Complex Numbers

The shortest path between two truths in the real domain passes through the complex domain. ......Hadamard, Jacques

The complex number system

A complex number (z) is a number that can be expressed in the form z = a + ib where a and b are real numbers and $i^2 = -1$. Here 'a' is called as real part of z which is denoted by (Re z) and 'b' is called as imaginary part of z, which is denoted by (Im z).

Any complex number is :

(i) Purely real, if b = 0 ;

(ii) Imaginary, if $b \neq 0$.

(iii) Purely imaginary, if a = 0

Note :

(a) The set R of real numbers is a proper subset of the Complex Numbers. Hence the complete number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(b) Zero is purely real as well as purely imaginary but not imaginary.

(c) $i = \sqrt{-1}$ is called the imaginary unit.

Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.

(d) $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ only if atleast one of a or b is non - negative.

(e) If z = a + ib, then a – ib is called complex conjugate of z and written as $\bar{z} = a - ib$

(f) Real numbers satisfy order relations where as imaginary numbers do not satisfy order relations i.e. i > 0, 3 + i < 2 are meaningless.

Self Practice Problems

(1) Write the following as complex number

(i) $\sqrt{-16}$ (ii) $\sqrt{x}$, (x > 0) (iii) $-b + \sqrt{-4ac}$, (a, c> 0)

(2) Write the following as complex number

(i) $\sqrt{x}$ (x < 0) (ii) roots of $x^2 - (2\cos\theta)x + 1 = 0$

Answers : (1) (i) 0 + 4i (ii) $\sqrt{x} + 0i$ (iii) $-b + i\sqrt{4ac}$

(2) (i) $0 + i\sqrt{x}$ (ii) $\cos\theta + i\sin\theta$, $\cos\theta - i\sin\theta$

Algebraic Operations:

Fundamental operations with complex numbers

In performing operations with complex numbers we can proceed as in the algebra of real numbers, replacing i^2 by – 1 when it occurs.

1. Addition $(a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d)i$
2. Subtraction $(a + bi) - (c + di) = a + bi - c - di = (a - c) + (b - d)i$
3. Multiplication $(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$
4. Division $\dfrac{a+bi}{c+di} = \dfrac{a+bi}{c+di}\cdot\dfrac{c-di}{c-di} = \dfrac{ac-adi+bci-bdi^2}{c^2-d^2i^2} = \dfrac{ac+bd+(bc-ad)i}{c^2+d^2} = \dfrac{ac+bd}{c^2+d^2} + \dfrac{bc-ad}{c^2+d^2}i$

Inequalities in imaginary numbers are not defined. There is no validity if we say that imaginary number is positive or negative.

e.g. z > 0, 4 + 2i < 2 + 4 i are meaningless.

In real numbers if $a^2 + b^2 = 0$ then $a = 0 = b$ however in complex numbers,
$z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

Example # 1 : Find the multiplicative inverse of 4 + 3i .
Solution : Let z be the multiplicative inverse of 4 + 3i then
$z(4 + 3i) = 1$
$z = \frac{1}{4+3i} \times \frac{4-3i}{4-3i} = \frac{4-3i}{16+9} = \frac{4-3i}{25}$. **Ans.** $\frac{4-3i}{25}$

Self Practice Problem :

(3) Simplify $i^n + i^{n+1} + i^{n+2} + i^{n+3}$, $n \in I$. **Ans.** 0

Equality In Complex Number :

Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real and imaginary parts are equal respectively
i.e. $z_1 = z_2 \Leftrightarrow Re(z_1) = Re(z_2)$ and $I_m(z_1) = I_m(z_2)$.

Example # 2 : Find the value of x and y for which
$(x^4 + 2xi) - (3x^2 + yi) = (3 - 5i) + (1 + 2iy)$, where x, yR
Solution : $(x^4 + 2xi) - (3x^2 + yi) = (3 - 5i) + (1 + 2iy)$
$\Rightarrow$ $x^4 - 3x^2 - 4 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$
and $2x - y = -5 + 2y$
$2x + 5 = 3y$
when $x = 2 \Rightarrow y = 3$
and $x = -2 \Rightarrow y = 1/3$ **Ans.** (2,3) or (–2,1/3)

Example # 3 : Find the value of expression $x^4 + 4x^3 + 5x^2 + 2x + 3$, when $x = -1 + i$.
Solution : $x = -1 + i$
$(x + 1)^2 = i^2$
$x^2 + 2x + 2 = 0$
now, $x^4 + 4x^3 + 5x^2 + 2x + 3 = (x^2 + 2x + 2)(x^2 + 2x - 1) + 5 = 5$

