Conic Section

Everything should be made as simple as possible, but not simpler...... Einstein, Albert

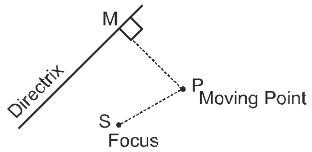
This chapter focusses on parabolic curves, which constitutes one category of various curves obtained by slicing a cone by a plane, called conic sections. A cone (not necessarily right circular) can be out in various ways by a plane, and thus different types of conic sections are obtained.

Let us start with the definition of a conic section and then we will see how are they obtained by slicing a right circular cone.

1. Definition of Conic Sections:

A conic section or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

- The fixed point is called the **Focus.**
- The fixed straight line is called the **Directrix.**
- The constant ratio is called the **Eccentricity** denoted by e.



$$\frac{PS}{PM} = e$$

- The line passing through the focus & perpendicular to the directrix is called the **Axis**.
- A point of intersection of a conic with its axis is called a **Vertex**.

If S is (p, q) & directrix is $\ell x + my + n = 0$

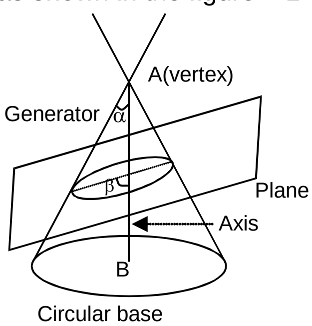
Then PS = $\sqrt{(x-\alpha)^2 + (y-\beta)^2}$ & PM = $\frac{|\ell x + my + n|}{\sqrt{\ell^2 + m^2}}$

$\frac{PS}{PM} = e \Rightarrow (\ell^2 + m^2)\,[(x - p)^2 + (y - q)^2] = e^2\,(\ell x + my + n)^2$

Which is of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

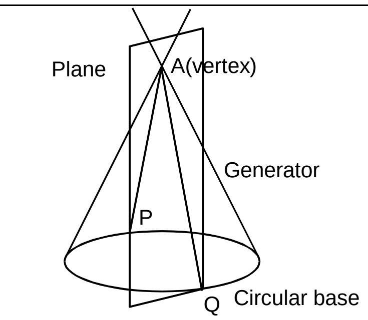
1.1 Section of right circular cone by different planes

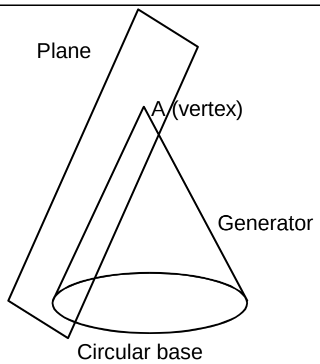
A right circular cone is as shown in the figure – 1

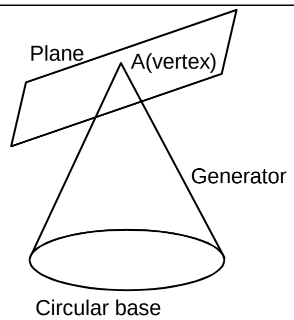


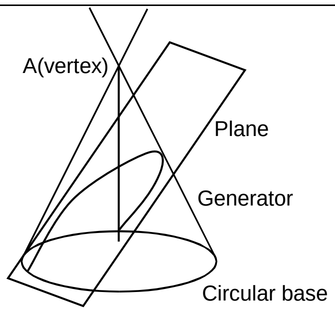
α is angle between generator and axis.
β is angle between plane and axis

Section of a right circular cone by a plane passing through its vertex is a pair of straight lines. Section of a right circular cone by a plane not passing through vertex is either circle or parabola or ellipse or hyperbola which is shown in table below :

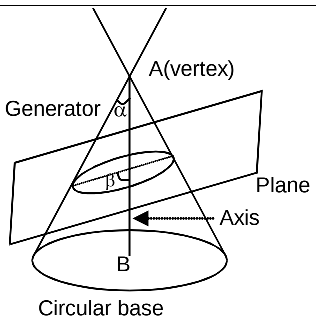
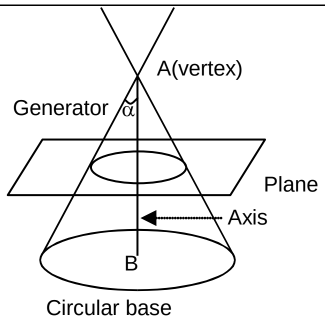
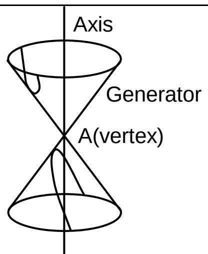
<table>
<tr><th>Type of conic section</th><th>3-D view of section of right circular cone with plane</th><th>Condition of conic in definition of conic</th><th>condition of conic in $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$</th></tr>
<tr><td>Two distinct real lines</td><td>

Plane passes through vertex A and $0 \le \beta < \alpha$</td><td>$e > 1$, focus lies on directrix</td><td>$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0,\ h^2 > ab$</td></tr>
<tr><td>Two real same lines</td><td>

Plane passes through vertex A and $\beta = \alpha$</td><td>$e = 1$, focus lies on directrix</td><td>$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0,\ h^2 = ab,$ (either $g^2 = ac$ or $f^2 = bc$)</td></tr>
<tr><td>Two imaginary lines/point</td><td>

Plane passes through vertex A and $\beta > \alpha$</td><td>$0 < e < 1$, focus lies on directrix</td><td>$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0,\ h^2 < ab$</td></tr>
<tr><td>Parabola</td><td>

Plane does not passes through vertex A and $\beta = \alpha$</td><td>$e = 1$, focus does not lies on directrix</td><td>$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \ne 0,\ h^2 = ab$</td></tr>
</table>

Ellipse	 Plane does not passes through vertex A and $\alpha < \beta < 90$	$0 < e < 1$, focus does not lies on directrix	$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$, $h^2 < ab$, (either $a \neq b$ or $h \neq 0$)
Circle	 Plane does not passes through vertex A and $\beta = 90$	$e = 0$, focus does not lies on directrix	$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$, $a = b$, $h = 0$
Hyperbola	 Plane does not passes through vertex A and $0 \le \beta < \alpha$	$e > 1$, focus does not lies on directrix	$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$, $h^2 > ab$

Note : (i) Pair of real parallel lines is not the part of conic but it is part of general two degree equation.

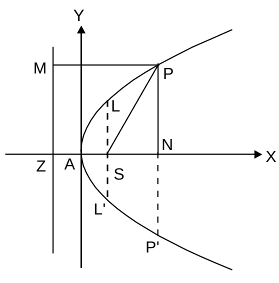
For it $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$, $h^2 = ab$, (either $g^2 > ac$ or $f^2 > bc$)

$\Rightarrow$ General two degree equation can represent real curve other than conic section.

(ii) For rectangular hyperbola $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$, $h^2 > ab$, $a + b = 0$

2. Elementary Concepts of Parabola

2.1 Definition and terminology of parabola



$y^2 = 4ax$

A parabola is the locus of a point, whose distance from a fixed point (focus) is equal to perpendicular distance from a fixed straight line (directrix). Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$

For parabola $y^2 = 4ax$.

(i) Vertex is (0, 0) **(ii)** focus is (a, 0)

(iii) Axis is y = 0 **(iv)** Directrix is x + a = 0

Focal Distance: The distance of a point on the parabola from the focus.

Focal Chord : A chord of the parabola, which passes through the focus.

Double Ordinate: A chord of the parabola perpendicular to the axis of the symmetry.

Latus Rectum: A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the Latus Rectum (L.R.).

For $y^2 = 4ax$. $\Rightarrow$ Length of the latus rectum = 4a.

$\Rightarrow$ ends of the latus rectum are L(a, 2a) & L' (a, – 2a).

NOTE :

(i) Perpendicular distance from focus on directrix = half the latus rectum.

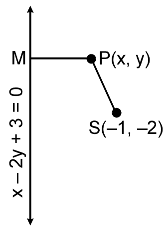
(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

(iii) Two parabolas are said to be equal if they have the same latus rectum.

Example # 1: Find the equation of the parabola whose focus is at (– 1, – 2) and the directrix is $x - 2y + 3 = 0$.

Solution : Let P(x, y) be any point on the parabola whose focus is S(– 1, – 2) and the directrix $x - 2y + 3 = 0$. Draw PM perpendicular to directrix $x - 2y + 3 = 0$. Then by definition,

SP = PM



$\Rightarrow$ $SP^2 = PM^2$

$\Rightarrow$ $(x + 1)^2 + (y + 2)^2 = \left(\frac{x - 2y + 3}{\sqrt{1+4}}\right)^2$

$\Rightarrow$ $5\,[(x + 1)^2 + (y + 2)^2] = (x - 2y + 3)^2$

$\Rightarrow$ $5(x^2 + y^2 + 2x + 4y + 5) = (x^2 + 4y^2 + 9 - 4xy + 6x - 12y)$

$\Rightarrow$ $4x^2 + y^2 + 4xy + 4x + 32y + 16 = 0$

This is the equation of the required parabola.

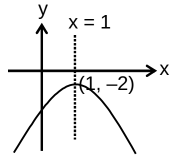
Example # 2 : Find the vertex, axis, focus, directrix, latusrectum of the parabola, also draw their rough sketches. $x^2 - 2x + 4y + 9 = 0$.

Solution : The given equation is $x^2 - 2x + 4y + 9 = 0$

$\Rightarrow (x - 1)^2 = -4(y + 2)$

which of the form $X^2 = -4bY$

Vertex -



$(X, Y) \equiv (0, 0)$

$(x, y) \equiv (1, -2)$

Axis

$X = 0 \Rightarrow x = 1$

Focus-

$(X, Y) = (0, -b)$

$(x, y) \equiv (1, -1 -2) = (1, -3)$

Directrix -

$Y = b \Rightarrow y + 2 = 1$

$y = -1$

Latusrectum -

The length of the latusrectum of the given parabola is 4b = 4.

Self Practice Problems :

(1) Find the equation of the parabola whose focus is the point (0, 0)and whose directrix is the straight line $4x – 3y – 2 = 0$.

(2) Find the extremities of latus rectum of the parabola $y = x^2 – 2x + 3$.

(3) Find the latus rectum & equation of parabola whose vertex is origin & directrix is $x + y = 2$.

(4) Find the equation of the parabola whose focus is (–1, 1) and whose vertex is (1, 2). Also find its axis and latusrectum.

Ans. (1) $9x^2 + 16y^2 + 24xy + 16x – 12y – 4 = 0$ (2) $\left(\frac{1}{2},\frac{9}{4}\right)\left(\frac{3}{2},\frac{9}{4}\right)$

(3) $4\sqrt{2}$, $x^2 + y^2 – 2xy + 8x + 8y = 0$

(4) $(2y – x – 3)^2 = – 20 (y + 2x – 4)$, Axis $2y – x – 3 = 0$. $LL' = 4\sqrt{5}$.

2.2 Parametric representation of parabola

The simplest & the best form of representing the co–ordinates of a point on the parabola is $(at^2, 2at)$ i.e. the equations $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter.

Parametric form for : $y^2 = – 4ax$ $(–at^2, 2at)$
$x^2 = 4ay$ $(2at , at^2)$
$x^2 = – 4ay$ $(2at , – at^2)$

Example # 3: Find the parametric equation of the parabola $(x + 1)^2 = –6 (y + 2)$

Solution: $\because$ $4a = – 6$ $\Rightarrow$ $a = \frac{-3}{2}$, $y + 2 = at^2$

$x + 1 = 2 at$ $\Rightarrow$ $x = –1 – 3t, y = –2 – \frac{3}{2}t^2$

Self Practice Problems:

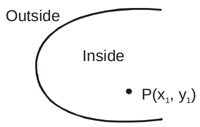
(5) Find the parametric equation of the parabola $x^2 = 4a(y – 1)$

Ans. $x = 2at, y = 1 + at^2$

.

2.3 Position of a point relative to a parabola:

The point (x_1 , y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 – 4ax_1$ is positive, zero or negative.



$S_1 : y_1^2 – 4ax_1$
$S_1 < 0 \rightarrow$ Inside
$S_1 > 0 \rightarrow$ Outside

Example # 4 : Check whether the point (4, 5) lies inside or outside the parabola $y^2 = 4x$.

Solution : $y^2 – 4x = 0$

$\because$ $S_1 \equiv y_1^2 – 4x_1 = 25 – 16 = 9 > 0$

$\therefore$ (3, 4) lies outside the parabola.

Self Practice Problems :

(6) Find the set of value's of α for which $(\alpha, – 2 – \alpha)$ lies inside the parabola $y^2 + 4x = 0$.

Ans. $\alpha \in (– 4 – 2\sqrt{3} , – 4 + 2\sqrt{3})$

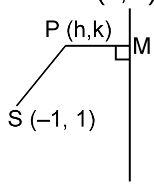
3. Elementary Concepts of Ellipse

3.1 Definition of Ellipse

It is locus of a point which moves in such a way that the ratio of its distance from a fixed point called focus and a fixed line called directrix (not passes through fixed point and all points and line lies in same plane) is constant (e = eccentricity), which is less than one.

Example # 5 : Find the equation to the ellipse whose focus is the point (– 1, 1), whose directrix is the straight line $x - y + 3 = 0$ and eccentricity is $\frac{1}{2}$.

