pi}{2}} \left(\sin^{m-2} x.\cos^n x - \sin^m x.\cos^n x\right) \ dx$

$= \left(\frac{m-1}{n+1}\right) I_{m-2,n} - \left(\frac{m-1}{n+1}\right) I_{m,n} \ \Rightarrow \ \left(1+\frac{m-1}{n+1}\right) I_{m,n} = \left(\frac{m-1}{n+1}\right) I_{m-2,n}$

$I_{m,n} = \left(\frac{m-1}{m+n}\right) I_{m-2,n}$

Note : 1. $I_{m,n} = \left(\frac{m-1}{m+n}\right)\left(\frac{m-3}{m+n-2}\right)\left(\frac{m-5}{m+n-4}\right)$ .... $I_{0,n}$ or $I_{1,n}$ according as m is even or odd.

$I_{0,n} = \int_0^{\frac{\pi}{2}} \cos^n x dx$ and $I_{1,n} = \int_0^{\frac{\pi}{2}} \sin x.\cos^n x dx = \frac{1}{n+1}$

2. Walli's Formula

$$I_{m,n} = \begin{cases} \dfrac{(m-1)\ (m-3)\ (m-5)\ ....(n-1)\ (n-3)\ (n-5).......}{(m+n)\ (m+n-2)\ (m+n-4)........}\ \dfrac{\pi}{2} & \text{when both } m, n \text{ are even} \\[2ex] \dfrac{(m-1)\ (m-3)\ (m-5)\ ....(n-1)\ (n-3)\ (n-5).......}{(m+n)\ (m+n-2)\ (m+n-4)........} & \text{otherwise} \end{cases}$$

Example # 20 : Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \ \cos^2 x(\sin x \ + \ \cos x)\ dx$.

Solution : Given integral = $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x \ \cos^2 x \ dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \ \cos^3 x \ dx$

$= 0 + 2\int_0^{\frac{\pi}{2}} \sin^2 x \ \cos^3 x \ dx$ ($\because$ $\sin^3x \cos^2x$ is odd and $\sin^2x \cos^3x$ is even)

$= 2.\ \dfrac{1.2}{5.3.1} = \dfrac{4}{15}$

Example # 21 : Evaluate $\int_0^{\pi} x \sin^7 x \cos^6 x \ dx$

Solution : Let I = $\int_0^{\pi} x \sin^7 x \cos^6 x \ dx$

I = $\int_0^{\pi} (\pi - x)\ \sin^7(\pi - x)\ \cos^6(\pi - x)\ dx = \pi \int_0^{\pi} \sin^7 x \cos^6 x\, dx - \int_0^{\pi} x \sin^7 x \cos^6 x \ dx$

$\Rightarrow 2I = \pi\ .\ 2 \int_0^{\frac{\pi}{2}} \sin^7 x.\cos^6 x dx \Rightarrow I = \pi\ \dfrac{6.4.2.5.3.1}{13.11.9.7.5.3.1} \ \Rightarrow\ I = \dfrac{48\pi}{9009}$

Example # 22 : Evaluate : $\int_0^a x^{5/2}\sqrt{a-x}\,dx$

Solution : Put x = a $\sin^2\theta$ $\Rightarrow$ dx = 2a $\sin\theta \cos\theta\ d\theta$

Lower limit : x = 0 $\Rightarrow$ $\theta = 0$

Upper limit x = a $\Rightarrow \theta = \dfrac{\pi}{2}$.

$$\int_0^a x^{5/2}\sqrt{a-x}\,dx = \int_0^{\frac{\pi}{2}} 2a^4 \sin^6\theta \cos^2\theta\, d\theta = 2a^4 x \frac{\pi}{2}.\frac{(5.3.1)(1)}{8.6.4.2} = \frac{5\pi a^4}{128}$$

Self Practice Problems:

Evaluate the following

(28) $\int_0^{\frac{\pi}{2}} \sin^{11} x \ dx$.

(29) $\int_0^{\frac{\pi}{2}} \sin^5 x \ \cos^4 x \ dx$.

(30) $\int_0^1 x^6 \ \sin^{-1} x \ dx$.

(31) $\int_0^a x \ \left(a^2 - x^2\right)^{\frac{7}{2}} \ dx$.

(32) $\int_0^2 x^{3/2} \ \sqrt{2-x} \ dx$.