Example # 4 : Find the square root of – 21 – 20i
Solution : Let $x + iy = \sqrt{-21-20i}$
$(x + iy)^2 = -21 - 20i$
$x^2 - y^2 = -21$ ----- (i)
$xy = -10$ ----- (ii)
From (i) & (ii)
$x^2 = 4 \Rightarrow x = \pm 2$
when $x = 2, y = -5$ and $x = -2, y = 5$
$x + iy = (2 - i5)$ or $(-2 + i5)$

Self Practice Problem

(4) Solve for z : $\bar{z} = i z^2$

(5) Given that $x, y \in R$, solve : $4x^2 + 3xy + (2xy - 3x^2)i = 4y^2 - (x^2/2) + (3xy - 2y^2)i$

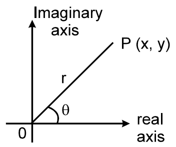
Answers : (4) $\pm\frac{\sqrt{3}}{2} - \frac{1}{2}i, 0, i$ (5) $x = K, y = \frac{3K}{2}$ $K \in R$

Representation of a complex number :

To each complex number there corresponds one and only one point in plane, and conversely to each point in the plane there corresponds one and only one complex number. Because of this we often refer to the complex number z as the point z.

(a) Cartesian Form (Geometric Representation) :

Every complex number $z = x + i\,y$ can be represented by a point on the Cartesian plane known as complex plane (Argand diagram) by the ordered pair (x, y).



Length OP is called modulus of the complex number which is denoted by $|z|$ & θ is called argument or amplitude.

$|z| = \sqrt{x^2 + y^2}$ and $\tan\theta = \left(\frac{y}{x}\right)$ (angle made by OP with positive x–axis)

Note :

(i) Argument of a complex number is a many valued function. If θ is the argument of a complex number then $2n\pi + \theta$; $n \in I$ will also be the argument of that complex number. Any two arguments of a complex number differ by $2n\pi$.

(ii) The unique value of θ such that $-\pi < \theta \le \pi$ is called the principal value of the argument. Unless otherwise stated, amp z implies principal value of the argument.

(iii) By specifying the modulus & argument a complex number is defined completely. For the complex number 0 + 0 i the argument is not defined and this is the only complex number which is only given by its modulus.

(b) Trignometric/Polar Representation :

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

Note : $\cos\theta + i\sin\theta$ is also written as CiS θ

(c) Euler's Formula :

$z = re^{i\theta}$, $|z| = r$, $\arg z = \theta$

$\bar{z} = re^{-i\theta}$

$e^{i\theta} = \cos\theta + i\sin\theta$.

Note : If θ is real then $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$; $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

(d) **Vectorial Representation :**

Every complex number can be considered as the position vector of a point. If the point P represents the complex number z then, $\vec{OP} = z$ & $|\vec{OP}| = |z|$.

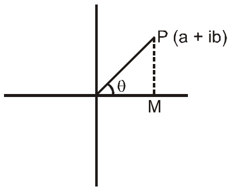
Agrument of a Complex Number :

Argument of a non-zero complex number P(z) is denoted and defined by arg(z) = angle which OP makes with the positive direction of real axis.

If $OP = |z| = r$ and $\arg(z) = \theta$, then obviously $z = r(\cos\theta + i\sin\theta)$, called the polar form of z.

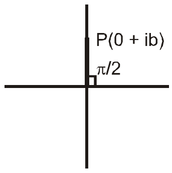
'Argument of z' would mean principal argument of z(i.e. argument lying in $(-\pi, \pi]$ unless the context requires otherwise. Thus argument of a complex number $z = a + ib = r(\cos\theta + i\sin\theta)$ is the value of θ satisfying $r\cos\theta = a$ and $r\sin\theta = b$. Let $\theta = \tan^{-1}\left|\frac{b}{a}\right|$