Solution : Let $P \equiv (h, k)$ be moving point,



$e = \frac{PS}{PM} = \frac{1}{2} \quad \Rightarrow \quad (h + 1)^2 + (k - 1)^2 = \frac{1}{4}\left(\frac{h-k+3}{\sqrt{2}}\right)^2$

$\Rightarrow$ locus of P(h, k) is

$8\{x^2 + y^2 + 2x - 2y + 2\} = (x^2 + y^2 - 2xy + 6x - 6y + 9)$

$7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$.

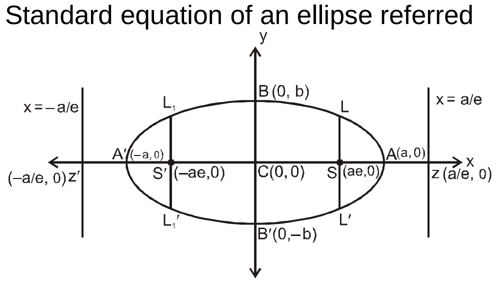
Self Practice Problems :

(7) Find the equation to the ellipse whose focus is (0, 0) directrix is $x + y - 1 = 0$ and $e = \frac{1}{\sqrt{2}}$.

Ans. $3x^2 + 3y^2 - 2xy + 2x + 2y - 1 = 0$.

3.2 Standard Equation of Ellipse

Standard equation of an ellipse referred



to its principal axes along the co–ordinate axes is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$ & $b^2 = a^2(1 - e^2)$.

Eccentricity: $e = \sqrt{1 - \frac{b^2}{a^2}}$, $(0 < e < 1)$

Focii : $S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

Equations of Directrices: $x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

Major Axis: The line segment A′A in which the focii S′ & S lie is of length 2a & is called the major axis (a > b) of the ellipse. Point of intersection of major axis with directrix is called the foot of the directrix (Z).

Minor Axis: The y–axis intersects the ellipse in the points $B' \equiv (0, -b)$ & $B \equiv (0, b)$. The line segment B′B is of length 2b (b < a) is called the minor axis of the ellipse.

Principal Axis : The major & minor axes together are called principal axis of the ellipse.

Vertices: Point of intersection of ellipse with major axis. $A' \equiv (-a, 0)$ & $A \equiv (a, 0)$.

Focal Chord: A chord which passes through a focus is called a focal chord.

Double Ordinate: A chord perpendicular to the major axis is called a double ordinate.

Latus Rectum: The focal chord perpendicular to the major axis is called the latus rectum.

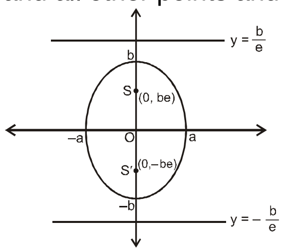
Length of latus rectum **(LL′)** = $\frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a(1-e^2)$

= 2e (distance from focus to the corresponding directrix)

Centre: The point which bisects every chord of the conic drawn through it, is called the centre of the conic. C ≡ (0, 0) the origin is the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Note : (i) If the equation of the ellipse is given as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and nothing is mentioned, then the rule is to assume that a > b.

(ii) If b > a is given, then the y–axis will become major axis and x-axis will become the minor axis and all other points and lines will change accordingly.



Equation : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Foci (0, ± be) Directrices : $y = \pm\frac{b}{e}$

$a^2 = b^2(1 - e^2)$, a < b. ⇒ $e = \sqrt{1 - \frac{a^2}{b^2}}$

Vertices (0, ± b) ; L.R. y = ± be

ℓ (L·R.) = $\frac{2a^2}{b}$, centre : (0, 0)

Example # 6: Find the equation to the ellipse whose centre is origin, axes are the axes of co-ordinate and passes through the points (2, 2) and (3, 1).

Solution: Let the equation to the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Since it passes through the points (2, 2) and (3, 1)

$\therefore \frac{4}{a^2} + \frac{4}{b^2} = 1$.......... (i) and $\frac{9}{a^2} + \frac{1}{b^2} = 1$ ....(ii)

from (i) – 4 (ii), we get

$\frac{4-36}{a^2} = 1 - 4 \Rightarrow a^2 = \frac{32}{3}$

from (i), we get

$\frac{1}{b^2} = \frac{1}{4} - \frac{3}{32} = \frac{8-3}{32} \Rightarrow b^2 = \frac{32}{5}$

∴ Ellipse is $3x^2 + 5y^2 = 32$

Example # 7 : Find the equation of the ellipse whose focii are (4, 0) and (– 4, 0) and eccentricity is $\frac{1}{3}$

Solution: Since both focus lies on x-axis, therefore x-axis is major axis and mid point of focii is origin which is centre and a line perpendicular to major axis and passes through centre is minor axis which is y-axis.

Let equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

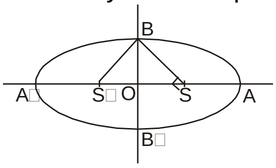
$\because$ ae = 4 and $e = \frac{1}{3}$ (Given)

$\therefore$ a = 12 and and $b^2 = a^2(1 - e^2)$

$\Rightarrow$ $b^2 = 144\left(1 - \frac{1}{9}\right)$ $\Rightarrow$ $b^2 = 16 \times 8$ $\Rightarrow$ $b = 8\sqrt{2}$

Equation of ellipse is $\frac{x^2}{144} + \frac{y^2}{128} = 1$

Example # 8 : In the given figure find the eccentricity of the ellipse if SS' subtends right angle at B.

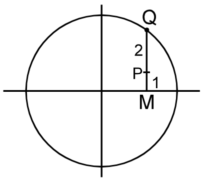


Solution: here b = ae --- (i)
in ellipse $b^2 = a^2 - a^2e^2$ ----- (ii)
from (i) & (ii) $a^2e^2 = a^2 - a^2e^2$

$2e^2 = 1 \Rightarrow e = \frac{1}{\sqrt{2}}$

Example # 9 : From a point Q on the circle $x^2 + y^2 = a^2$, perpendicular QM are drawn to x-axis, find the locus of point 'P' dividing QM in ratio 2 : 1.

Solution :



Let $Q \equiv (a\cos\theta, a\sin\theta)$
$M \equiv (a\cos\theta, 0)$
Let $P \equiv (h, k)$

$\therefore h = a\cos\theta,\ k = \frac{a\sin\theta}{3}$

$\therefore$ $\left(\frac{3k}{a}\right)^2 + \left(\frac{h}{a}\right)^2 = 1$ $\Rightarrow$ Locus of P is $\frac{x^2}{a^2} + \frac{y^2}{(a/3)^2} = 1$

Example # 10 : Find the equation of axes, directrix, co-ordinate of focii, centre, vertices, length of latus - rectum and eccentricity of an ellipse $16x^2 + 25y^2 - 96x - 100y + 156 = 0$.

Solution : The given ellipse is $\frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1$.

Let x – 3 = X, y – 2 = Y, so equation of ellipse becomes as $\frac{X^2}{5^2} + \frac{Y^2}{4^2} = 1$.

equation of major axis is Y = 0 $\Rightarrow$ y = 2.
equation of minor axis is X = 0 $\Rightarrow$ x = 3.
centre (X = 0, Y = 0) $\Rightarrow$ x = 3, y = 2
$C \equiv (3, 2)$

Length of semi-major axis a = 5
Length of major axis 2a = 10
Length of semi-minor axis b = 4
Length of minor axis = 2b = 8.
Let 'e' be eccentricity

$\therefore$ $b^2 = a^2(1 - e^2)$

$e = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{25 - 16}{25}} = \frac{3}{5}$.

Length of latus rectum = LL′ = $\frac{2b^2}{a} = \frac{2\times16}{5} = \frac{32}{5}$

Co-ordinates focii are X = ± ae, Y = 0

⇒ S ≡ (X = 3, Y = 0) & S′ ≡ (X = –3, Y = 0)

⇒ S ≡ (6, 2) & S′ ≡ (0, 2)

Co-ordinate of vertices

Extremities of major axis A ≡ (X = a, Y = 0) & A′ ≡ (X = – a, Y = 0)

⇒ A ≡ (x = 8, y = 2) & A′ = (x = – 2, 2)

A ≡ (8, 2) & A′ ≡ (– 2, 2)

Extremities of minor axis B ≡ (X = 0, Y = b) & B′ ≡ (X = 0, Y = – b)

B ≡ (x = 3, y = 6) & B′ ≡ (x = 3, y = – 2)

B ≡ (3, 6) & B′ ≡ (3, – 2)

Equation of directrix $X = \pm\frac{a}{e}$ $x - 3 = \pm\frac{25}{3}$ ⇒ $x = \frac{34}{3}$ & $x = -\frac{16}{3}$

Self Practice Problems:

(8) Find the equation to the ellipse whose axes are of lengths 6 and 2 and their equations are x – 3y + 3 = 0 and 3x + y – 1 = 0 respectively.

(9) Find the co-ordinates of the focii of the ellipse $4x^2 + 9y^2 = 1$.

(10) A point moves so that the sum of the squares of its distances from two intersecting lines is constant (given that the lines are neither perpendicular nor they make complimentry angle). Prove that its locus is an ellipse.

Hint. : Assume the lines to be y = mx and y = – mx.

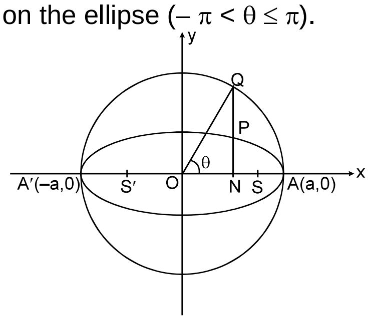
Ans. (8) $3(x - 3y + 3)^2 + 2(3x + y - 1)^2 = 180$, $21x^2 - 6xy + 29y^2 + 6x - 58y - 151 = 0$.

(9) $\left(\pm\frac{\sqrt{5}}{6}, 0\right)$

3.3 Auxiliary Circle / Eccentric Angle of Ellipse

A circle described on major axis of ellipse as diameter is called the **auxiliary circle**.

Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that line through Q perpendicular to the x – axis on the way intersects the ellipse at P, then P & Q are called as the **Corresponding Points** on the ellipse & the auxiliary circle respectively. 'θ' is called the **Eccentric Angle** of the point P on the ellipse ($-\pi < \theta \le \pi$). Q ≡ (a cosθ , a sinθ)



P ≡ (a cosθ , b sinθ)

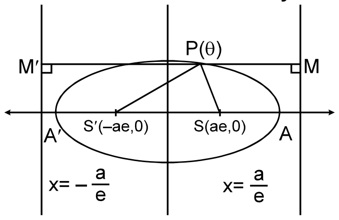
Note that :

$$\frac{\ell(PN)}{\ell(QN)} = \frac{b}{a} = \frac{\text{Semi minor axis}}{\text{Semi major axis}}$$

NOTE : If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle.

Example # 11 : Find the focal distance of a point P(θ) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (a > b)

Solution : Let 'e' be the eccentricity of ellipse.



$\therefore$ PS = e . PM = $e\left(\frac{a}{e} - a\cos\theta\right)$

PS = (a – a e cosθ)

and PS′ = e. PM′ = $e\left(a\cos\theta + \frac{a}{e}\right)$

PS′ = a + ae cosθ

$\therefore$ focal distance are (a ± ae cosθ)

Note : PS + PS′ = 2a

PS + PS′ = AA′

Example # 12 : Find the distance from centre of the point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose radius makes angle α with y – axis in clockwise direction.

Solution : Let P ≡ (a cosθ, b sinθ)

$\therefore$ $m_{(op)} = \frac{b}{a}$ tanθ = tan(π/2 – α) ⇒ tanθ = $\frac{a}{b}$ tan (π/2 – α)

$$OP = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} = \sqrt{\frac{a^2 + b^2\tan^2\theta}{\sec^2\theta}}$$

$$= \sqrt{\frac{a^2 + b^2\tan^2\theta}{1 + \tan^2\theta}} = \sqrt{\frac{a^2 + b^2 \times \frac{a^2}{b^2}\tan^2(\pi/2 - \alpha)}{1 + \frac{a^2}{b^2}\tan^2(\pi/2 - \alpha)}} \Rightarrow OP = \frac{ab}{\sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}}$$

Self Practice Problems :

(11) Find the distance from centre of the point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose eccentric angle is α

(12) Find the eccentric angle of a point on the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$whose distance from the centre is 3.

(13) Show that the area of triangle inscribed in an ellipse bears a constant ratio to the area of the triangle formed by joining points on the auxiliary circle corresponding to the vertices of the first triangle.

Ans. (11) $r = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}$ (12) $\pm\frac{\pi}{2}$

3.4 Parametric Representation of Ellipse

The equations x = a cos θ & y = b sin θ together represent the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Where θ is a parameter. Note that if P(θ) ≡ (a cos θ, b sin θ) is on the ellipse then;
Q(θ) ≡ (a cos θ, a sin θ) is on the auxiliary circle.
The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by $\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$

Example # 13 : Write the equation of chord of an ellipse $\frac{x^2}{25}+\frac{y^2}{16}$ =1 joining two points $P\left(\frac{\pi}{4}\right)$ and $Q\left(\frac{5\pi}{4}\right)$.