Ans. (28) $\frac{128}{693}$ (29) $\frac{8}{315}$ (30) $\frac{\pi}{14} - \frac{16}{245}$ (31) $\frac{a^9}{9}$ (32) $\frac{\pi}{2}$

Property (8) If $\psi(x) \le f(x) \le \phi(x)$ for $a \le x \le b$, then

$$\int_a^b \psi(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b \phi(x)\,dx$$

Property (9) If $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

Further if f(x) is monotonically decreasing in (a, b), then $f(b)(b-a) < \int_a^b f(x)\,dx < f(a)(b-a)$ and if f(x) is monotonically increasing in (a, b), then $f(a)(b-a) < \int_a^b f(x)\,dx < f(b)(b-a)$

Property (10) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

Property (11) If $f(x) \ge 0$ on [a, b], then $\int_a^b f(x)\,dx \ge 0$

Example # 23 : For $x \in (0, 1)$ arrange $f_1(x) = \frac{1}{9-x^2}$, $f_2(x) = \frac{1}{9-2x^2}$ and $f_3(x) = \frac{1}{9-x^2-x^3}$ in ascending order and hence prove that $\frac{1}{6}\ell n2 < \int_0^1 \frac{1}{9-x^2-x^3}dx < \frac{1}{6\sqrt{2}}\ell n5$

Solution : $\because$ $0 < x^3 < x^2$, for all $x \in (0,1)$ $\Rightarrow$ $x^2 < x^2 + x^3 < 2x^2$

$\Rightarrow$ $-2x^2 < -x^2 - x^3 < -x^2$ $\Rightarrow$ $9 - 2x^2 < 9 - x^2 - x^3 < 9 - x^2$

$\Rightarrow$ $\frac{1}{9-x^2} < \frac{1}{9-x^2-x^3} < \frac{1}{9-2x^2}$

$f_1(x) < f_3(x) < f_2(x)$ for $x \in (0, 1)$

$\Rightarrow$ $\int_0^1 f_1(x)\,dx < \int_0^1 f_3(x)\,dx < \int_0^1 f_2(x)\,dx$

$\Rightarrow$ $\int_0^1 \frac{dx}{9-x^2} < \int_0^1 \frac{dx}{9-x^2-x^3} < \int_0^1 \frac{dx}{9-2x^2}$

$\Rightarrow$ $\frac{1}{6}\left(\ell n\left|\frac{3+x}{3-x}\right|\right)_0^1 < \int_0^1 \frac{dx}{9-x^2-x^3} < \frac{1}{6\sqrt{2}}\left(\ell n\left|\frac{3+2x}{3-2x}\right|\right)_0^1$

$\Rightarrow$ $\frac{1}{6}\ell n2 < \int_0^1 \frac{1}{9-x^2-x^3}dx < \frac{1}{6\sqrt{2}}\ell n5$

Example # 24 : Prove that $1 < \int_0^2 \left(\frac{5-x}{9-x^2}\right)dx < \frac{6}{5}$

Solution : Let $f(x) = \frac{5-x}{9-x^2}$

$\therefore f'(x) = -\frac{(x-9)(x-1)}{(9-x^2)^2}$ $\Rightarrow f'(x) = 0$ or not defined $\Rightarrow x = 1$

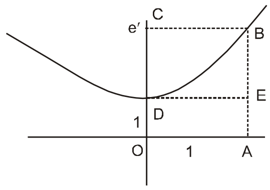
Then f(0) = 5/9, f(1) = $\frac{1}{2}$, f(2) = 3/5 The greatest and least values of the integrand in the interval [0,2] are respectively, equal to f(2) = 3/5 and f(1) = $\frac{1}{2}$

$(2-0)\frac{1}{2} < \int_0^2 \left(\frac{5-x}{9-x^2}\right) dx < (2-0)\,3/5$ Hence $1 < \int_0^2 \left(\frac{5-x}{9-x^2}\right)dx < \frac{6}{5}$

Example # 25 : Estimate the value of $\int_0^1 e^{x^2} dx$ using (i) rectangle, (ii) triangle.

Solution : (i) By using rectangle

Area OAED < $\int_0^1 e^{x^2} dx$ < Area OABC



$1 < \int_0^1 e^{x^2} dx < 1 . e$

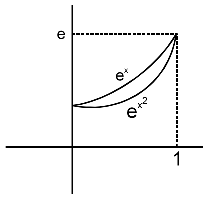
$1 < \int_0^1 e^{x^2} dx < e$

(ii) By using triangle

Area OAED < $\int_0^1 e^{x^2} dx$ < Area OAED + Area of triangle DEB

$1 < \int_0^1 e^{x^2} dx < 1 + \frac{1}{2}.1.(e-1)$ $\qquad 1 < \int_0^1 e^{x^2} dx < \frac{e+1}{2}$

Example # 26 : Estimate the value of $\int_0^1 e^{x^2} dx$ by using $\int_0^1 e^{x} dx$.



Solution : For $x \in (0, 1)$, $e^{x^2} < e^x$

$\Rightarrow \quad 1 \times 1 < \int_0^1 e^{x^2} dx < \int_0^1 e^{x} dx$

$1 < \int_0^1 e^{x^2} dx < e - 1$

Self practice problems :

(33) Prove the following : $\int_0^1 e^{-x} \cos^2 x\, dx < \int_0^1 e^{-x^2} \cos^2 x\, dx$

(34) Prove the following : $0 < \int_0^{\frac{\pi}{2}} \sin^{n+1} x\, dx < \int_0^{\frac{\pi}{2}} \sin^2 x\, dx$, n > 1