(i) $a > 0, b > 0$



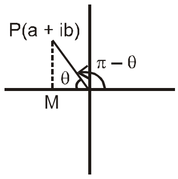
p.v. arg $z = \theta$

(ii) $a = 0, b > 0$



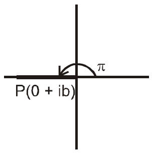
p.v. arg $z = \frac{\pi}{2}$

(iii) $a < 0, b > 0$



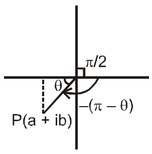
p.v. arg $z = \pi - \theta$

(iv) $a < 0, b = 0$



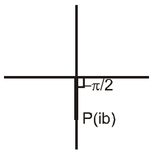
p.v. arg $z = \pi$

(v) $a < 0, b < 0$



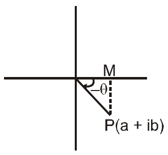
p.v. arg $z = -(\pi - \theta)$

(vi) $a = 0, b < 0$



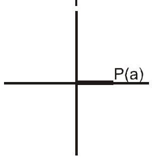
p.v. arg $z = -\frac{\pi}{2}$

(vii) $a > 0, b < 0$



p.v. arg $z = -\theta$

(viii) $a > 0, b = 0$



p.v. arg $z = 0$

Example # 5 : Solve for z if $(\overline{z})^2 + 2|z| = 0$.

Solution : Let $z = x + iy$

$\Rightarrow \quad x^2 - y^2 - 2ixy + 2\sqrt{x^2 + y^2} = 0 \quad \Rightarrow \quad x^2 - y^2 + 2\sqrt{x^2 + y^2} = 0$ and $2xy = 0$

when $x = 0$, $\Rightarrow \quad -y^2 + 2|y| = 0$

$\Rightarrow \quad y = 0, 2, -2$

$\Rightarrow \quad z = 0, 2i, -2i$

when $y = 0$ $\Rightarrow \quad x^2 + 2|x| = 0$

$\Rightarrow \quad x = 0 \Rightarrow z = 0$

Ans. $z = 0, 2i, -2i$.

Example # 6 : Find the modulus and principal argument of complex number $z = 1 + i\tan\alpha$, $\pi < \alpha < \dfrac{3\pi}{2}$

Solution : $|z| = \sqrt{1+\tan^2\alpha} = |\sec\alpha| = -\sec\alpha$, where $\pi < \alpha < \dfrac{3\pi}{2}$

$\text{Arg}(z) = \tan^{-1}\left|\dfrac{\tan\alpha}{1}\right| = \tan^{-1}(\tan\alpha) = \alpha - \pi$

Ans. $-\sec\alpha$, $\alpha - \pi$

Self Practice Problems

(6) Find the principal argument and |z|. If $z = \dfrac{(2+i)\ (3-4i)}{3+i}$

(7) Find the |z| and principal argument of the complex number
$z = -8(\cos 310^\circ - i\sin 310^\circ)$

Answers : (6) $-\pi/4$, $\dfrac{5\sqrt{2}}{2}$ (7) 8, –130°

Demoivre's Theorem :

(i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ where $n \in I$

(ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3) \ldots(\cos\theta_n + i\sin\theta_n)$
$= \cos(\theta_1 + \theta_2 + \theta_3 + \ldots\ldots\ldots \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \ldots\ldots. + \theta_n)$ where $n \in N$

(iii) If $p, q \in Z$ and $q \neq 0$, then $(\cos\theta + i\sin\theta)^{p/q}$ can take 'q' distinct values which are equal to
$\cos\left(\dfrac{2k\pi + p\theta}{q}\right) + i\sin\left(\dfrac{2k\pi + p\theta}{q}\right)$ where k = 0, 1, 2, 3, ..., q – 1

Note : Continued product of the roots of a complex quantity should be determined using theory of equations.

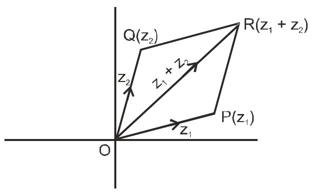
Self practice problems :

(8) Prove the identity: $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$;

(9) Prove that identity: $\cos^4\theta = \dfrac{1}{8}\cos 4\theta + \dfrac{1}{2}\cos 2\theta + \dfrac{3}{8}$

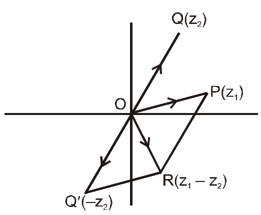
Geometrical Representation of Fundamental Operations :

(i) Geometrical representation of addition.



If two points P and Q represent complex numbers z_1 and z_2 respectively in the Argand plane, then the sum $z_1 + z_2$ is represented by the extremity R of the diagonal OR of parallelogram OPRQ having OP and OQ as two adjacent sides.

(ii) Geometric representation of substraction.



(iii) Modulus and argument of multiplication of two complex numbers.

Theorem : For any two complex numbers z_1, z_2 we have $|z_1 z_2| = |z_1| |z_2|$ and $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$.