Solution : Equation of chord is $\frac{x}{5}\cos\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2}+\frac{y}{4}\cdot\sin\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2}=\cos\frac{\left(\frac{\pi}{4}-\frac{5\pi}{4}\right)}{2}$

$\frac{x}{5}\cdot\cos\left(\frac{3\pi}{4}\right)+\frac{y}{4}\cdot\sin\left(\frac{3\pi}{4}\right)=0 \Rightarrow -\frac{x}{5}+\frac{y}{4}=0 \Rightarrow 4x=5y$

Example # 14 : If P(α) and P(β) are extremities of a chord of ellipse which passes through the mid-point of the line segment joining focus & centre then prove that its eccentricity

$e = 2.\left|\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}\right|$

Solution : Let the equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

∴ equation of chord is $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$

above chord passes through (ae/2, 0) or (– ae/2, 0)

∴ $\pm e\cos\left(\frac{\alpha+\beta}{2}\right)=2\cos\left(\frac{\alpha-\beta}{2}\right)$ ∴ $e=2\left|\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}\right|$ **Ans.**

Self Practice Problems :

(14) Find the locus of the foot of the perpendicular from the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ on the chord joining two points whose eccentric angles differ by $\frac{\pi}{2}$.

Ans. (14) $2(x^2+y^2)^2=a^2x^2+b^2y^2$.

3.5 Position of a Point w.r.t. an Ellipse :

The point $P(x_1, y_1)$ lies outside, inside or on the ellipse according as $S_1>0$, $S_1<0$ or $S_1=0$ where $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$.

Example # 15 : Check whether the point P(1, –1) lies inside or outside of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$.

Solution : $S_1 \equiv \frac{1}{25}+\frac{1}{16}-1<0$

∴ Point P ≡ (1, –1) lies inside the ellipse.

Example # 16 : Find the set of value(s) of 'α' for which the point P(2α, – 3α) lies inside the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$.

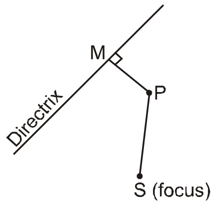
Solution : If P(2α, – 3α) lies inside the ellipse

∴ $S_1<0$

$\Rightarrow \frac{\alpha^2}{4}+\frac{\alpha^2}{1}-1<0 \Rightarrow -\frac{2}{\sqrt5}<\alpha<\frac{2}{\sqrt5}$ ∴ $\alpha\in\left(-\frac{2}{\sqrt5},\frac{2}{\sqrt5}\right)$.

4. Elementary Concepts of Hyperbola

Hyperbolic curves are of special importance in the field of science and technology especially astronomy and space studies. In this chapter we are going to study the characteristics of such curves.

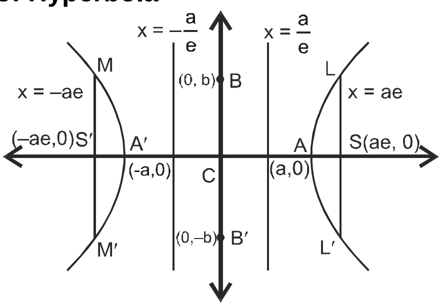


4.1 Definition of Hyperbola

A hyperbola is defined as the locus of a point moving in a plane in such a way that the ratio of its distance from a fixed point to that from a fixed line (the point does not lie on the line) is a fixed constant greater than 1.

$\frac{PS}{PM} = e > 1$,   e – eccentricity

4.2 Standard equation of Hyperbola



Standard equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = a^2(e^2 - 1)$.

- **Eccentricity (e) :** $e^2 = 1 + \frac{b^2}{a^2}$
- **Foci :** $S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.
- **Equations of directrices :** $x = \frac{a}{e}$ & $x = -\frac{a}{e}$.
- **Transverse axis :**
  The line segment A′A of length 2a in which the foci S′ & S both lie is called the transverse axis of the hyperbola.
- **Conjugate axis :**
  The line segment B′B of length 2b between the two points $B' \equiv (0, -b)$ & $B \equiv (0, b)$ is called as the conjugate axis of the hyperbola.
- **Principal axes :**
  The transverse & conjugate axis together are called principal axes of the hyperbola.
- **Vertices :**
  $A \equiv (a, 0)$ & $A' \equiv (-a, 0)$
- **Focal chord :**
  A chord which passes through a focus is called a focal chord.
- **Double ordinate :**
  A chord perpendicular to the transverse axis is called a double ordinate.
- **Latus rectum :**
  Focal chord perpendicular to the transverse axis is called latus rectum. Its length (ℓ) is

given by $\ell = \frac{2b^2}{a} = \frac{(C.A.)^2}{T.A.} = 2a(e^2 - 1)$.

Note : (i) Length of latus rectum = 2 e × (distance of focus from corresponding directrix)

(ii) End points of latus rectum are $L \equiv \left(ae, \frac{b^2}{a}\right)$, $L' \equiv \left(ae, -\frac{b^2}{a}\right)$, $M \equiv \left(-ae, \frac{b^2}{a}\right)$, $M' \equiv \left(-ae, -\frac{b^2}{a}\right)$

- **Centre:**
The point which bisects every chord of the conic, drawn through it, is called the centre of the conic. C ≡ (0,0) the origin is the centre of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

General note :
Since the fundamental equation to hyperbola only differs from that to ellipse in having $-b^2$ instead of b^2 it will be found that many propositions for hyperbola are derived from those for ellipse by simply changing the sign of b^2.

Example #17: Find the equation of the hyperbola whose directrix is x + 2y = 1, focus (2,1) and eccentricity $\sqrt{3}$.

Solution: Let P(x,y) be any point on the hyperbola.
Draw PM perpendicular from P on the directrix.
Then by definition SP = e PM

⇒ $(SP)^2 = e^2 (PM)^2$

⇒ $(x-2)^2 + (y-1)^2 = 3\left\{\frac{x+2y-1}{\sqrt{4+1}}\right\}^2$ ⇒ $2x^2 - 7y^2 - 12xy - 14x + 2y + 22 = 0$

Which is the required hyperbola.

Example # 18: Find the eccentricity of the hyperbola whose latus rectum is half of its transverse axis.

Solution: Let the equation of hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Then transverse axis = 2a and latus–rectum = $\frac{2b^2}{a}$. According to question $\frac{2b^2}{a} = \frac{1}{2}$ (2a)

⇒ $2b^2 = a^2$ $(\because b^2 = a^2 (e^2 - 1))$

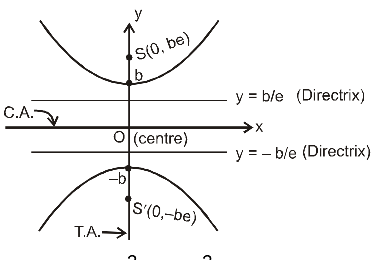
⇒ $2a^2 (e^2 - 1) = a^2$ ⇒ $2e^2 - 2 = 1$ ⇒ $e^2 = \frac{3}{2}$

∴ $e = \sqrt{\frac{3}{2}}$ Hence the required eccentricity is $\sqrt{\frac{3}{2}}$.

4.3 Conjugate hyperbola :

Two hyperbolas such that transverse & conjugate axes of one hyperbola are respectively the conjugate & the transverse axes of the other are called **conjugate hyperbolas** of each other.

eg. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are conjugate hyperbolas of each other.



Equation : $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

$a^2 = b^2 (e^2 - 1)$ ⇒ $e = \sqrt{1 + \frac{a^2}{b^2}}$

Vertices(0, ± b) ; ℓ (L.R.) = $\frac{2a^2}{b}$

Note : (a) If e_1 & e_2 are the eccentrcities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.

(b) The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

(c) Two hyperbolas are said to be similar if they have the same eccentricity.

(d) Two similar hyperbolas are said to be equal if they have same latus rectum.

(e) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

Example # 19 : Find the lengths of transverse axis and conjugate axis, eccentricity, the co-ordinates of foci, vertices, length of the latus-rectum and equations of the directrices of the following hyperbola $16x^2 - 9y^2 = -144$.

Solution : The equation $16x^2 - 9y^2 = -144$ can be written as $\frac{x^2}{9} - \frac{y^2}{16} = -1$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

$\therefore$ $a^2 = 9$, $b^2 = 16$ $\Rightarrow$ $a = 3$, $b = 4$

Length of transverse axis : The length of transverse axis = 2b = 8

Length of conjugate axis : The length of conjugate axis = 2a = 6

Eccentricity : $e = \sqrt{\left(1+\frac{a^2}{b^2}\right)} = \sqrt{\left(1+\frac{9}{16}\right)} = \frac{5}{4}$

Foci : The co-ordinates of the foci are $(0, \pm be)$ i.e., $(0, \pm 5)$

Vertices : The co–ordinates of the vertices are $(0, \pm b)$ i.e., $(0, \pm 4)$

Length of latus–rectum : The length of latus–rectum $= \frac{2a^2}{b} = \frac{2(3)^2}{4} = \frac{9}{2}$

Equation of directrices : The equation of directrices are

$$y = \pm \frac{b}{e} \Rightarrow y = \pm \frac{4}{(5/4)} \Rightarrow y = \pm \frac{16}{5}$$

Self Practice Problems :

(15) Find the equation of the hyperbola whose foci are (6, 4) and (– 4, 4) and eccentricity is 2.

(16) Obtain the equation of a hyperbola with coordinates axes as principal axes given that the distances of one of its vertices from the foci are 9 and 1 units.

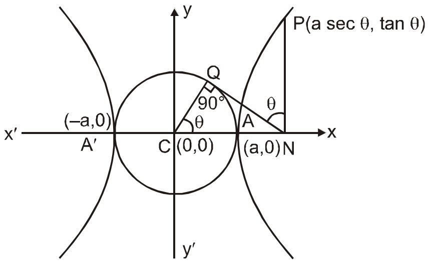
(17) The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Find the equation of the hyperbola if its eccentricity is 2.

Ans. (15) $12x^2 - 4y^2 - 24x + 32y - 127 = 0$ (16) $\frac{x^2}{16} - \frac{y^2}{9} = 1$, $\frac{y^2}{16} - \frac{x^2}{9} = 1$

(17) $3x^2 - y^2 - 12 = 0$.

4.4 Auxiliary Circle of Hyperbola

A circle drawn with centre C and transverse axis as a diameter is called the **auxiliary circle** of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.
Note from the following figure that P & Q are called the **"corresponding points"** of the hyperbola & the auxiliary circle.



4.5 Parametric representation of Hyperbola

The equations x = a sec θ & y = b tan θ together represent the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where θ is a parameter.

Note that if P(θ) ≡ (a sec θ, b tan θ) is on the hyperbola then,
Q(θ) ≡ (a cos θ, a sin θ) is on the auxiliary circle.

The equation to the chord of the hyperbola joining the two points P(α) & Q(β) is given by

$$.\ \frac{x}{a} \cos \frac{\alpha - \beta}{2} - \frac{y}{b} \sin \frac{\alpha + \beta}{2} = \cos\frac{\alpha + \beta}{2}$$

4.6 Position of a point 'P' w.r.t. a hyperbola :

The quantity $S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ is positive, zero or negative according as the point (x_1, y_1) lies inside, on or outside the curve.

Example # 20 : Find the position of the point (5, – 4) relative to the hyperbola $9x^2 - y^2 = 1$.

Solution : Since $9(5)^2 - (-4)^2 - 1 = 225 - 16 - 1 = 208 > 0$,
So the point (5,–4) lies inside the hyperbola $9x^2 - y^2 = 1$..

5. Rectangular hyperbola (equilateral hyperbola) :

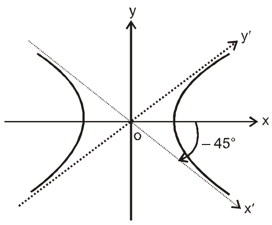
The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an Equilateral Hyperbola. Note that the eccentricity of the rectangular hyperbola is.

Since a = b

equation becomes $x^2 - y^2 = a^2$

whose asymptotes are y = ± x.

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1+1} = \sqrt{2}$$



Rotation of this system through an angle of 45° in clockwise direction gives another form to the equation of rectangular hyperbola.

which is $xy = c^2$ where $c^2 = \frac{a^2}{2}$.

It is referred to its asymptotes as axes of co–ordinates.

Vertices : (c, c) & (– c, – c);

Foci : $\left(\sqrt{2}\,c, \sqrt{2}\,c\right)$ & $\left(-\sqrt{2}\,c, -\sqrt{2}\,c\right)$,

Directrices : $x + y = \pm \sqrt{2}\,c$

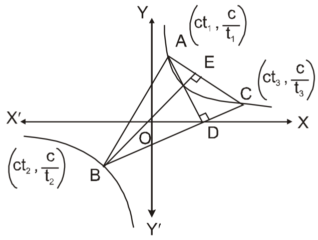
Latus Rectum (l) : $\ell = 2\sqrt{2}\,c$ = T.A. = C.A.

Parametric equation x = ct, y = c/t, t ∈ R – {0}

Example # 21 : A triangle has its vertices on a rectangular hyperbola. Prove that the orthocentre of the triangle also lies on the same hyperbola.

Solution : Let "t_1", "t_2" and "t_3" are the vertices of the triangle ABC, described on the rectangular hyperbola
$xy = c^2$.