(35) Prove the following : $e^{-\frac{1}{e}} < \int_0^1 x^x dx < 1$

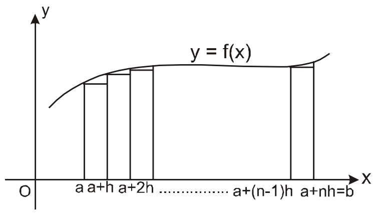
(36) Prove the following : $-\frac{1}{2} \le \int_0^1 \frac{x^3 \cos x}{2+x^2} dx < \frac{1}{2}$

(37) Prove the following : $1 < \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx < \sqrt{\frac{\pi}{2}}$

(38) Prove the following : $\frac{4}{\pi} < \int_{\pi/4}^{\pi/3} \frac{\tan x}{x} dx < \frac{3\sqrt{3}}{\pi}$

Definite Integral as a limit of sum

Let f(x) be a continuous real valued function defined on the closed interval [a, b] which is divided into n parts as shown in figure.



The point of division on x-axis are a, a + h, a + 2h ....a + (n – 1)h, a + nh, where $\frac{b-a}{n}$ = h.

Let S_n denotes the area of these n rectangles.

Then, S_n = hf(a) + hf(a + h) + hf(a + 2h) + ...+hf(a + (n – 1)h)

Clearly, S_n is area very close to the area of the region bounded by curve y = f(x), x–axis and the ordinates x = a, x = b.

Hence $\int_a^b f(x)\ dx = \underset{n\to\infty}{\text{Lt}}\ S_n$

$$\int_a^b f(x)\ dx = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} h\ f(a+rh) = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} \left(\frac{b-a}{n}\right) f\left(a+\frac{(b-a)\ r}{n}\right)$$

Note :

1. We can also write

S_n = hf(a + h) + hf (a + 2h) + ...+ hf(a + nh) and $\int_a^b f(x)\ dx = \underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{n} \left(\frac{b-a}{n}\right) f\left(a+\left(\frac{b-a}{n}\right) r\right)$

2. If a = 0, b = 1, $\int_0^1 f(x)\ dx = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} \frac{1}{n}\ f\left(\frac{r}{n}\right)$

Steps to express the limit of sum as definite integral :

Step 1. Replace $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx and $\underset{n\to\infty}{\text{Lt}}$ by $\int$

Step 2. Evaluate $\underset{n\to\infty}{\text{Lt}}\left(\frac{r}{n}\right)$ by putting least and greatest values of r as lower and upper limits respectively.

For example $\underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{pn} \frac{1}{n}\ f\left(\frac{r}{n}\right) = \int_0^p f(x)\ dx$ $\left(\because \underset{n\to\infty}{\text{Lt}}\left(\frac{r}{n}\right)\Big|_{r=1} = 0,\ \underset{n\to\infty}{\text{Lt}}\left(\frac{r}{n}\right)\Big|_{r=np} = p\right)$

Example # 27 : Evaluate $\underset{n\to\infty}{\text{Lt}}\left[\frac{1}{1+n}+\frac{1}{2+n}+\frac{1}{3+n}+.........+\frac{1}{10n}\right]$

Solution : $\underset{n\to\infty}{\text{Lt}}\left[\frac{1}{1+n}+\frac{1}{2+n}+\frac{1}{3+n}+.........+\frac{1}{10n}\right] = \underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{9n} \frac{1}{r+n}$

$$= \underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{9n} \frac{1}{n} \frac{1}{\left(\frac{r}{n}\right)+1} = \int_0^9 \frac{dx}{x+1} = \left[\ell n\ (x+1)\right]_0^9 = \ell n\ 10$$

Example # 28 : Evaluate $\underset{n\to\infty}{\text{Lt}}\left[\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\frac{n+3}{n^2+3^2}+........+\frac{1}{n}\right]$.

Solution : $\underset{n\to\infty}{\text{Lt}}\sum_{r=1}^{n}\frac{n+r}{n^2+r^2} = \underset{n\to\infty}{\text{Lt}}\sum_{r=1}^{n}\frac{1}{n}\frac{1+\frac{r}{n}}{1+\left(\frac{r}{n}\right)^2}$ $\because$ $\underset{n\to\infty}{\text{Lt}}\left(\frac{r}{n}\right)$ = 0, when r = 1, lower limit = 0

and $\underset{n\to\infty}{\text{Lt}}\left(\frac{r}{n}\right) = \underset{n\to\infty}{\text{Lt}}\left(\frac{n}{n}\right)$ = 1, when r = n, upper limit = 1

$$\int_0^1\frac{1+x}{1+x^2}\,dx = \int_0^1\frac{1}{1+x^2}\,dx + \frac{1}{2}\int_0^1\frac{2x}{1+x^2}\,dx = [\tan^{-1}x]_0^1 + \left[\frac{1}{2}\log_e(1+x^2)\right]_0^1$$

$$= \frac{\pi}{4}+\frac{1}{2}\ell n\ 2$$

Example # 29 : Evaluate : $\lim\limits_{n\to\infty}\left(\frac{(2n)\ !}{n\ !\ n^n}\right)^{\frac{1}{n}}$