Proof : $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$z_1z_2 = r_1r_2 e^{i(\theta_1+\theta_2)} \Rightarrow |z_1z_2| = |z_1| |z_2|$

$\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

i.e. to multiply two complex numbers, we multiply their absolute values and add their arguments.

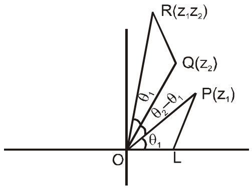
Note : (i) P.V. $\arg(z_1z_2) \neq$ P.V. $\arg(z_1)$ + P.V. $\arg(z_2)$

(ii) $|z_1 z_2 \ldots z_n| = |z_1| |z_2| \ldots |z_n|$

(iii) $\arg(z_1z_2 \ldots z_n) = \arg z_1 + \arg z_2 + \ldots + \arg z_n$

(iv) Geometrical representation of multiplication of complex numbers.

Let P, Q be represented by $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$ repectively. To find point R representing complex number z_1z_2, we take a point L on real axis such that OL = 1 and draw triangle OQR similar to triangle OLP. Therefore



$\frac{OR}{OQ} = \frac{OP}{OL} \Rightarrow OR = OP.OQ$ i.e. $OR = r_1r_2$ and $Q\hat{O}R = \theta_1$

$L\hat{O}R = L\hat{O}P + P\hat{O}Q + Q\hat{O}R = \theta_1 + \theta_2 - \theta_1 + \theta_1 = \theta_1 + \theta_2$

Hence, R is represented by $z_1z_2 = r_1r_2 e^{i(\theta_1+\theta_2)}$

(v) Modulus and argument of division of two complex numbers.

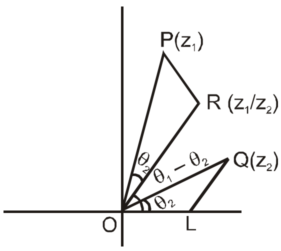
Theorem : If z_1 and z_2 ($\neq 0$) are two complex numbers, then $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Note : P.V. $\arg\left(\frac{z_1}{z_2}\right) \neq$ P.V. $\arg(z_1)$ – P.V. $\arg(z_2)$

(vi) Geometrical representation of the division of complex numbers.

Let P, Q be represented by $z_1 = r_1e^{i\theta_1}$, $z_2 = r_2e^{i\theta_2}$ respectively. To find point R representing complex number $\frac{z_1}{z_2}$, we take a point L on real axis such that OL = 1 and draw a triangle OPR similar to OQL.

Therefore $\frac{OP}{OQ} = \frac{OR}{OL} \Rightarrow OR = \frac{r_1}{r_2}$ and $L\hat{O}R = L\hat{O}P - R\hat{O}P = \theta_1 - \theta_2$



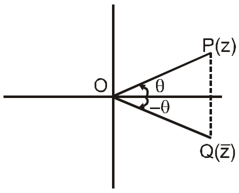
Hence, R is represented by $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

Conjugate of a complex Number :

Conjugate of a complex number $z = a + ib$ is denoted and defined by $\bar{z} = a - ib$.

In a complex number if we replace i by – i, we get conjugate of the complex number. $\bar{z}$ is the mirror image of z about real axis on Argand's Plane.

Geometrical representation of conjugate of complex number.



$|z| = |\bar{z}|$

$\arg(\bar{z}) = -\arg(z)$

General value of $\arg(\bar{z}) = 2n\pi$ – P.V. arg (z)

Properties

(i) If z = x + iy, then $x = \dfrac{z+\bar{z}}{2}$, $y = \dfrac{z-\bar{z}}{2i}$

(ii) $z = \bar{z}$ $\Leftrightarrow$ z is purely real

(iii) $z + \bar{z} = 0$ $\Leftrightarrow$ z is purely imaginary

(iv) Relation between modulus and conjugate. $|z|^2 = z\bar{z}$

(v) $\bar{\bar{z}} = z$

(vi) $\overline{(z_1 \pm z_2)} = \bar{z}_1 \pm \bar{z}_2$

(vii) $\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2$, In general $\overline{(z^n)} = (\bar{z})^n$

(viii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{(\bar{z}_1)}{(\bar{z}_2)}$ $(z_2 \neq 0)$

Theorem : Imaginary roots of polynomial equations with real coefficients occur in conjugate pairs