$\therefore$ Co–ordinates of A,B and C are $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ respectively



Now slope of BC is $\frac{c(t_3 - t_2)}{c(t_2 - t_3)t_2 t_3} = -\frac{1}{t_2 t_3}$

$\therefore$ Slope of AD is t_2t_3

Equation of Altitude AD is $y - \frac{c}{t_1} = t_2t_3(x - ct_1)$

or $t_1y - c = x\, t_1t_2t_3 - ct_1^2t_2t_3$ .....(1)

Similarly equation of altitude BE is

$t_2y - c = x\, t_1t_2t_3 - ct_1t_2^2t_3$ ......(2)

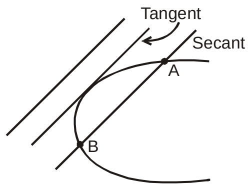
Solving (1) and (2),

we get the orthocentre $\left(-\frac{c}{t_1t_2t_3}, -ct_1t_2t_3\right)$ Which lies on $xy = c^2$.

6. Line & a parabola :

The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a > cm$, $a = cm$, $a < cm$ respectively.

$\Rightarrow$ condition of tangency is, $c = a/m$.



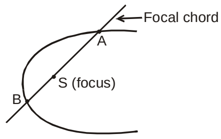
Length of the chord intercepted by the parabola on the line $y = m\,x + c$ is :

$\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a - mc)}$

NOTE :

1. The equation of a chord joining t_1 & t_2 is $2x - (t_1 + t_2)\, y + 2\, at_1t_2 = 0$.

2. If t_1 & t_2 are the ends of a focal chord of the parabola $y^2 = 4ax$ then $t_1t_2 = -1$.Hence the co–ordinates at the extremities of a focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$



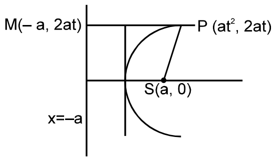
3. Length of the focal chord making an angle α with the x– axis is $4a\text{cosec}^2\,\alpha$.

Example # 22 : Discuss the position of line $y = x + 3$ with respect to parabola $y^2 = 4(x + 2)$.

Solution : Solving we get $(x + 3)^2 = 4(x + 2)$ $\Rightarrow$ $(x - 1)^2 = 0$
so $y = x + 3$ is tangent to the parabola.

Example # 23 : Prove that focal distance of a point $P(at^2, 2at)$ on parabola $y^2 = 4ax$ $(a > 0)$ is $a(1 + t^2)$.

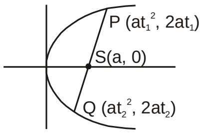
Solution :



$\because$ PS = PM = a + at^2 PS = a (1 + t^2).

Example # 24 : If t_1, t_2 are end points of a focal chord then show that $t_1 t_2 = -1$.

Solution : Let parabola is $y^2 = 4ax$



since P, S & Q are collinear $\therefore$ $m_{PQ} = m_{PS}$

$\Rightarrow \quad \dfrac{2}{t_1 + t_2} = \dfrac{2t_1}{t_1^2 - 1} \quad \Rightarrow \quad t_1^2 - 1 = t_1^2 + t_1t_2 \;\Rightarrow\; t_1t_2 = -1$

Example # 25 : If the endpoint t_1, t_2 of a chord satisfy the relation $t_1\, t_2 = -3$, then prove that the chord of $y^2 = 4x$ always passes through a fixed point. Find the point ?

Solution : Equation of chord joining (at_1^2, $2at_1$) and (at_2^2, $2at_2$) is

$y - 2at_1 = \dfrac{2}{t_1 + t_2}(x - at_1^2) \qquad \Rightarrow \qquad (t_1 + t_2)\, y - 2at_1^2 - 2at_1t_2 = 2x - 2at_1^2$

$y = \dfrac{2}{t_1 + t_2}(x - 3)$ ($\because$ $t_1t_2 = -3$) $\therefore$ This line passes through a fixed point (3, 0).

Self Practice Problems :

(18) If the line $y = 3x + \lambda$ intersect the parabola $y^2 = 4x$ at two distinct point's then set of value's of 'λ' is

(19) Find the midpoint of the chord $x + y = 2$ of the parabola $y^2 = 4x$.

(20) If one end of focal chord of parabola $y^2 = 16x$ is (16, 16) then coordinate of other end is.

(21) If PSQ is focal chord of parabola $y^2 = 4ax$ ($a > 0$), where S is focus then prove that $\dfrac{1}{PS} + \dfrac{1}{SQ} = \dfrac{1}{a}$

(22) Find the length of focal chord whose one end point is (ap^2, $2ap$)

Ans. (18) $(-\infty, 1/3)$ (19) (4, – 2) (20) (1, – 4) (22) $a\left(p + \dfrac{1}{p}\right)^2$

6.1 Tangents to the parabola $y^2 = 4ax$:

Equation of tangent at a point on the parabola can be obtained by replacement method or using derivatives.

In replacement method, following changes are made to the second degree equation to obtain T.

$x^2 \to x\,x_1$, $y^2 \to y\,y_1$, $2xy \to xy_1 + x_1y$, $2x \to x + x_1$, $2y \to y + y_1$

So, it follows that the targents are :

(i) $y\,y_1 = 2a\,(x + x_1)$ at the point (x_1, y_1) ;

(ii) $y = mx + \dfrac{a}{m}$ $(m \neq 0)$ at $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$

(iii) $t\,y = x + a\,t^2$ at (at^2, $2at$).

(iv) Point of intersection of the tangents at the point t_1 & t_2 is { at_1t_2, $a(t_1 + t_2)$ }.

Example # 26 : Prove that the straight line $y = mx + c$ touches the parabola $y^2 = 4a(x + a)$ if $c = ma + \dfrac{a}{m}$

Solution: Equation of tangent of slope 'm' to the parabola $y^2 = 4a(x + a)$ is

$y = m(x + a) + \dfrac{a}{m} \qquad \Rightarrow \qquad y = mx + a\left(m + \dfrac{1}{m}\right)$

But the given tangent is $y = mx + c$ $\quad \therefore c = am + \dfrac{a}{m}$

Example # 27 : A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line $y = 3x + 5$. Find its equation and its point of contact.

Solution : Slope of required tangent's are $m = \dfrac{3 \pm 1}{1 \mp 3} \;\Rightarrow\; m_1 = -2,\; m_2 = \dfrac{1}{2}$

$\because$ Equation of tangent of slope m to the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$.

$\therefore$ tangent's $y = -2x - 1$ at $\left(\frac{1}{2}, -2\right)$ $\Rightarrow y = \frac{1}{2}x + 4$ at (8, 8)

Example # 28 : Find the equation to the tangents to the parabola $y^2 = 9x$ which goes through the point (4, 10).

Solution : Equation of tangent to parabola $y^2 = 9x$ is $y = mx + \frac{9}{4m}$

Since it passes through (4, 10)

$\therefore 10 = 4m + \frac{9}{4m}$ $\Rightarrow$ $16m^2 - 40m + 9 = 0$ $m = \frac{1}{4}, \frac{9}{4}$

$\therefore$ equation of tangent's are $y = \frac{x}{4} + 9$ & $y = \frac{9}{4}x + 1$.

Example # 29 : Find the equations to the common tangents of the parabolas $(y - 1)^2 = 4ax$ and $x^2 = 4b(y - 1)$.

Solution : Equation of tangent to $(y - 1)^2 = 4ax$ is

$(y - 1) = mx + \frac{a}{m}$ ....(i)

Equation of tangent to $x^2 = 4b(y - 1)$ is

$x = m_1(y - 1) + \frac{b}{m_1}$ $\Rightarrow$ $(y - 1) = \frac{1}{m_1}x - \frac{b}{(m_1)^2}$ ....(ii)

for common tangent, (i) & (ii) must represent same line.

$\therefore$ $\frac{1}{m_1} = m$ & $\frac{a}{m} = -\frac{b}{m_1^2}$ $\Rightarrow$ $\frac{a}{m} = -bm^2 \Rightarrow$ $m = \left(-\frac{a}{b}\right)^{1/3}$

$\therefore$ equation of common tangent is $y = \left(-\frac{a}{b}\right)^{1/3} x + a\left(-\frac{a}{b}\right)^{1/3} + 1$.

Self Practice Problems:

(23) Find equation tangent to parabola $y^2 = 4x$ whose intercept on y–axis is 2.

(24) Prove that perpendicular drawn from focus upon any tangent of a parabola lies on the tangent at the vertex.

(25) Prove that image of focus in any tangent to parabola lies on its directrix.

(26) Prove that the area of triangle formed by three tangents to the parabola $y^2 = 4ax$ is half the area of triangle formed by their points of contacts..

Ans. (23) $y = \frac{x}{2} + 2$

7. Line and an Ellipse :

The line $y = mx + c$ meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in two points real, coincident or imaginary according as $c^2 < a^2m^2 + b^2$, $c^2 = a^2m^2 + b^2$ or $c^2 > a^2m^2 + b^2$

Hence $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

NOTE: The equation to the chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ joining two points with eccentric angles α & β is given by $\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$

Example # 30 : Find the set of value(s) of 'λ' for which the line $x + y + \lambda = 0$ intersect the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at two distinct points.

Solution : Solving given line with ellipse, we get $\frac{x^2}{16} + \frac{(x+\lambda)^2}{9} = 1$

$25x^2 + 39\lambda x + 16\lambda^2 - 144 = 0$

Since, line intersect the parabola at two distinct points,

$\therefore$ roots of above equation are real & distinct

$\therefore$ $D > 0$

$\therefore$ $(32\lambda)^2 - 4.25(16\lambda^2 - 144) > 0 \quad \Rightarrow \quad \lambda \in (-5, 5)$

Self Practice Problems :

(27) Find the value of 'λ' for which $2x - y + \sqrt{109}\,\lambda = 0$ touches the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$

Ans. (27) $\lambda = \pm 1$

7.1 Tangents to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(a) Slope form: $y = mx \pm \sqrt{a^2m^2 + b^2}$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for all values of m.

(b) Point form : $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1).

(c) Parametric form: $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a\cos\theta, b\sin\theta)$.

Note : (i) There are two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction.These tangents touches the ellipse at extremities of a diameter.

(ii) Point of intersection of the tangents at the point α & β is, $\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

(iii) The eccentric angles of the points of contact of two parallel tangents differ by π.

Example # 31 : Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are parallel to the line $x - 2y + \sqrt{7} = 0$

Solution: Slope of tangent = $m = \frac{1}{2}$

Given ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1$

Equation of tangent whose slope is 'm' is $y = mx \pm \sqrt{4m^2 + 3}$

$\because$ $m = \frac{1}{2}$ $\quad \therefore y = \frac{1}{2}x \pm \sqrt{1+3} \quad \Rightarrow \quad 2y = x \pm 4$

Example # 32 : A tangent to the ellipse $9x^2 + 16y^2 - 144 = 0$ touches at the point P on it in the first quadrant and meets the co-ordinate axes in A and B respectively. If P divides AB in the ratio 3 : 1, find the equation of the tangent.

Solution: The given ellipse is $\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1 \Rightarrow \quad a = 4, b = 3$

Let $P \equiv (a\cos\theta, b\sin\theta)$ $\quad \therefore$ equation of tangent is

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

$A \equiv (a\sec\theta, 0)$

$B \equiv (0, b\,\mathrm{cosec}\theta)$

$\because$ P divide AB internally in the ratio 3 : 1

$\therefore$ $a\cos\theta = \frac{a\sec\theta}{4} \quad \Rightarrow \quad \cos^2\theta = \frac{1}{4} \quad \Rightarrow \quad \cos\theta = \frac{1}{2}$

and $\quad b \sin \theta = \frac{3b\,\text{cosec}\theta}{4} \quad \Rightarrow \quad \sin\theta = \frac{\sqrt{3}}{2}$

$\therefore \quad$ tangent is $\frac{x}{2a} + \frac{\sqrt{3}y}{2b} = 1 \quad \Rightarrow \quad bx + \sqrt{3}\,ay = 2ab \quad \Rightarrow \quad 3x + 4\sqrt{3}\,y = 24$

Example # 33 : Prove that the locus of the point of intersection of tangents to an ellipse at two points whose eccentric angle differ by $\frac{\pi}{3}$ is an ellipse having the same eccentricity.

Solution : Let P (h, k) be the point of intersection of tangents at A(θ) and B(β) to the ellipse.

$$\therefore \quad h = \frac{a\cos\left(\frac{\theta+\beta}{2}\right)}{\cos\left(\frac{\theta-\beta}{2}\right)} \ \& \ k = \frac{b\sin\left(\frac{\theta+\beta}{2}\right)}{\cos\left(\frac{\theta-\beta}{2}\right)} \Rightarrow \left(\frac{h}{a}\right)^2 + \left(\frac{k}{b}\right)^2 = \sec^2\left(\frac{\theta-\beta}{2}\right)$$

but given that $\theta - \beta = \frac{\pi}{3}$

$\therefore \quad$ locus is $\frac{x^2}{a^2\sec^2\left(\frac{\pi}{6}\right)} + \frac{y^2}{b^2\sec^2\left(\frac{\pi}{6}\right)} = 1$ which is ellipse having same eccentricity.