Solution : Let y = $\lim\limits_{n\to\infty}\left(\frac{(2n)!}{n!\,n^n}\right)^{\frac{1}{n}}$ $\Rightarrow$ $\ell n\ y = \lim\limits_{n\to\infty}\frac{1}{n}\ell n\left(\frac{(2n)!}{n!\,n^n}\right)$

$$= \lim_{n\to\infty}\frac{1}{n}\ell n\left(\frac{2n(2n-1)(2n-2)....(n+1)}{n^n}\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}[\ell n(1 + r/n)] = \int_0^1\ell n(1+x)dx = (x\ell n(1+x))_0^1 - \int_0^1\frac{x}{1+x}dx$$

$$= (x\ell n(1+x))_0^1 - (x-\ell n(1+x))_0^1 = \ell n2 - (1-\ell n2) = \ell n4/e \Rightarrow y = 4/e$$

Self Practice Problems :

Evaluate the following limits

(39) $\lim\limits_{n\to\infty}\left\{\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+.....+\frac{1}{8n}\right\}$

(40) $\lim\limits_{n\to\infty}\left[\frac{1}{1+n}+\frac{1}{2+n}+\frac{1}{3+n}+.......+\frac{1}{5n}\right]$

(41) $\lim\limits_{n\to\infty}\frac{1}{n^2}\left[\sin^3\frac{\pi}{4n}+2\sin^3\frac{2\pi}{4n}+3\sin^3\frac{3\pi}{4n}+........+n\sin^3\frac{n\pi}{4n}\right]$

(42) $\lim\limits_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{\sqrt{n^2-r^2}}$

(43) $\lim\limits_{n\to\infty}\left(\tan\frac{\pi}{2n}\ \tan\frac{2\pi}{2n}\ \tan\frac{3\pi}{2n}......\tan\frac{n\pi}{2n}\right)^{\frac{1}{n}}$

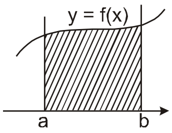
Ans. (39) $\frac{3}{8}$ (40) $\ell n\ 5$ (41) $\frac{\sqrt{2}}{9\pi^2}(52-15\pi)$

(42) $\frac{\pi}{2}$ (43) 1

Area Between The Curve :

Area included between the curve y = f(x), x-axis and the ordinates x = a, x = b

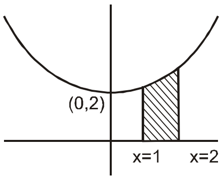
(a) If $f(x) \geq 0$ for $x \in [a, b]$, then area bounded by curve y = f(x), x-axis, x = a and x = b is $\int_a^b f(x)\, dx$



Graph of y = f(x)

Example # 30 : Find the area enclosed between the curve $y = x^2 + 2$, x-axis, x = 1 and x = 2.

Solution :



Graph of $y = x^2 + 2$

$$\text{Area} = \int_1^2 (x^2 + 2)\, dx = \left[\frac{x^3}{3} + 2x\right]_1^2 = \frac{13}{3}$$

Example # 31 : Find area bounded by the curve $y = \ell n\, x + \tan^{-1} x$ and x-axis between ordinates x = 1 and x = 2.

Solution : $y = \ell n\, x + \tan^{-1}x$

Domain $x > 0$, $\frac{dy}{dx} = \frac{1}{x} + \frac{1}{1+x^2} > 0$

y is increasing and x = 1, $y = \frac{\pi}{4}$ $\Rightarrow$ y is positive in [1, 2]

$\therefore$ Required area $= \int_1^2 (\ell n\ x + \tan^{-1} x)\ dx = \left[x \ell n x - x + x \tan^{-1} x - \frac{1}{2} \ell n(1 + x^2)\right]_1^2$

$= 2\, \ell n\, 2 - 2 + 2 \tan^{-1}2 - \frac{1}{2}\ \ell n\, 5 - 0 + 1 - \tan^{-1} 1 + \frac{1}{2}\ \ell n\, 2$

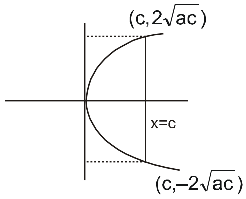
$= \frac{5}{2}\ \ell n\, 2 - \frac{1}{2}\ \ell n\, 5 + 2 \tan^{-1}2 - \frac{\pi}{4} - 1$

Note : If a function is known to be positive valued then graph is not necessary.

Example # 32 : The area cut off from a parabola by any double ordinate is k times the corresponding rectangle contained by the double ordinate and its distance from the vertex. Find the value of k ?