Note : If w = f(z), then $\bar{w} = f(\bar{z})$

Theorem : $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm (z_1\bar{z}_2 + \bar{z}_1 z_2) = |z_1|^2 + |z_2|^2 \pm 2\,\mathrm{Re}(z_1\bar{z}_2)$

$= |z_1|^2 + |z_2|^2 \pm 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

Example # 7 : If $\dfrac{z-1}{z+1}$ is purely imaginary, then prove that **| z | = 1**

Solution : $\mathrm{Re}\left(\dfrac{z-1}{z+1}\right) = 0$ $\Rightarrow$ $\dfrac{z-1}{z+1} + \overline{\left(\dfrac{z-1}{z+1}\right)} = 0$

$\Rightarrow$ $\dfrac{z-1}{z+1} + \dfrac{\bar{z}-1}{\bar{z}+1} = 0$ $\Rightarrow$ $z\bar{z} - \bar{z} + z - 1 + z\bar{z} - z + \bar{z} - 1 = 0$

$\Rightarrow$ $z\bar{z} = 1$ $\Rightarrow$ $|z|^2 = 1$ $\Rightarrow$ $|z| = 1$ **Hence proved**

Example # 8 : If z_1 and z_2 are two complex numbers and c > 0, then prove that

$|z_1 + z_2|^2 \leq (1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2$

Solution : We have to prove : $|z_1 + z_2|^2 \leq (1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2$

i.e. $|z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 \leq (1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2$

or $z_1\bar{z}_2 + \bar{z}_1 z_2 \leq c|z_1|^2 + c^{-1}|z_2|^2$ or $c|z_1|^2 + \dfrac{1}{c}|z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2 \geq 0$

(using $\mathrm{Re}(z_1\bar{z}_2) \leq |z_1\bar{z}_2|$) or $\left(\sqrt{c}\,|z_1| - \dfrac{1}{\sqrt{c}}|z_2|\right)^2 \geq 0$ which is always true.

Example # 9 : Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then show that $\dfrac{z_1 + z_2}{z_1 - z_2}$ is purely imaginary.

Solution : $z_1 = r(\cos\theta + i\sin\theta)$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$z_2 = r(\cos\phi + i\sin\phi)$, $-\pi < \phi < 0$

$\Rightarrow \quad \frac{z_1+z_2}{z_1-z_2} = -i\cot\left(\frac{\theta-\phi}{2}\right)$, $\quad -\frac{\pi}{4} < \frac{\theta-\phi}{2} < \frac{3\pi}{4}$

Hence purely imaginary.

Self Practice Problem

(10) If $|z + \alpha| > |\bar{\alpha}z + 1|$ and $|\alpha| > 1$, then show that $|z| < 1$.

(11) If $z = x + iy$ and $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, then show that $f(z) = \bar{z}^2 + 2iz$

Distance, Triangular Inequality

If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then distance between points z_1, z_2 in argand plane is

$|z_1 - z_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

In triangle OAC

OC ≤ OA + AC

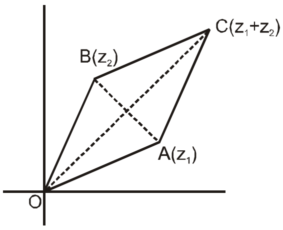
OA ≤ AC + OC

AC ≤ OA + OC

using these in equalities we have $||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$

Similarly from triangle OAB

we have $||z_1| - |z_2|| \le |z_1 - z_2| \le |z_1| + |z_2|$



Note :

(a) $||z_1| - |z_2|| = |z_1 + z_2|$, $|z_1 - z_2| = |z_1| + |z_2|$ iff origin, z_1 and z_2 are collinear and origin lies between z_1 and z_2.

(b) $|z_1 + z_2| = |z_1| + |z_2|$, $||z_1| - |z_2|| = |z_1 - z_2|$ iff origin, z_1 and z_2 are collinear and z_1 and z_2 lies on the same side of origin.

Example # 10 : If $|z - 5 - 7i| = 9$, then find the greatest and least values of $|z - 2 - 3i|$.

Solution : We have $9 = |z - (5 + 7i)|$ = distance between z and 5 + 7i.
Thus locus of z is the circle of radius 9 and centre at 5 + 7i. For such a z (on the circle), we have to find its greatest and least distance as from 2 + 3i, which obviously 14 and 4.