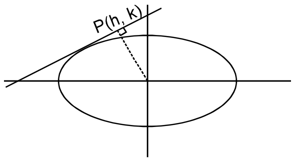
Example # 34 : If the locus of foot of perpendicular drawn from centre to any tangent to the ellipse $3x^2 + 4y^2 = 12$ is $(x^2 + y^2)^2 = ax^2 + by^2$, then find a + b.

Solution : Let P(h, k) be the foot of perpendicular to a tangent $y = mx + \sqrt{4m^2+3}$ ....(i)
from centre

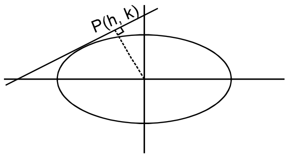
$\therefore$ $\frac{k}{h} . m = -1$ $\Rightarrow$ $m = -\frac{h}{k}$ ....(ii)

$\because$ P(h, k) lies on tangent

$\therefore$ $k = mh + \sqrt{4m^2+3}$ ....(iii)



from equation (ii) & (iii), we get



$\left(k + \frac{h^2}{k}\right)^2 = \frac{4h^2}{k^2} + 3$ $\Rightarrow$ locus is $(x^2 + y^2)^2 = 4x^2 + 3y^2$

Self Practice Problems :

(28) Show that the locus of the point of intersection of the tangents at the extremities of any focal chord of an ellipse is the directrix corresponding to the focus.

(29) Show that the locus of the foot of the perpendicular on a varying tangent to an ellipse from either of its foci is a concentric circle.

(30) Prove that the portion of the tangent to an ellipse intercepted between the ellipse and the directrix subtends a right angle at the corresponding focus.

(31) Find the area of parallelogram formed by tangents at the extremities of latera recta of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(32) If y_1 is ordinate of a point P on the ellipse then show that the angle between its focal radius and tangent at it, is $\tan^{-1}\left(\frac{b^2}{aey_1}\right)$.

(33) Find the eccentric angle of the point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ tangent at which, is equally inclined to the axes.

Ans. (31) $\frac{2a^3}{\sqrt{a^2 - b^2}}$ (33) $\theta = \pm \tan^{-1}\left(\frac{b}{a}\right)$, $\pi - \tan^{-1}\left(\frac{b}{a}\right)$, $-\pi + \tan^{-1}\left(\frac{b}{a}\right)$

8. Line and a hyperbola :

The straight line y = mx + c is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as : $c^2 > a^2m^2 - b^2$ or $c^2 = a^2m^2 - b^2$ or $c^2 < a^2m^2 - b^2$, respectively.

NOTE: The equation to the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ joining the two points $P(\alpha)$ & $Q(\beta)$ is given by

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}$$

8.1 Tangents to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

(i) Slope form : $y = mx \pm \sqrt{a^2m^2 - b^2}$ can be taken as the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, having slope 'm'.

(ii) Point form: Equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

(iii) Parametric form: Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point.

$(a \sec\theta, b\tan\theta)$ is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

Note : **(i)** Point of intersection of the tangents at $P(\theta_1)$ & $Q(\theta_2)$ is $\left(a\frac{\cos\frac{\theta_1-\theta_2}{2}}{\cos\frac{\theta_1+\theta_2}{2}},\ b\tan\left(\frac{\theta_1+\theta_2}{2}\right)\right)$

(ii) If $|\theta_1 + \theta_2| = \pi$, then tangents at these points (θ_1 & θ_2) are parallel.

(iii) There are two parallel tangents having the same slope m. These tangent touches the hyperbola at the extremities of a diameter.

Example # 35 : Find c, if x + y = c touch the hyperbola $\frac{x^2}{4} - y^2 = 1$.

Solution: Solving line and hyperbola we get

$x^2 - 4(c-x)^2 = 4$

$3x^2 + 8cx + 4c^2 + 4 = 0$

$D = 0$

$64c^2 - 4.3.4(c^2-1) = 0$

$c^2 - 3 = 0$

$c = \pm\sqrt{3}$

Example # 36 : Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $\sqrt{3}x + y + \sqrt{5} = 0$

Solution : $y = mx \pm \sqrt{36m^2 - 9}$, where $m = \frac{1}{\sqrt{3}}$

$\therefore$ equation of tangents are $y = \frac{x}{\sqrt{3}} \pm \sqrt{3}$ $\Rightarrow \sqrt{3}y = x \pm 3$

Example # 37 : Find the point of contact if $3x - \sqrt{7}y - 9 = 0$ is tangent to $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Solution : Let the point of contact is (x_1, y_1). The equation of tangent is

$\frac{xx_1}{16} - \frac{yy_1}{9} - 1 = 0$...........(i)

The given equation of tangent is $3x - \sqrt{7}y - 9 = 0$...........(ii)

From Equ (i) & (ii)

$\frac{x_1}{16\times3} = \frac{y_1}{9\sqrt{7}} = \frac{1}{9}$ $\Rightarrow$ $(x_1, y_1) = \left(\frac{16}{3}, \sqrt{7}\right)$

9. Line and Rectangular hyperbola :

Equation of a chord joining the points $P(t_1)$ & $Q(t_2)$ is $x + t_1t_2y = c(t_1 + t_2)$.

Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ & at $P(t)$ is $\frac{x}{t} + ty = 2c$.

Example # 38 : A, B, C are three points on the rectangular hyperbola $xy = c^2$, find
(i) The area of the triangle ABC
(ii) The area of the triangle formed by the tangents at A, B and C.

Solution : Let co–ordinates of A,B and C on the hyperbola $xy = c^2$ are $\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right)$, and $\left(ct_3, \frac{c}{t_3}\right)$ respectively.

(i) $\therefore$ Area of triangle ABC $= \frac{1}{2}\left[\begin{vmatrix} ct_1 & \frac{c}{t_1} \\ ct_2 & \frac{c}{t_2} \end{vmatrix} + \begin{vmatrix} ct_2 & \frac{c}{t_2} \\ ct_3 & \frac{c}{t_3} \end{vmatrix} + \begin{vmatrix} ct_3 & \frac{c}{t_3} \\ ct_1 & \frac{c}{t_1} \end{vmatrix}\right]$

$= \frac{c^2}{2}\left|\frac{t_1}{t_2} - \frac{t_2}{t_1} + \frac{t_2}{t_3} - \frac{t_3}{t_2} + \frac{t_3}{t_1} - \frac{t_1}{t_3}\right| = \frac{c^2}{2t_1t_2t_3}\left|t_1^2t_3 - t_2^2t_3 + t_1t_2^2 - t_3^2t_1 + t_2t_3^2 - t_1^2t_2\right|$

$= \frac{c^2}{2t_1t_2t_3} \; |(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|$

(ii) Equations of tangents at A,B,C are

$x + yt_1^2 - 2ct_1 = 0$

$x + yt_2^2 - 2ct_2 = 0$

and $x + yt_3^2 - 2ct_3 = 0$

$\therefore$ Required Area $= \frac{1}{2|C_1C_2C_3|}\begin{vmatrix} 1 & t_1^2 & -2ct_1 \\ 1 & t_2^2 & -2ct_2 \\ 1 & t_3^2 & -2ct_3 \end{vmatrix}^2$ ....(1)

where $C_1 = \begin{vmatrix} 1 & t_2^2 \\ 1 & t_3^2 \end{vmatrix}$, $C_2 = -\begin{vmatrix} 1 & t_1^2 \\ 1 & t_3^2 \end{vmatrix}$ and $C_3 = \begin{vmatrix} 1 & t_1^2 \\ 1 & t_2^2 \end{vmatrix}$

$\therefore$ $C_1 = t_3^2 - t_2^2$, $C_2 = t_1^2 - t_3^2$ and $C_3 = t_2^2 - t_1^2$

From (1) $\frac{1}{2\left|(t_3^2 - t_2^2)(t_1^2 - t_3^2)(t_2^2 - t_1^2)\right|} = 4c^2.(t_1 - t_2)^2 (t_2 - t_3)^2 (t_3 - t_1)^2$

$= 2c^2\left|\frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}\right|$

$\therefore$ Required area is, $2c^2\left|\frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}\right|$

Example # 39 : Prove that the perpendicular focal chords of a rectangular hyperbola are equal.

Solution : Let rectangular hyperbola is $x^2 - y^2 = a^2$

Let equations of PQ and DE are

$y = mx + c$ ......(1)

and $y = m_1x + c_1$ ......(2)

respectively.

Be any two focal chords of any rectangular hyperbola $x^2 - y^2 = a^2$ through its focus. We have to prove PQ = DE. Since PQ $\perp$ DE.

$\therefore$ $mm_1 = -1$ ......(3)

Also PQ passes through S $(a\sqrt{2}, 0)$ then from (1),

$0 = ma\sqrt{2} + c$

or $c^2 = 2a^2m^2$ ......(4)

Let (x_1,y_1) and (x_2,y_2) be the co–ordinates of P and Q then

$(PQ)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ ......(5)

Since (x_1,y_1) and (x_2,y_2) lie on (1)

$\therefore$ $y_1 = mx_1 + c$ and $y_2 = mx_2 + c$

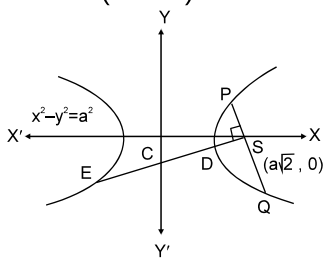
$\therefore$ $(y_1 - y_2) = m(x_1 - x_2)$ ....(6)

From (5) and (6)

$(PQ)^2 = (x_1 - x_2)^2 (1 + m^2)$ ....(7)

Now solving $y = mx + c$ and $x^2 - y^2 = a^2$ then $x^2 - (mx + c)^2 = a^2$

or $(m^2 - 1) x^2 + 2mcx + (a^2 + c^2) = 0$



$\therefore$ $x_1 + x_2 = -\frac{2mc}{m^2-1}$ and $x_1x_2 = = \frac{a^2+c^2}{m^2-1}$

$\Rightarrow$ $(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2 = \frac{4m^2c^2}{(m^2-1)^2} - \frac{4(a^2+c^2)}{(m^2-1)}$

$= \frac{4\{a^2+c^2-a^2m^2\}}{(m^2-1)^2} = \frac{4a^2(m^2+1)}{(m^2-1)^2}$ $\{\because c^2 = 2a^2m^2\}$

From (7), $(PQ)^2 = 4a^2 \left(\frac{m^2+1}{m^2-1}\right)^2$

Similarly, $(DE)^2 = 4a^2\left(\frac{m_1^2+1}{m_1^2-1}\right)^2 = 4a^2\left(\frac{\left(-\frac{1}{m}\right)^2+1}{\left(-\frac{1}{m}\right)^2-1}\right)^2 = 4a^2\left(\frac{m^2+1}{m^2-1}\right)^2 = (PQ)^2$

$(\because mm_1 = -1)$ Thus $(PQ)^2 = (DE)^2 \Rightarrow PQ = DE$.

Hence perpendicular focal chords of a rectangular hyperbola are equal.

Self Practice Problems :

(34) Show that the line $x \cos\alpha + y \sin\alpha = p$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $a^2\cos^2\alpha - b^2\sin^2\alpha = p^2$.

(35) For what value of λ does the line $y = 2x + \lambda$ touches the hyperbola $16x^2 - 9y^2 = 144$?

(36) Find the equation of the tangent to the hyperbola $x^2 - y^2 = 1$which is parallel to the line $4y = 5x + 7$.

Ans. (35) $\lambda = \pm 2\sqrt{5}$ (36) $4y = 5x \pm 3$

10 Pair of tangents : The equation to the pair of tangents which can be drawn from any point (x_1, y_1) to the curve $S = 0$ is $SS_1 = T^2$

Curve(S=0)	T for point (x_1, y_1) & S = 0	S_1 for point (x_1, y_1) & S = 0	Combined equation of tangents from external point (x_1, y_1) to S=0
Parabola $(y^2 - 4ax = 0)$	$T \equiv y y_1 - 2a(x + x_1)$	$S_1 = y_1^2 - 4ax_1$	$SS_1 = T^2$
Ellipse $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0\right)$	$T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$	$S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$	$SS_1 = T^2$

Hyperbola $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0\right)$	$T \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$	$S_1 = \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$	$SS_1 = T^2$

Example # 40 : Write the equation of pair of tangents to the parabola $y^2 = 4x$ drawn from a point P(–1, 2)

Solution : We know the equation of pair of tangents are given by $SS_1 = T^2$

$\therefore \quad (y^2 - 4x)(4 + 4) = (2y - 2(x - 1))^2$

$\Rightarrow \quad 8y^2 - 32x = 4y^2 + 4x^2 + 4 - 8xy + 8y - 8x \Rightarrow \quad y^2 - x^2 + 2xy - 6x - 2y = 1$

Example # 41 : Find the locus of the point P from which tangents are drawn to parabola $y^2 = 4ax$ having slopes m_1, m_2 such that

(i) $|m_1 - m_2| = 2$ (ii) $\theta_1 + \theta_2 = \pi/3$

Solution : Equation of tangent to $y^2 = 4ax$, is $y = mx + \frac{a}{m}$

Let it passes through P(h, k)

$\therefore \quad m^2h - mk + a = 0$

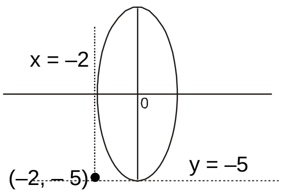
(i) $m_1 + m_2 = \frac{k}{h}$ and $m_1 . m_2 = \frac{a}{h}$ $\Rightarrow$ $|m_1 - m_2| = 2 \Rightarrow (m_1 + m_2)^2 - 4m_1m_2 = 4$

$\frac{k^2}{h^2} - 4\frac{a}{h} = 4 \Rightarrow 4ax = 4x^2$

(ii) $\tan \pi/3 = \frac{m_1 + m_2}{1 - m_1 m_2} = \frac{k/h}{1 - a/h}$ $\Rightarrow$ $y = (x - a)\sqrt{3}$

Example # 42 : How many real tangents can be drawn from the point (–2, –5) to the ellipse $\frac{x^2}{4}+\frac{y^2}{25} = 1$. Find the equation of these tangents & angle between them.