Solution : Consider $y^2 = 4ax$, $a > 0$ and $x = c$

Area by double ordinate $= 2\int_0^c 2\sqrt{a}\sqrt{x}\ \ dx = \frac{8}{3}\sqrt{a}\ \ c^{3/2}$

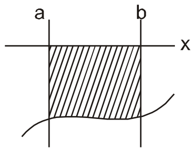


Figure

Area by double ordinate = k (Area of rectangle)

$\frac{8}{3}\sqrt{a}\ \ c^{3/2} = k\ \ 4\sqrt{a}\ \ c^{3/2} \Rightarrow k = \frac{2}{3}$

(b) If f(x) < 0 for $x \in [a, b]$, then area bounded by curve y = f(x), x-axis, x = a and x = b is $-\int_a^b f(x)\,dx$

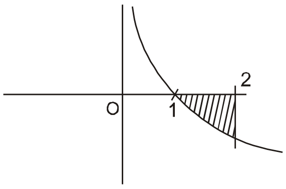


Graph of y = f(x)

Example # 33 : Find area bounded by $y = \log_{\frac{1}{2}} x$ and x-axis between x = 1 and x = 2

Solution : A rough graph of $y = \log_{\frac{1}{2}} x$ is as follows

$$\text{Area} = -\int_1^2 \log_{\frac{1}{2}} x\,dx = -\int_1^2 \log_e x \,.\, \log_{\frac{1}{2}} e\,dx$$



$$= -\log_{\frac{1}{2}} e \,.\, [x \log_e x - x]_1^2 = -\log_{\frac{1}{2}} e \,.\, (2\log_e 2 - 2 - 0 + 1) = -\log_{\frac{1}{2}} e \,.\, (2\log_e 2 - 1)$$

Note : If y = f(x) does not change sign in [a, b], then area bounded by y = f(x), x-axis between ordinates x = a, x = b is $\left|\int_a^b f(x)\,dx\right|$

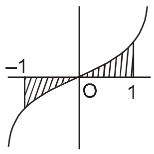
(c) If $f(x) \geq 0$ for $x \in [a,c]$ and $f(x) \leq 0$ for $x \in [c,b]$ (a < c < b) then area bounded by curve y = f(x) and x-axis between x = a and x = b is $\int_a^c f(x)\,dx - \int_c^b f(x)\,dx$

Example # 34 : Find the area bounded by $y = x^3$ and x- axis between ordinates x = –1 and x = 1

Solution : Required area = $\int_{-1}^0 -x^3 dx + \int_0^1 x^3 dx$

$$= \left[-\frac{x^4}{4}\right]_{-1}^0 + \left[\frac{x^3}{4}\right]_0^1$$

$$= 0 - \left(-\frac{1}{4}\right) + \frac{1}{4} - 0 = \frac{1}{2}$$



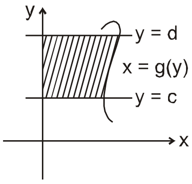
Graph of y = x^3

Note : Most general formula for area bounded by curve y = f(x) and x- axis between ordinates x = a and x = b is $\int_a^b |f(x)|\,dx$

Area included between the curve x = g(y), y-axis and the abscissas y = c, y = d

(a) If $g(y) \geq 0$ for $y \in [c,d]$ then area bounded by curve $x = g(y)$ and y–axis between abscissa $y = c$ and

$y = d$ is $\int\limits_{y=c}^{d} g(y)dy$

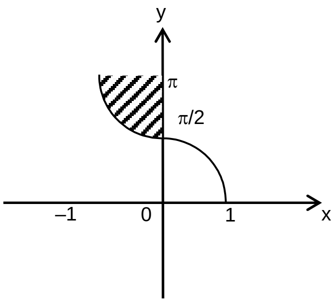


Graph of x = g(y)

Example # 35 : Find area bounded between $y = \cos^{-1}x$ and y–axis between $y = \frac{\pi}{2}$ and $y = \pi$.

Solution : $y = \cos^{-1} x \quad \Rightarrow \quad x = \cos y$

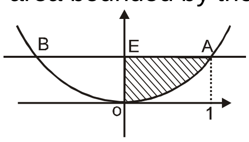
Required area $= -\int\limits_{\frac{\pi}{2}}^{\pi} \cos y \, dy$



Graph of $y = \cos^{-1}x$

$= - \sin y\Big]_{\frac{\pi}{2}}^{\pi} = 1$

Example # 36 : Find the area bounded by the parabola $x^2 = y$, y-axis and the line $y = 1$.

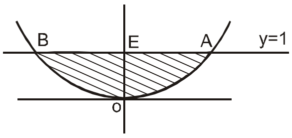


Solution : Graph of $y = x^2$

Area OEBO = Area OAEO = $\int\limits_{0}^{1} |x| \, dy = \int\limits_{0}^{1} \sqrt{y} \, dy = \frac{2}{3}$

Example # 37 : Find the area bounded by the parabola $x^2 = y$ and line $y = 1$.

Solution :



Graph of $y = x^2$

Required area is area OABO

$= 2$ area (OAEO) $= 2\int\limits_{0}^{1} |x| \, dy = 2\int\limits_{0}^{1} \sqrt{y} \, dy = \frac{4}{3}$.

Example # 38 : For any real t, $x = \frac{1}{2}(e^t + e^{-t})$, $y = \frac{1}{2}(e^t - e^{-t})$ is point on the hyperbola $x^2 - y^2 = 1$.Show that the area bounded by the hyperbola and the lines joining its centre to the points corresponding to t_1 and $-t_1$ is t_1.