Example # 11 : Find the minimum value of $|z| + |z - 2|$

Solution : $|z| + |z - 2| \ge |z + 2 - z|$

$|z| + |z - 2| \ge 2$

Example # 12 : If $\theta_i \in [\pi/6, \pi/3]$, i = 1, 2, 3, 4, 5, and $z^4\cos\theta_1 + z^3\cos\theta_2 + z^2\cos\theta_3. + z\cos\theta_4 + \cos\theta_5 = 2\sqrt{3}$,

then show that $|z| > \frac{3}{4}$

Solution : Given that $\cos\theta_1 . z^4 + \cos\theta_2 . z^3 + \cos\theta_3 . z^2 + \cos\theta_4 . z + \cos\theta_5 = 2\sqrt{3}$

or $\quad |\cos\theta_1 . z^4 + \cos\theta_2 . z^3 + \cos\theta_3 . z^2 + \cos\theta_4 . z + \cos\theta_5| = 2\sqrt{3}$

$2\sqrt{3} \le |\cos\theta_1 . z^4| + |\cos\theta_2 . z^3| + |\cos\theta_3 . z^2| + \cos\theta_4 . z| + |\cos\theta_5|$

$\because$ $\theta_i \in [\pi/6, \pi/3]$

$\therefore$ $\frac{1}{2} \le \cos\theta_i \le \frac{\sqrt{3}}{2}$

$2\sqrt{3} \le \frac{\sqrt{3}}{2}|z|^4 + \frac{\sqrt{3}}{2}|z|^3 + \frac{\sqrt{3}}{2}|z|^2 + \frac{\sqrt{3}}{2}|z| + \frac{\sqrt{3}}{2}$

$3 \le |z|^4 + |z|^3 + |z|^2 + |z|$

Case I : If $|z| \ge 1$, then above result is automatically true

Case II : If $|z| < 1$, then

$3 < |z| + |z|^2 + |z|^3 + |z|^4 + |z|^5 + ...\infty$

$3 < \frac{|z|}{1-|z|} \Rightarrow 3 - 3|z| < |z| \Rightarrow |z| > \frac{3}{4}$ Hence by both cases, $|z| > \frac{3}{4}$

Example # 13 : $\left|z - \frac{3}{z}\right| = 2$, then find maximum and minimum value of $|z|$.

Solution : $\left||z| - \left|\frac{3}{z}\right|\right| \le \left|z - \frac{3}{z}\right|$

Let $|z| = r$

$\left|r - \frac{3}{r}\right| \le 2 \Rightarrow -2 \le r - \frac{3}{r} \le 2$

$r^2 + 2r - 3 \ge 0$ ....(i) and $r^2 - 2r - 3 \le 0$ ....(ii)

$\Rightarrow$ $r \in [1, 3]$

from (i) and (ii)

$|z|_{max} = 3$ and $|z|_{min} = 1$.

Self Practice Problem

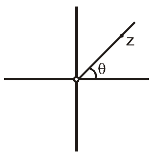
(12) $|z - 3| < 1$ and $|z - 4i| > M$ then find the positive real value of M for which there exist at least one complex number z satisfying both the equation.

(13) If z lies on circle $|z| = 2$, then show that $\left|\frac{1}{z^4 - 4z^2 + 3}\right| \le \frac{1}{3}$

Answers : (12) $M \in (0, 6)$

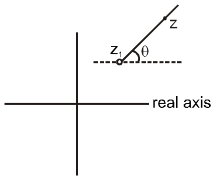
Important results :

(i) arg $z = \theta$ represnets points (non-zero) on ray



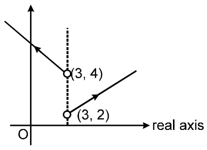
eminating from origin making an angle θ with positive direction of real axis

(ii) $\arg(z - z_1) = \theta$ represents points ($\neq z_1$) on ray eminating from z_1 making an angle θ with positive direction of real axis



Example # 14 : Solve for z, which satisfy $\text{Arg}(z - 3 - 2i) = \frac{\pi}{6}$ and $\text{Arg}(z - 3 - 4i) = \frac{2\pi}{3}$.