Solution :



By direct observation
x = –2, y = –5 are tangents.

Example # 43 : Find the locus of point of intersection of perpendicular tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

Solution : Let P(h, k) be the point of intersection of two perpendicular tangents
equation of pair of tangents is $SS_1 = T^2$

$$\Rightarrow \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right) = \left(\frac{hx}{a^2}+\frac{ky}{b^2}-1\right)^2$$

$$\Rightarrow \frac{x^2}{a^2}\left(\frac{k^2}{b^2}-1\right) + \frac{y^2}{b^2}\left(\frac{h^2}{a^2}-1\right) + ... = 0 \quad ....(i)$$

Since equation (i) represents two perpendicular lines

$$\therefore \frac{1}{a^2}\left(\frac{k^2}{b^2}-1\right) + \frac{1}{b^2}\left(\frac{h^2}{a^2}-1\right) = 0$$

$\Rightarrow \quad k^2 - b^2 + h^2 - a^2 = 0 \Rightarrow$ locus is $x^2 + y^2 = a^2 + b^2$

Example # 44 : How many real tangents can be drawn from the point (2, 1) to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$. Find the equation of these tangents.

Solution : Given point $P \equiv (2, 1)$ Hyperbola $S \equiv \frac{x^2}{16} - \frac{y^2}{9} - 1 = 0$

$\because$ $S_1 \equiv \frac{4}{16} - \frac{1}{9} - 1 = -\frac{31}{36} < 0$ $\Rightarrow$ Point $P \equiv (2, 1)$ lies outside the hyperbola.

$\therefore$ Two tangents can be drawn from the point P(2, 1).

Equation of pair of tangents is $SS_1 = T_2$

$$\Rightarrow \left(\frac{x^2}{16} - \frac{y^2}{9} - 1\right)\left(\frac{1}{4} - \frac{1}{9} - 1\right) = \left(\frac{2x}{16} - \frac{y}{9} - 1\right)^2 \Rightarrow 144\,(9x^2 - 16y^2 - 144) + (9x - 9y - 72)^2 = 0$$

Example # 45 : Find the locus of point of intersection of perpendicular tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Solution : Let P(h, k) be the point of intersection of two perpendicular tangents. Equation of pair of tangents is $SS_1 = T^2$

$$\Rightarrow \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{h^2}{a^2} - \frac{k^2}{b^2} - 1\right) = \left(\frac{hx}{a^2} - \frac{ky}{b^2} - 1\right)^2$$

$$\Rightarrow \frac{x^2}{a^2}\left(-\frac{k^2}{b^2} - 1\right) - \frac{y^2}{b^2}\left(\frac{h^2}{a^2} - 1\right) + \text{........} = 0 \qquad \text{.........(i)}$$

Since equation (i) represents two perpendicular lines

$$\therefore \frac{1}{a^2}\left(-\frac{k^2}{b^2} - 1\right) - \frac{1}{b^2}\left(\frac{h^2}{a^2} - 1\right) = 0 \Rightarrow -k^2 - b^2 - h^2 + a^2 = 0 \Rightarrow \text{locus is } x^2 + y^2 = a^2 - b^2$$

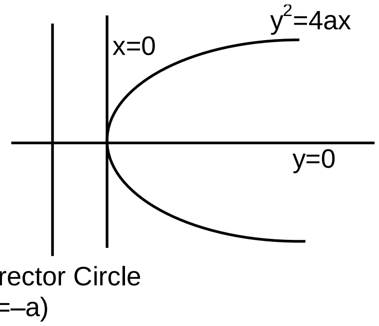
Self Practice Problems :

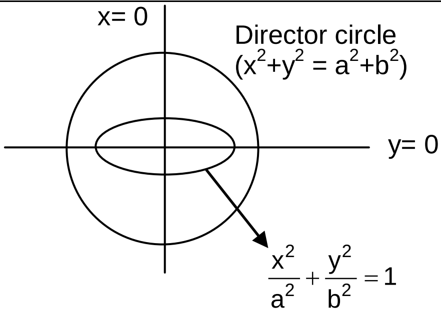
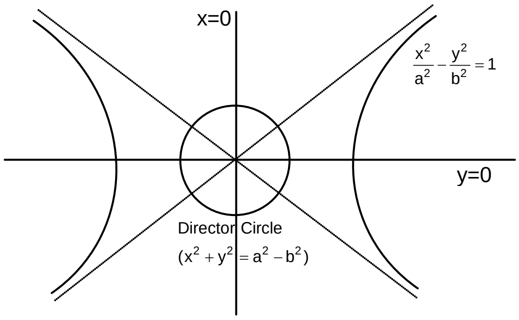
(37) If two tangents to the parabola $y^2 = 4ax$ from a point P make angles θ_1 and θ_2 with the axis of the parabola, then find the locus of P in each of the following cases.
(i) $\tan^2\theta_1 + \tan^2\theta_2 = \lambda$ (a constant) (ii) $\cos\theta_1 \cos\theta_2 = \lambda$ (a constant)

(38) Find the locus of point of intersection of the tangents drawn at the extremities of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Ans. (37) (i) $y^2 - 2ax = \lambda x^2$, (ii) $x^2 = \lambda^2\{(x - a)^2 + y^2\}$ (38) $x = \pm\frac{a}{e}$

11. **Director circle:** Locus of the point of intersection of the tangents which meet at right angles is called the Director Circle**.**

Curve(S=0)	Locus of Director Circle of (S=0)	Figure
Parabola ($y^2 - 4ax = 0$)	$x + a = 0$	

Ellipse $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0\right)$	$x^2 + y^2 = a^2 + b^2$	
Hyperbola $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0\right)$	$x^2 + y^2 = a^2 - b^2$	

Note: For hyperbola, if $b^2 < a^2$, then the director circle is real.
If $b^2 = a^2$ (i.e. rectangular hyperbola), then the radius of the director circle is zero and it reduces to a point circle at the origin. In this case centre is the only point from which two perpendicular tangents can be drawn on the curve.
If $b^2 > a^2$, then the radius of the director circle is imaginary, so that there is no such circle and so no pair of tangents at right angle can be drawn to the curve.

Example # 46 : Find the point of the line x – y = 0 for from where perpendicular tangent can be drawn to $\frac{x^2}{9} + y^2 = 1$

Solution : Solving director circle $x^2 + y^2 = 10$ & $x - y = 0$ $\Rightarrow$ $(\sqrt{5}, \sqrt{5}), (-\sqrt{5}, -\sqrt{5})$

Self Practice Problems :

(39) Find the angle between the tangent drawn from (–2, 1) to $x^2 + 4y^2 = 4$

Ans. (39) $\frac{\pi}{2}$

12. **Chord of contact:** Equation to the chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the curve S = 0 is T = 0

Curve(S=0)	T for point (x_1, y_1) & S = 0	equation of chord of contact from external point (x_1, y_1) to S=0 is T = 0
Parabola ($y^2 - 4ax = 0$)	$T \equiv y y_1 - 2a(x + x_1)$	$yy_1 - 2a(x + x_1) = 0$
Ellipse $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0\right)$	$T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$	$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$
Hyperbola $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0\right)$	$T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$	$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$

Rectangular Hyperbola ($xy - c^2 = 0$)	$T = \frac{xy_1 + yx_1}{2} - c^2$	$\frac{xy_1 + yx_1}{2} - c^2 = 0$

NOTE : The area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is $\frac{1}{2a}$ $(y_1^2 - 4ax_1)^{3/2}$

Example # 47 : Find the length of chord of contact of the tangents drawn from point (–2, 3) to the parabola $y^2 = 8x$.

Solution : Let tangent at $P(t_1)$ & $Q(t_2)$ meet at (–2, 3)

$\therefore$ $2t_1t_2 = -2$ & $2(t_1 + t_2) = 3$

$\because$ $PQ = \sqrt{(2t_1^2 - 2t_2^2)^2 + (4(t_1 - t_2))^2}$

$= 2\sqrt{((t_1 + t_2)^2 - 4t_1t_2)((t_1 + t_2)^2 + 4)} = \sqrt{\frac{(3^2 - 4.2(-2))(3^2 + 4.2^2)}{2^2}} = 25/2$

Example # 48 : If the line $x - y - 1 = 0$ intersect the parabola $y^2 = 8x$ at P & Q, then find the point of intersection of tangents at P & Q.

Solution : Let (h, k) be point of intersection of tangents then chord of contact is

$yk = 4(x + h)$

$4x - yk + 4h = 0$ .....(i)

But given is $x - y - 1 = 0$

$\therefore$ $\frac{4}{1} = \frac{-k}{-1} = \frac{4h}{-1}$ $\Rightarrow h = -1, k = 4$ $\therefore$ point $\equiv (-1, 4)$

Example # 49 : Find the locus of point whose chord of contact w.r.t to the parabola $y^2 = 4bx$ is the tangents of the circle $x^2 + y^2 = a^2$.

Solution : Let it is chord of contact for parabola $y_2 = 4bx$ w.r.t. the point P(h, k)

$\therefore$ Equation of chord of contact is $yk = 2b(x + h)$

$y = \frac{2b}{k}x + \frac{2bh}{k}$ .....(i)

(i) is tangents to $x^2 + y^2 = a^2$ $\Rightarrow$ $\left|\frac{\frac{2bh}{k}}{\sqrt{1 + \frac{4b^2}{k^2}}}\right| = a$ $\Rightarrow 4b^2x^2 = a^2(y^2 + 4b^2)$

Example # 50 : If tangents to the circle $x^2 + y^2 = b^2$ intersect the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at A and B, then find the locus of point of intersection of tangents at A and B.

Solution : Let $P \equiv (h, k)$ be the point of intersection of tangents at A & B

$\therefore$ equation of chord of contact AB is $\frac{xh}{a^2} + \frac{yk}{b^2} = 1$ ....(i)

which touches the circle $x^2 + y^2 = b^2$

$\therefore$ $\frac{1}{\sqrt{\frac{h^2}{a^4} + \frac{k^2}{b^4}}} = b$ $\Rightarrow$ required locus is $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{b^2}$

Example # 51: If tangents to the parabola $y^2 = 4ax$ intersect the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at A and B, then find the locus of point of intersection of tangents at A and B.

Solution: Let P ≡ (h, k) be the point of intersection of tangents at A & B

$\therefore$ Equation of chord of contact AB is $\frac{xh}{a^2} - \frac{yk}{b^2} = 1$ ......(i)

Which touches the parabola

Equation of tangent to parabola $y^2 = 4ax$

$y = mx + \frac{a}{m}$ $\Rightarrow$ $mx - y = -\frac{a}{m}$ ......(ii)

equation (i) & (ii) as must be same

$$\therefore \quad \frac{m}{\left(\frac{h}{a^2}\right)} = \frac{-1}{\left(-\frac{k}{b^2}\right)} = \frac{-\frac{a}{m}}{1} \Rightarrow m = \frac{h}{k}\frac{b^2}{a^2} \ \& \ m = -\frac{ak}{b^2}$$

$$\therefore \quad \frac{hb^2}{ka^2} = -\frac{ak}{b^2} \quad \Rightarrow \text{ locus of P is } y^2 = -\frac{b^4}{a^3} \cdot x$$

Self Practice Problems :

(40) Prove that locus of a point whose chord of contact w.r.t. parabola passes through focus is directrix

(41) If from a variable point 'P' on the line $2x - y - 1 = 0$ pair of tangent's are drawn to the parabola $x^2 = 8y$ then prove that chord of contact passes through a fixed point, also find that point.

(42) Find the locus of point of intersection of tangents at the extremities of normal chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(43) Find the locus of point of intersection of tangents at the extremities of the chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ subtending a right angle at its centre.

Ans. (41) (8,1) (42) $\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 - b^2)^2$ (43) $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}$

13. Chord with a given middle point:

Equation of the chord of the curve S = 0 whose middle point is (x_1, y_1) is $T = S_1$.