Solution : It is a point on hyperbola $x^2 - y^2 = 1$.

$$\text{Area (PQRP)} = 2\int_{1}^{\frac{e^{t_1}+e^{-t_1}}{2}} y\,dx = 2\int_{1}^{\frac{e^{t_1}+e^{-t_1}}{2}} \sqrt{x^2-1}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ell n(x+\sqrt{x^2-1})\right]_1^{\frac{e^{t_1}+e^{-t_1}}{2}} = \frac{e^{2t_1}-e^{-2t_1}}{4} - t_1$$

$$\text{Area of } \Delta OPQ = 2\times \frac{1}{2}\left(\frac{e^{t_1}+e^{-t_1}}{2}\right)\left(\frac{e^{t_1}-e^{-t_1}}{2}\right) = \frac{e^{2t_1}-e^{-2t_1}}{4}$$

$\therefore$ Required area = area ΔOPQ – area (PQRP) = t_1

(**b**) If $g(y) \le 0$ for $y \in [c,d]$ then area bounded by curve $x = g(y)$ and y–axis between abscissa $y = c$ and $y = d$ is $-\int_{y=c}^{d} g(y)\,dy$

Note : General formula for area bounded by curve $x = g(y)$ and y–axis between abscissa $y = c$ and $y = d$ is $\int_{y=c}^{d} |g(y)|\,dy$

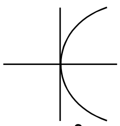
Curve-tracing :

To find approximate shape of a curve, the following phrases are suggested :

(a) Symmetry:

(i) Symmetry about x-axis :

If all the powers of 'y' in the equation are even then the curve (graph) is symmetrical about the x-axis.

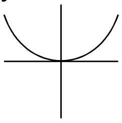


a > 0

E.g. : $y^2 = 4ax$.

(ii) Symmetry about y-axis :

If all the powers of 'x' in the equation are even then the curve (graph) is symmetrical about the y-axis.

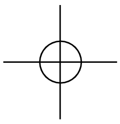


a > 0

E.g. : $x^2 = 4ay$.

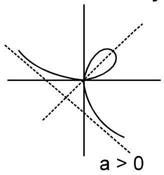
(iii) Symmetry about both axis :

If all the powers of 'x' and 'y' in the equation are even, then the curve (graph) is symmetrical about the axis of 'x' as well as 'y' .



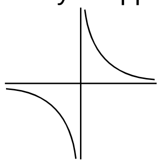
E.g. : $x^2 + y^2 = a^2$.

(iv) Symmetry about the line y = x :
If the equation of the curve remain unchanged on interchanging 'x' and 'y', then the curve (graph) is symmetrical about the line y = x.



E.g. : $x^3 + y^3 = 3axy$.

(v) Symmetry in opposite quadrants :
If the equation of the curve (graph) remain unaltered when 'x' and 'y' are replaced by '–x' and '–y' respectively, then there is symmetry in opposite quadrants.



E.g. : $xy = c^2$

(b) Find the points where the curve crosses the x-axis and the y-axis.

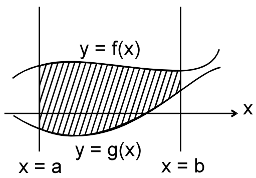
(c) Find $\frac{dy}{dx}$ and equate it to zero to find the points on the curve where you have horizontal tangents.

(d) Examine intervals when f(x) is increasing or decreasing

(e) Examine what happens to 'y' when $x \to \infty$ or $x \to -\infty$

Area between two curves

If $f(x) \geq g(x)$ for $x \in [a,b]$ then area bounded by curves (graph) y = f(x) and y = g(x) between ordinates x = a and x = b is $\int_a^b \left(f(x) - g(x)\right) dx$.



Example # 39 : Find the area enclosed by curve (graph) $y = x^2 + x + 1$ and its tangent at (1,3) between ordinates x = – 2 and x = 1.

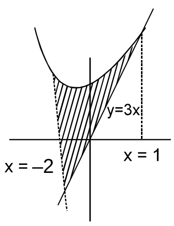
Solution : $\frac{dy}{dx} = 2x + 1$

$\frac{dy}{dx} = 3$ at x = 1

Equation of tangent is

$y - 3 = 3(x - 1)$

$y = 3x$



Required area $= \int_{-2}^{1} (x^2 + x + 1 - 3x)\ dx$

$$= \int_{-2}^{1} (x^2 - 2x + 1)\ dx = \left.\frac{x^3}{3} - x^2 + x\right]_{-2}^{1} = \left(\frac{1}{3} - 1 + 1\right) - \left[-\frac{8}{3} - 4 - 2\right] = 9$$

Note : Area bounded by curves y = f(x) and y = g (x) between ordinates x = a and x = b is $\int_a^b |f(x) - g(x)|\, dx$.