Solution : From the figure, it is clear that there is no z, which satisfy both ray

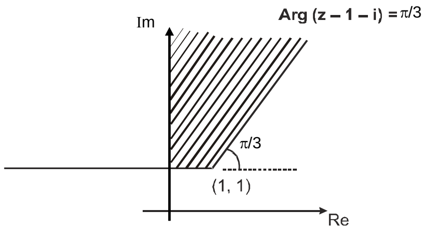


Example # 15 : Sketch the region given by

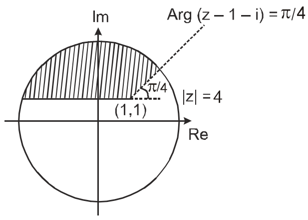
(i) $\pi/2 \geq \text{Arg}(z - 1 - i) \geq \pi/3$ (ii) $|z| \leq 4$ & $\text{Arg}(z - i - 1) > \pi/4$

Solution :

(i)



(ii)



Self Practice Problems

(14) Sketch the region given by

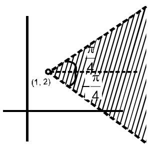
(i) $|\text{Arg}(z - i - 2)| < \pi/4$ (ii) $\text{Arg}(z + 1 - i) \leq \pi/6$

(15) Consider the region $|z - 4 - 3i| \leq 3$. Find the point in the region which has

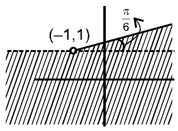
(i) max $|z|$ (ii) min $|z|$

(iii) max arg (z) (iv) min arg (z)

Answers : (14) (i)



(ii)

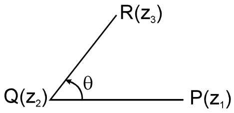


(15) (i) $\frac{32}{5} + i\frac{24}{5}$ (ii) $\frac{8}{5} + i\frac{6}{5}$ (iii) $\frac{28}{25} + i\frac{96}{25}$ (iv) 4 + 0i

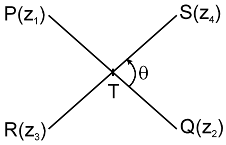
Rotation theorem :

(i) If $P(z_1)$ and $Q(z_z)$ are two complex numbers such that $|z_1| = |z_2|$, then $z_2 = z_1 e^{i\theta}$ where $\theta = \angle POQ$

(ii) If $P(z_1)$, $Q(z_2)$ and $R(z_3)$ are three complex numbers and $\angle PQR = \theta$, then $\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \left|\frac{z_3 - z_2}{z_1 - z_2}\right| e^{i\theta}$



(iii) If $P(z_1)$, $Q(z_2)$, $R(z_3)$ and $S(z_4)$ are four complex numbers and $\angle STQ = \theta$, then $\frac{z_3 - z_4}{z_1 - z_2} = \left|\frac{z_3 - z_4}{z_1 - z_2}\right| e^{i\theta}$



Example # 16 : If $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$ then interpret the locus.

Solution : $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4} \Rightarrow \arg\left(\frac{1-z}{-1-z}\right) = \frac{\pi}{4}$

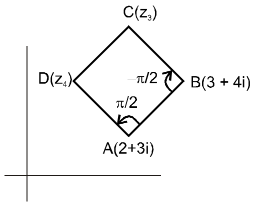
Here $\arg\left(\frac{1-z}{-1-z}\right)$ represents the angle between lines joining –1 and z, and 1 and z. As this angle is constant, the locus of z will be a larger segment of circle. (angle in a segment is constant).

Example # 17 : If A(2 + 3i) and B(3 + 4i) are two vertices of a square ABCD (take in anticlock wise order) then find C and D.

Solution : Let affix of C and D are z_3 and z_4 respectively.

Considering $\angle DAB = 90^\circ$ and AD = AB

we get $\frac{z_4 - (2+3i)}{(3+4i)-(2+3i)} = \frac{AD}{AB} e^{\frac{i\pi}{2}}$



$\Rightarrow \quad z_4 - (2 + 3i) = (1 + i)\, i \qquad \Rightarrow \quad z_4 = 2 + 3i + i - 1 = 1 + 4i$

and $\quad \frac{z_3 - (3+4i)}{(2+3i)-(3+4i)} = \frac{CB}{AB} e^{-\frac{i\pi}{2}} \qquad \Rightarrow \quad z_3 = 3 + 4i - (1 + i)(-i)$

$\Rightarrow \quad z_3 = 3 + 4i + i - 1 = 2 + 5i$

Self Practice Problems

(16) Let ABC be an isosceles triangle inscribed in the circle |z|= r with AB = AC. If z_1, z_2, z_3 represent the points A, B, C respectively, show that $z_2 z_3 = z_1^2$

(17) Check that z_1z_2 and z_3z_4 are parallel or, not

where, $z_1 = 1 + i$ $z_3 = 4 + 2i$

$z_2 = 2 - i$ $z_4 = 1 - i$

(18) P is a point on the argand diagram on the circle with OP as diameter, two point Q and R are taken such that $\angle POQ = \angle QOR = \theta$. If O is the origin and P, Q, R are represented by complex z_1, z_2, z_3 respectively then show that $z_2^2 \cos 2\theta = z_1 z_3 \cos^2\theta$

(19) If a, b, c ; u, v, w are complex numbers representing the vertices of two triangles such that $c = (1 - r)a + rb$, $w = (1 - r)u + rv$ where r is a complex number show that the two triangles are similiar.