Curve(S=0)	T for point (x_1, y_1) & S = 0	S_1 for point (x_1, y_1) & S = 0	Chord with middle point (x_1, y_1) for S=0 is $T = S_1$
Parabola ($y^2 - 4ax = 0$)	$T = y y_1 - 2a(x + x_1)$	$S_1 = y_1^2 - 4ax_1$	$y y_1 - 2a(x + x_1) = y_1^2 - 4ax_1$
Ellipse $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0\right)$	$T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$	$S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$	$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{xx_1}{a^2} + \frac{yy_1}{b^2}$
Hyperbola $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0\right)$	$T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$	$S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$	$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$
Rectangular Hyperbola ($xy - c^2 = 0$)	$T = \frac{xy_1 + yx_1}{2} - c^2$	$S_1 = x_1y_1 - c^2$	$xy_1 + yx_1 = 2x_1y_1$

Example # 52 : Find the locus of middle point of the chord of the parabola $y^2 = 16x$ which pass through a given point (7, –2).

Solution : Let P(h, k) be the mid point of chord of parabola $y^2 = 16x$

so equation of chord is $yk - 8(x + h) = k^2 - 16h$.

Since it passes through (7, –2)

$-2k - 8(7 + h) = k^2 - 16h$

∴ Required locus is

$y^2 + 2y - 8x + 56 = 0$

Example # 53 : Find the locus of middle point of the chord of the parabola $y^2 = 4ax$ which is parallel to line $y = mx + c$

Solution : Let P(h, k) be the mid point of chord of parabola $y^2 = 4ax$,

so equation of chord is $yk - 2a(x + h) = k^2 - 4ah$.

but slope $= \frac{2a}{k} = m$ ∴ locus is $y = \frac{2a}{m}$

Example # 54 : Find the locus of the mid - point of chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Which are focal chords of $y^2 = 4ax$

Solution : Let P ≡ (h, k) be the mid-point

∴ equation of chord whose mid-point is given $\frac{xh}{a^2}+\frac{yk}{b^2} - 1 = \frac{h^2}{a^2}+\frac{k^2}{b^2} - 1$

since it is a focal chord,

∴ it passes through focus (a, 0)

$\Rightarrow \frac{h}{a} = \frac{h^2}{a^2}+\frac{k^2}{b^2}$ ⇒ required locus is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{x}{a}$

Example # 55 : Find the mid point of chord x + 2y = 4 of ellipse $9x^2 + 36y^2 = 324$

Solution : Let (h,k) be mid point of chord . So $T = S_1$

$9xh + 36yk = 9h^2 + 36k^2$ ------ (i) $x + 2y = 4$ ------ (ii)

From (i) and (ii)

$\frac{9h}{1} = \frac{36k}{2} = \frac{9h^2+36k^2}{4}$ ⇒ (h, k) = (2,1)

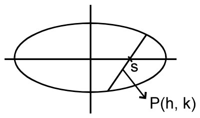
Example # 56 : Find the locus of the mid - point of focal chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solution : Let P ≡ (h, k) be the mid-point

∴ equation of chord whose mid-point is given is $\frac{xh}{a^2} - \frac{yk}{b^2} - 1 = \frac{h^2}{a^2} - \frac{k^2}{b^2} - 1$

since it is a focal chord,

∴ it passes through focus, either (ae, 0) or (–ae, 0)



If it passes through (ae, 0)

∴ locus is $\frac{ex}{a} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

If it passes through (–ae, 0)

∴ locus is $-\frac{ex}{a} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

Example # 57 : Find the condition on 'a' and 'b' for which two distinct chords of the hyperbola $\frac{x^2}{2a^2} - \frac{y^2}{2b^2} = 1$ passing through (a, b) are bisected by the line x + y = b.

Solution : Let the line x + y = b bisect the chord at P(α, b – α)

∴ equation of chord whose mid-point is P(α, b – α)

$$\frac{x\alpha}{2a^2} - \frac{y(b-\alpha)}{2b^2} = \frac{\alpha^2}{2a^2} - \frac{(b-\alpha)^2}{2b^2}$$

Since it passes through (a, b)

$$\therefore \quad \frac{\alpha}{2a} - \frac{(b-\alpha)}{2b} = \frac{\alpha^2}{2a^2} - \frac{(b-\alpha)^2}{2b^2}$$

$$\alpha^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + \alpha\left(\frac{1}{b} - \frac{1}{a}\right) = 0 \quad \alpha = 0, \quad \alpha = \frac{1}{\frac{1}{a} + \frac{1}{b}} \qquad \therefore \qquad a \neq \pm b$$

Example # 58 : Find the locus of the mid point of the chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which subtend a right angle at the origin.

Solution : let (h,k) be the mid–point of the chord of the hyperbola. Then its equation is

$$\frac{hx}{a^2} - \frac{ky}{b^2} - 1 = \frac{h^2}{b^2} - \frac{k^2}{b^2} - 1 \qquad \text{or} \qquad \frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2} \qquad ....(1)$$

The equation of the lines joining the origin to the points of intersection of the hyperbola and the chord (1) is obtained by making homogeneous hyperbola with the help of (1)

$$\therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{\left(\frac{hx}{a^2} - \frac{ky}{b^2}\right)^2}{\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2}$$

$$\Rightarrow \frac{1}{a^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 x^2 - \frac{1}{b^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 y^2 = \frac{h^2}{a^4} x^2 + \frac{k^2}{b^4} y^2 - \frac{2hk}{a^2b^2} xy \qquad ....(2)$$

The lines represented by (2) will be at right angle if coefficient of x^2 + coefficient of y^2 = 0

$$\Rightarrow \frac{1}{a^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 - \frac{h^2}{a^4} - \frac{1}{b^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 - \frac{k^2}{b^4} = 0 \Rightarrow \left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{h^2}{a^4} + \frac{h^2}{a^4}$$

hence, the locus of (h,k) is $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{x^2}{a^4} + \frac{y^2}{b^4}$

Self Practice Problems :

(44) Find the mid point of chord $x - y - 2 = 0$ of parabola $y^2 = 4x$.

(45) Find the locus of mid - point of chord of parabola $y^2 = 4ax$ which touches the parabola $x^2 = 4by$.

(46) Find the equation of the chord $\frac{x^2}{36} + \frac{y^2}{9} = 1$ which is bisected at (2, 1).

(47) Find the locus of the mid-points of normal chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(48) Find the equation of the chord $\frac{x^2}{36} - \frac{y^2}{9} = 1$ which is bisected at (2, 1).

(49) Find the point 'P' from which pair of tangents PA & PB are drawn to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ in such a way that (5, 2) bisect AB

(50) From the points on the circle $x^2 + y^2 = a^2$, tangent are drawn to the hyperbola $x^2 - y^2 = a^2$, prove that the locus of the middle points of the chords of contact is the curve $(x^2 - y^2)^2 = a^2 (x^2 + y^2)$.

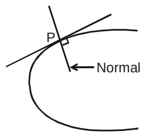
Ans. (44) (4, 2) (45) $y (2ax - y^2) = 4a^2b$ (46) $x + 2y = 4$

(47) $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 \left(\frac{a^6}{x^2} + \frac{b^6}{y^2}\right) = (a^2 - b^2)^2$ (48) $x = 2y$ (49) $\left(\frac{20}{3}, \frac{8}{3}\right)$

14 NORMAL

14.1 Normal to the parabola :

Normal is obtained using the slope of tangent.



Slope of tangent at $(x_1, y_1) = \frac{2a}{y_1}$ $\Rightarrow$ Slope of normal $= -\frac{y_1}{2a}$

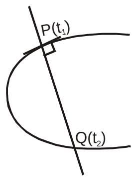
(i) $y - y_1 = -\frac{y_1}{2a}(x - x_1)$ at (x_1, y_1) ; **(ii)** $y = mx - 2am - am^3$ at $(am^2, -2am)$

(iii) $y + tx = 2at + at^3$ at $(at^2, 2at)$.

NOTE :

(i) Point of intersection of normals at t_1 & t_2 is $(a (t + t + t_1t_2 + 2), - a t_1 t_2 (t_1 + t_2))$.

(ii) If the normals to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point



t_2, then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.

(iii) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1 t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point (–2a, 0)

Example # 59 : If the normal at point 't_1' intersects the parabola again at 't_2' then find value of $|t_1.t_2 + t_1^2|$.

Solution : Slope of normal at P = $-t_1$ and slope of chord PQ = $\frac{2}{t_1 + t_2}$

$\therefore -t_1 = \frac{2}{t_1 + t_2} \Rightarrow t_2 = -t_1 - \frac{2}{t_1} \Rightarrow$ $t_2 = -t_1 - \frac{2}{t_1} \Rightarrow t_2.t_1 = -t_1^2 - 2 \Rightarrow |t_1.t_2 + t_1^2| = 2$

Example # 60 : If the normals at points t_1, t_2 meet at the point t_3 on the parabola then find value of $(t_1 + t_2 + t_3)^2 + (t_1 . t_2)^2$

Solution : Since normal at t_1 & t_2 meet the curve at t_3

$\therefore$ $t_3 = -t_1 - - \frac{2}{t_1}$ .....(i)

$t_3 = -t_2 - \frac{2}{t_2}$ .....(ii)

$\Rightarrow$ $(t_1^2 + 2)\, t_2 = t_1\, (t_2^2 + 2)$ $\Rightarrow$ $t_1t_2\,(t_1 - t_2) + 2\,(t_2 - t_1) = 0$

$\because$ $t_1 \neq t_2$, $t_1t_2 = 2$ ......(iii)

Hence (i) $t_1\, t_2 = 2$

from equation (i) & (iii), we get $t_3 = -t_1 - t_2$

Hence (ii) $t_1 + t_2 + t_3 = 0$ ....(iv)

from (iii) & (iv) $(t_1 + t_2 + t_3)^2 + (t_1 . t_2)^2 = 4$

Example # 61 : Find the locus of the point N from which 3 normals are drawn to the parabola $y^2 = 4ax$ are such that

(i) Two of them are equally inclined to y-axis

(ii) Two of them have product of their slops is equal to 2.

Solution : Equation of normal to $y^2 = 4ax$ is

$y = mx - 2am - am^3$

Let the normal passes through N(h, k)

$\therefore$ $k = mh - 2am - am^3$ $\Rightarrow$ $am^3 + (2a - h)\, m + k = 0$

For given value's of (h, k) it is cubic in 'm'.

Let m_1, m_2 & m_3 are root's of above equation

$\therefore$ $m_1 + m_2 + m_3 = 0$ ......(i)

$m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a - h}{a}$ ......(ii)

$m_1m_2m_3 = -\frac{k}{a}$ ......(iii)

(i) If two normal are equally inclined to x-axis, then $m_1 + m_2 = 0$

$\therefore$ $m_3 = 0$ $\Rightarrow$ $y = 0$

(ii) If two normal's have product of their slops = 2

$\therefore$ $m_1\, m_2 = 2$

from (3) $m_3 = -\frac{k}{2a}$ .....(iv)

from (2) $2 - \frac{k}{2a}\,(m_1 + m_2) = \frac{2a - h}{a}$ .....(v)

from (1) $m_1 + m_2 = \frac{k}{2a}$ .....(vi)

from (5) & (6), we get

$k_2 = 4ax$

Self Practice Problems:

(51) Find the points of the parabola $y^2 = 4ax$ at which the normal is inclined at 45° to the axis.
(52) If chord drawn from point P(9, –6) on the parabola $y^2 = 4x$ is normal at point Q then Q = ?
(53) Find the length of normal chord at point 't' to the parabola $y^2 = 4ax$.
(54) If the normals at 3 points P, Q & R on the parabola $(x - 3)^2 = y + 2$ are concurrent, then show that
(i) The sum of slopes of normals is zero,
(ii) The locus of centroid of ΔPQR is x – 3 = 0.

Ans. (51) (a, –2a), (a, 2a) (52) (9, –6) (53) $\ell = \dfrac{4a(t^2+1)^{\frac{3}{2}}}{t^2}$

14.2 Normal to Ellipse

(i) Equation of the normal at (x_1, y_1) to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $\dfrac{a^2x}{x_1} - \dfrac{b^2y}{y_1} = a^2 - b^2$.

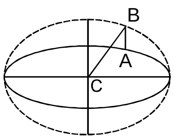
(ii) Equation of the normal at the point (acos θ, bsin θ) to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is;

$ax.\sec\theta - by.\operatorname{cosec}\theta = (a^2 - b^2)$.

(iii) Equation of a normal in terms of its slope 'm' is $y = mx - \dfrac{(a^2 - b^2)\,m}{\sqrt{a^2 + b^2m^2}}$.

Example # 62 : A and B are corresponding points on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and the auxiliary circles respectively. The normal at A to the ellipse meets CB in R, where C is the centre of the ellipse. Prove that locus of R is a circle of radius a + b.