Example # 40 : Find the area of the region bounded by y = sin x, y = cos x and ordinates x = 0, x = π/2

Solution : $\int_0^{\pi/2} |\sin x - \cos x|\, dx$

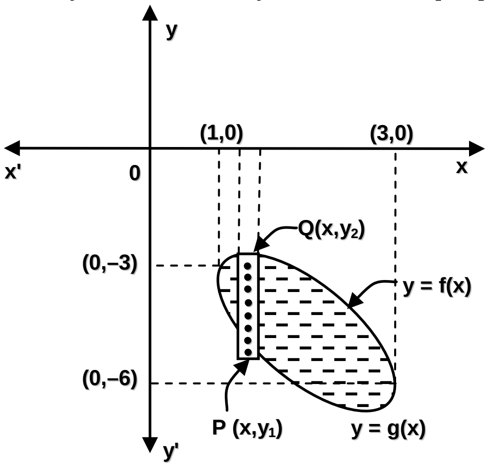
$\int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx = 2(\sqrt{2} - 1)$

Example # 41 : Find the area contained by ellipse $5x^2 + 6xy + 2y^2 + 7x + 6y + 6 = 0$

Solution : $5x^2 + 6xy + 2y^2 + 7x + 6y + 6 = 0$

$2y^2 + 6(1 + x)\, y + 5x^2 + 7x + 6 = 0 \qquad \Rightarrow y = \dfrac{-3(1+x) \pm \sqrt{(3-x)(x-1)}}{2}$

Clearly, the values of y are real for x [1,3]



when x = 1 , we get y = – 3 $\qquad$ and, x = 3 $\Rightarrow$ y = – 6

Let $f(x) = \dfrac{-3(1+x) + \sqrt{(3-x)(x-1)}}{2}$ $\qquad$ and , $g(x) = \dfrac{-3(1+x) - \sqrt{(3-x)(x-1)}}{2}$

required area = $\left|\int_1^3 \{g(x) - f(x)\}dx\right|$

$= \left|\int_1^3 \sqrt{-x^2 + 4x - 3}\right|dx = \left|\int_1^3 \sqrt{1^2 - (x-2)^2}\right|dx = \left|\left[\frac{1}{2}(x-2)\ \sqrt{-x^2+4x-3} + \frac{1}{2}\sin^{-1}\left(\frac{x-2}{1}\right)\right]_1^3\right|$

$= \left|\left[\left\{\frac{1}{2}\sin^{-1}1\right\} - \left\{\frac{1}{2}\sin^{-1}(-1)\right\}\right]\right| = \frac{\pi}{2}$ sq. unit

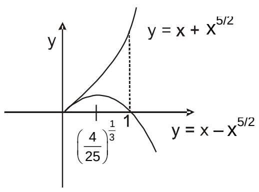
Miscellaneous examples

Example # 42 : Find the area contained between the two arms of curves $(y - x)^2 = x^5$ between x = 0 and x = 1.

Solution : $(y - x)^2 = x^5 \Rightarrow y = x \pm x^{5/2}$

For arm

$y = x + x^{5/2} \Rightarrow \dfrac{dy}{dx} = 1 + \dfrac{5}{2}x^{3/2} > 0 \qquad x \geq 0.$



y is increasing function.

For arm

$y = x - x^{5/2} \Rightarrow \dfrac{dy}{dx} = 1 - \dfrac{5}{2}x^{3/2}$

$\frac{5}{2}$ = 0 $\Rightarrow$ x = $\left(\frac{4}{25}\right)^{1/3}$, $\frac{d^2y}{dx^2} = -\frac{5}{4}$ $x^{\frac{1}{2}}$ < 0 at x = $\left(\frac{4}{25}\right)^{1/3}$

$\therefore$ at x = $\left(\frac{4}{25}\right)^{1/3}$, y = x – $x^{5/2}$ has maxima.

Required area = $\int_0^1 (x + x^{5/2} - x + x^{5/2})\ dx$ = 2 $\int_0^1 x^{5/2}dx = \left.\frac{2\ x^{7/2}}{7/2}\right]_0^1$ = $\frac{4}{7}$

Example # 43 : Let A (m) be area bounded by parabola y = x^2 + 2x – 3 and the line y = mx + 1. Find the least area A(m).

Solution : Solving we obtain

$x^2 + (2 - m)x - 4 = 0$

Let α,β be roots $\Rightarrow \alpha + \beta = m - 2,\ \alpha\beta = -4$

$$A(m) = \left|\int_\alpha^\beta (mx + 1 - x^2 - 2x + 3)dx\right| = \left|\int_\alpha^\beta (-x^2 + (m-2)x + 4)dx\right|$$

$$= \left|\left(-\frac{x^3}{3} + (m-2)\ \frac{x^2}{2} + 4x\right)_\alpha^\beta\right| = \left|\frac{\alpha^3 - \beta^3}{3} + \frac{m-2}{2}(\beta^2 - \alpha^2) + 4(\beta - \alpha)\right|$$