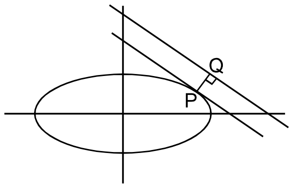
Solution : Let A ≡ (acos θ, b sinθ)
∴ B ≡ (a cosθ, a sinθ)



Equation of normal at A is
(a secθ) x – (b cosec θ) y = $a^2 - b^2$ ....(i)
equation of CB is y = tanθ . x ....(ii)
Solving equation (i) & (ii), we get (a – b) x = $(a^2 - b^2)$ cosθ
x = (a + b) cosθ, & y = (a + b) sinθ
∴ R ≡ ((a + b) cosθ, (a + b) sinθ) = (h, k)
$h^2 + k^2 = (a + b)^2 \Rightarrow x^2 + y^2 = (a + b)^2$

Example # 63 : Find the shortest distance between the line 3x + 4y = 12 and the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

Solution : Shortest distance occurs between two non-intersecting curve always along common normal.
Let 'P' be a point on ellipse and Q is a point on given line for which PQ is common normal.
∴ Tangent at 'P' is parallel to given line
∴ Equation of tangent parallel to given line is $(y = mx \pm \sqrt{a^2m^2 + b^2})$
$y = \dfrac{3x}{4} \pm 3\sqrt{2}$



$\therefore$ minimum distance = distance between

3x + 4y = 12 & 3x + 4y = $3\sqrt{2}$

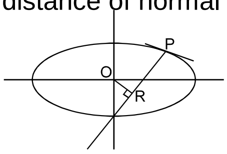
$\Rightarrow$ shortest distance = $\left|\dfrac{12-3\sqrt{2}}{5}\right|$

Example # 64 : Prove that, in an ellipse, the distance between the centre and any normal does not exceed the difference between the semi-axes of the ellipse.

Solution : Let the equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Equation of normal at P (θ) is $(a\sec\theta)x - (b\,\text{cosec}\,\theta)y - a^2 + b^2 = 0$

distance of normal from centre



$$= OR = \frac{|a^2-b^2|}{\sqrt{a^2+b^2+(a\tan\theta)^2+(b\cot\theta)^2}} = \frac{|a^2-b^2|}{\sqrt{(a+b)^2+(a\tan\theta-b\cot\theta)^2}}$$

$\because (a+b)^2 + (a\tan\theta - b\cot\theta)^2 \geq (a+b)^2$ or $\leq \dfrac{|a^2-b^2|}{\sqrt{(a+b)^2}} \Rightarrow |OR| \leq (a-b)$

Self Practice Problems :

(55) Find the value(s) of 'a' for which the line x + y = a is a normal to the ellipse $3x^2 + 4y^2 = 12$

(56) If the normal at the point P(θ) to the ellipse $\frac{x^2}{14}+\frac{y^2}{5}$ = 1 intersects it again at the point Q(2θ) then find the value of cosθ

Ans. (55) $a = \pm\frac{1}{\sqrt{7}}$ (56) $-\frac{2}{3}$

14.3 Normal to Hyperbola

(a) The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point P (x_1, y_1) on it is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1} = a^2 + b^2 = a^2e^2.$$

(b) The equation of the normal at the point P (a sec θ, b tan θ) on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2.$$

(c) Equation of normals in terms of its slope 'm' are $y = mx \pm \dfrac{(a^2+b^2)m}{\sqrt{a^2-b^2m^2}}$.

14.4 Normal to Rectangular hyperbola

Equation of the normal at P (t) is $x\,t^3 - y\,t = c\,(t^4-1)$

Example # 65 : A normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the axes in M and N. find a locus of point R on segment MN such that NR : RM = 2 :1.

Solution : The equation of normal at the point Q (a sec ϕ, b tan ϕ) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$ax \cos\phi + by \cot\phi = a^2 + b^2$ ....(1)

The normal (1) meets the x–axis in

$M\left(\frac{a^2+b^2}{a}\sec\phi,\ 0\right)$ and y–axis in $N\left(0, \frac{a^2+b^2}{b}\tan\phi\right)$

Let R (h, k) is point whose locus we have to find. as NR : RM = 2 :1.

$\Rightarrow \quad h = \frac{2}{3}\frac{(a^2+b^2)}{a}\sec\phi,\ \ k = \frac{(a^2+b^2)}{3b}\tan\phi$

we know that

$\sec^2\phi - \tan^2\phi = 1 \ \Rightarrow \ \frac{9a^2}{4(a^2+b^2)^2}x^2 - \frac{9b^2}{(a^2+b^2)^2}y^2 = 1 \Rightarrow \frac{a^2x^2}{4} - b^2y^2 = \frac{(a^2+b^2)^2}{9}$

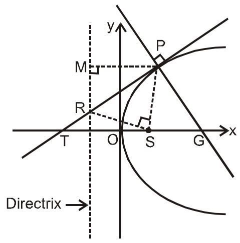
Self Practice Problems :

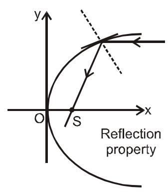
(57) Prove that the line lx + my – n = 0 will be a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

if $\frac{a^2}{\ell^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$.

15. Important Highlights of Parabola :

(i) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then ST = SG = SP where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of theparabola after reflection.

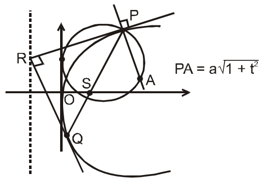




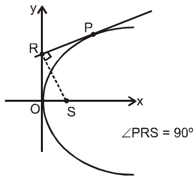
(ii) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the focus.

See figure above.

(iii) The tangents at the extremities of a focal chord intersect at right angles on the directrix, and hence a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P (at^2, 2at) as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.



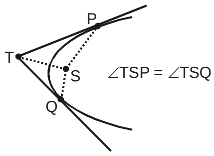
(iv) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.



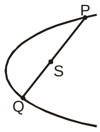
(v) If the tangents at P and Q meet in T, then:

$\Rightarrow$ TP and TQ subtend equal angles at the focus S.

$\Rightarrow$ $ST^2 = SP. SQ$ & $\Rightarrow$ The triangles SPT and STQ are similar.



(vi) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola.



$$\frac{2(PS)(SQ)}{PS + SQ} = 2a$$

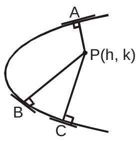
(vii) The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

(viii) If normal are drawn from a point P(h, k) to the parabola $y^2 = 4ax$ then

$k = mh - 2am - am^3$ i.e. $am^3 + m(2a - h) + k = 0.$

$m_1 + m_2 + m_3 = 0$; $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a - h}{a}$; $m_1 m_2 m_3 = -\frac{k}{a}$.

Where m_1, m_2, & m_3 are the slopes of the three concurrent normals. Note that



A, B, C $\rightarrow$ Conormal points

$\Rightarrow$ algebraic sum of the slopes of the three concurrent normals is zero.

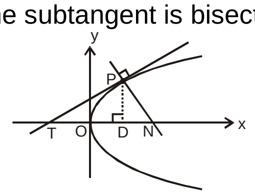
$\Rightarrow$ algebraic sum of the ordinates of the three conormal points on the parabola is zero

$\Rightarrow$ Centroid of the Δ formed by three co–normal points lies on the x–axis.

$\Rightarrow$ Condition for three real and distinct normals to be drawn froma point P (h, k) is

$h > 2a$ & $k^2 < \frac{4}{27a}(h - 2a)^3$.

(ix) Length of subtangent at any point P(x, y) on the parabola $y^2 = 4ax$ equals twice the abscissa of the point P. Note that the subtangent is bisected at the vertex..



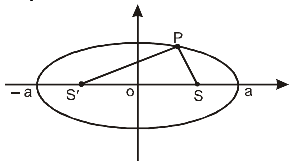
TD = 2(OD) , DN = 2a

(x) Length of subnormal is constant for all points on the parabola & is equal to the semi latus rectum. See figure above.

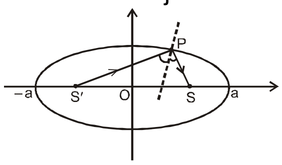
16. Important Highlights of Ellipse :

Refering to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

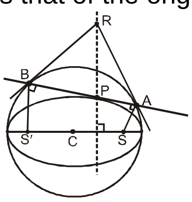
(i) If P be any point on the ellipse with S & S′ as its foci then ℓ (SP) + ℓ (S′P) = 2a.



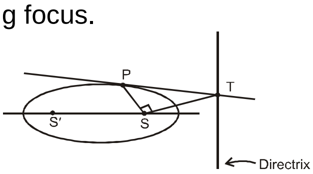
(ii) The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice–versa. Hence we can deduce that the straight lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it where G is the point where normal at P meets the major axis.



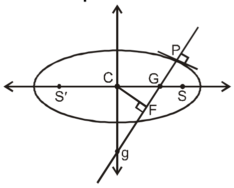
(iii) The product of the length's of the perpendicular segments from the foci on any tangent to the ellipse is b² and the feet of these perpendiculars lie on its auxiliary circle and the tangents at these feet to the auxiliary circle meet on the ordinate of P and that the locus of their point of Intersection is a similiar ellipse as that of the original one.



(iv) The portion of the tangent to an ellipse between the point of contact & the directrix subtends a right angle at the corresponding focus.



(v) If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively, & if CF be Perpendicular upon this normal, then



(i) PF. PG = b² (ii) PF. Pg = a²

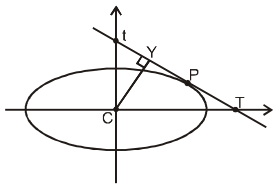
(iii) PG. Pg = SP. S′ P (iv) CG. CT = CS²

(v) Locus of the mid point of Gg is another ellipse having the same eccentricity as that of the original ellipse.

[Where S and S′ are the focii of the ellipse and T is the point where tangent at P meet the major axis]

(vi) The circle on any focal distance as diameter touches the auxiliary circle. Perpendiculars from the centre upon all chords which join the ends of any perpendicular diameters of the ellipse are of constant length.

(vii) If the tangent at the point P of a standard ellipse meets the axis in T and t and CY is the perpendicular on it from the centre then,

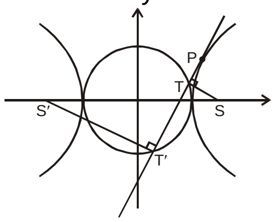


(i) $Tt. PY = a^2 - b^2$ and
(ii) least value of Tt is $a + b$.

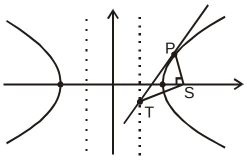
17. Important Highlights of Hyperbola:

(i) Difference of focal distances is a constant, i.e. $|PS - PS'| = 2a$

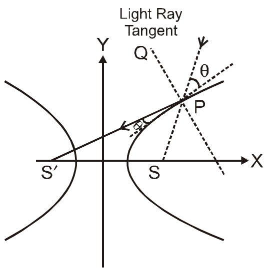
(ii) Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ & the product of these perpendiculars is b^2.



(iii) The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

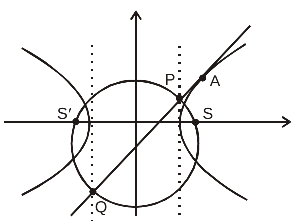


(iv) The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This explains the reflection property of the hyperbola as **"An incoming light ray "** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.



Note that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$& the hyperbola $\frac{x^2}{a^2 - k^2} - \frac{y^2}{k^2 - b^2} = 1$ $(a > k > b > 0)$ are confocal and therefore orthogonal.

(v) The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.



(vi) A rectangular hyperbola circumscribing a triangle also passes through the orthocentre of this triangle. If $\left(ct_i, \frac{c}{t_i}\right)$ i = 1, 2, 3 be the angular points P, Q, R then orthocentre is $\left(\frac{-c}{t_1\ t_2\ t_3}, -c\,t_1\ t_2\ t_3\right)$.

(vii) If a circle and the rectangular hyperbola $xy = c^2$ meet in the four points t_1, t_2, t_3 & t_4, then

(a) $t_1 t_2 t_3 t_4 = 1$

(b) the centre of the mean position of the four points bisects the distance between the centres of the two curves.

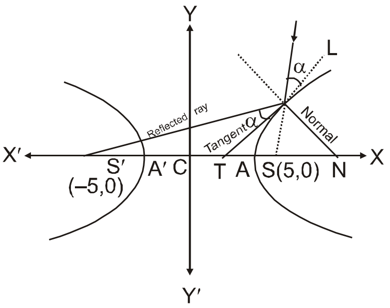
(c) the centre of the circle through the points t_1, t_2 & t_3 is :

$$\left\{\frac{c}{2}\left(t_1 + t_2 + t_3 + \frac{1}{t_1\ t_2\ t_3}\right), \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + t_1\ t_2\ t_3\right)\right\}$$

Example # 66 : A ray originating from the point (5, 0) is incident on the hyperbola $9x^2 - 16y^2 = 144$ at the point P with abscissa 8. Find the equation of the reflected ray after first reflection and point P lying in first quadrant.

Solution : Given hyperbola is $9x^2 - 16y^2 = 144$. This equation can be rewritten as $\frac{x^2}{16} - \frac{y^2}{9} = 1$ ....(1)

Since x co–ordinate of P is 8. Let y co–ordinate of P is α. $\because$ $(8,\alpha)$ lies on (1)



$\therefore$ $\frac{64}{16} - \frac{\alpha^2}{9} = 1 \Rightarrow \alpha^2 = 27 \Rightarrow \alpha = 3\sqrt{3}$ ($\because$ P lies in first quadrant)

Hence co-ordinate of point P is $(8, 3\sqrt{3})$.

$\because$ Equation of reflected ray passes through P $(8, 3\sqrt{3})$ and S′(–5,0)

$\therefore$ Its equation is $y - 3\sqrt{3} = \frac{0 - 3\sqrt{3}}{-5-8}(x - 8)$ or $13y - 39\sqrt{3} = 3\sqrt{3}\,x - 24\sqrt{3}$

or $3\sqrt{3}\,x - 13y + 15\sqrt{3} = 0$.