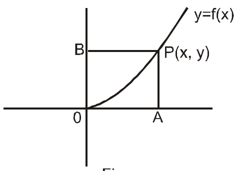
$$= |\beta - \alpha|.\left|-\frac{1}{3}(\beta^2 + \beta\alpha + \alpha^2) + \frac{(m-2)}{2}(\beta + \alpha) + 4\right|$$

$$= \sqrt{(m-2)^2 + 16}\ \left|-\frac{1}{3}\left((m-2)^2 + 4\right) + \frac{(m-2)}{2}(m-2) + 4\right| = \sqrt{(m-2)^2 + 16}\ \left|\frac{1}{6}(m-2)^2 + \frac{8}{3}\right|$$

$$A(m)\ \left((m-2)^2 + 16\right)^{3/2} = \frac{1}{6} \Rightarrow \text{ Least } A(m) = \frac{1}{6}\ (16)^{3/2} = \frac{32}{3}.$$

Example # 44 : A curve y = f(x) passes through the origin and lies entirely in the first quadrant. Through any point P(x, y) on the curve, lines are drawn parallel to the coordinate axes. If the curve divides the area formed by these lines and coordinate axes in m : n, then show that f(x) = $cx^{m/n}$ or f(x) = $cx^{n/m}$ (c-being arbitrary).

Solution : Area (OAPB) = xy $\Rightarrow$ Area (OAPO) = $\int_0^x f(t)\ dt$



Figure

Area (OPBO) = $xy - \int_0^x f(t)\ dt$ $\Rightarrow$ $\frac{\text{Area (OAPO)}}{\text{Area (OPBO)}} = \frac{m}{n}$

$n\int_0^x f(t)\ dt = m\left(xy - \int_0^x f(t)\ dt\right)$ $\Rightarrow$ $n\int_0^x f(t)\ dt = mx\ f(x) - m\int_0^x f(t)\ dt$

Differentiating w.r.t. x

nf(x) = m f(x) + mx f¢(x) – m f(x) $\Rightarrow$ $\frac{f'(x)}{f(x)} = \frac{n}{m}\frac{1}{x}$

f(x) = $cx^{n/m}$

similarly f(x) = $cx^{m/n}$

Self practice problems :

(44) Find the area bounded by the curves y = e^x, y = |x – 1| and x = 2.

(45) Compute the area of the region bounded by the parabolas $y^2 + 8x = 16$ and $y^2 - 24x = 48$.

(46) Find the area between the x-axis and the curve y = $\sqrt{1+\cos 4x}$, $0 \le x \le \pi$.

(47) What is geometrical significance of

(i) $\int_0^{\pi} |\cos x| \, dx$, (ii) $\int_0^{\frac{3\pi}{2}} \cos x \, dx$

(48) Find the area of the region bounded by the x-axis and the curves defined by y = tan x, $\left(\text{where } \frac{-\pi}{3} \le x \le \frac{\pi}{3}\right)$ and y = cot x . $\left(\text{where } \frac{\pi}{6} \le x \le \frac{2\pi}{3}\right)$

(49) Find the area bounded by the curves $x = y^2$ and $x = 3 - 2y^2$.

(50) Find the area bounded by the curve $y = x^2 - 2x + 5$, the tangent to it at the point (2, 5) and the axes of coordinates.

(51) Find the area of the region bounded by y = x – 1 and $(y - 1)^2 = 4(x + 1)$

(52) Find the area of the region lying in the first quadrant and included between the curves $x^2 + y^2 = 3a^2$. $x^2 = 2ay$ and $y^2 = 2ax$. a > 0

(53) Find the area enclosed by the curves $y = -x^2 + 6x - 5$, $y = -x^2 + 4x - 3$ and the straight line y = 3x – 15.

(54) Find the area bounded by the curves $4y = |4 - x^2|$, $y = 7 - |x|$

(55) Find the area bounded by the curves $x = |y^2 - 1|$ and y = x – 5.

(56) Find the area of the region formed by $x^2 + y^2 - 6x - 4y + 12 \le 0$, $y \le x$ and $2x \le 5$.

Ans. **44.** $(e^2 - 2)$ sq. units **45.** $32\sqrt{\frac{2}{3}}$ sq. units **46.** $2\sqrt{2}$ sq. units

47. (i) Area bounded by y = cos x, x-axis between x = 0, x = π.

(ii) Difference of area bounded by y = cos x, x-axis between x = 0, x = $\frac{\pi}{2}$ and area bounded by y = cos x, x-axis between x = $\frac{\pi}{2}$, x = $\frac{3\pi}{2}$.

48. $\ell n \frac{3}{2}$ **49.** 4 sq. units

50. 8/3 sq. units

51. 64/3 sq. units **52.** $a^2\left[\frac{\sqrt{2}}{3} + \frac{3}{2}\sin^{-1}\frac{1}{3}\right]$ sq. units **53.** $\frac{73}{6}$ sq. units

54. 32 sq. units **55.** $\frac{109}{6}$ sq. units **56.** $\left(\frac{\pi}{6} + \frac{1}{8} - \frac{\sqrt{3}}{8}\right)$sq. units