Answers : (17) z_1z_2 and z_3z_4 are not parallel.

Cube Root of Unity :

(i) The cube roots of unity are 1, $\frac{-1+i\sqrt{3}}{2}$, $\frac{-1-i\sqrt{3}}{2}$.

(ii) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3.

(iii) In polar form the cube roots of unity are :
$\cos 0 + i \sin 0$; $\cos \frac{2\pi}{3} + i \sin, \frac{2\pi}{3}$ $\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$

(iv) The three cube roots of unity when plotted on the argand plane constitute the verties of an equilateral triangle.

(v) The following factorisation should be remembered :
(a, b, c $\in$ R & ω is the cube root of unity)
$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$; $x^2 + x + 1 = (x - \omega)(x - \omega^2)$;
$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$; $a^2 + ab + b^2 = (a - b\omega)(a - b\omega^2)$
$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

Example # 18 : Find the value of $\omega^{200} + \omega^{198} + \omega^{193}$.
Solution : $\omega^{200} + \omega^{198} + \omega^{193}$
$\omega^2 + 1 + \omega = 0$.

Example # 19 If W is an imaginary cube root of unity then find the value of $\frac{1}{1+2w} + \frac{1}{2+w} - \frac{1}{1+w}$

Solution : $\frac{1}{1+w+w} + \frac{1}{1+(1+w)} - \frac{1}{1+w} = \frac{1}{-w^2+w} + \frac{1}{1-w^2} - \frac{1}{-w^2}$

$= \frac{1}{w(1-w)} + \frac{1}{(1-w^2)} + \frac{1}{w^2} = \frac{w(1+w)+w^2+1-w^2}{w^2(1-w^2)} = \frac{1+w+w^2}{w^2(1-w^2)} = 0$

Ans. 0

Self Practice Problem

(20) Find $\sum_{r=0}^{100}(1+\omega^r+\omega^{2r})$

(21) It is given that n is an odd integer greater than three, but n is not a multiple of 3. Prove that $x^3 + x^2 + x$ is a factor of $(x + 1)^n - x^n - 1$

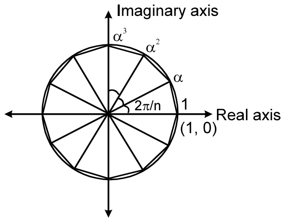
(22) If x = a + b, y = $a\alpha + b\beta$, z = $a\beta + b\alpha$ where α, β are imaginary cube roots of unity show that $xyz = a^3 + b^3$

(23) If $x^2 - x + 1 = 0$, then find the value of $\sum_{n=1}^{5}\left(x^n + \frac{1}{x^n}\right)^2$

Answers : (20) 102 (23) 8

n^{th} Roots of Unity :

If 1, α_1, α_2, α_3.... α_{n-1} are the n, n^{th} root of unity then :



(i) They are in G.P. with common ratio $e^{i(2\pi/n)}$

(ii) $1^p + \alpha_1^p + \alpha_2^p + ... + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n

= n if p is an integral multiple of n

(iii) $(1 - \alpha_1)(1 - \alpha_2)......(1 - \alpha_{n-1}) = n$ &

$(1 + \alpha_1)(1 + \alpha_2).......(1 + \alpha_{n-1}) = 0$ if n is even and 1 if n is odd.

(iv) $1.\ \alpha_1.\ \alpha_2.\ \alpha_3........\ \alpha_{n-1} = 1$ or -1 according as n is odd or even.

Example # 20 : Find the roots of the equation $z^5 = -32i$, whose real part is negative.

Solution : $z^5 = -32i$

$z^5 = 2^5 e^{i(4n-1)\frac{\pi}{2}}$, n = 0, 1, 2, 3, 4.

$z = 2e^{i(4n-1)\frac{\pi}{10}}$

$z = 2e^{-i\frac{\pi}{10}}, 2e^{i\frac{3\pi}{10}}, 2e^{i\frac{7\pi}{10}}, 2e^{i\frac{11\pi}{10}}, 2e^{i\frac{15\pi}{10}}$ roots with negative real part are $2e^{i\frac{7\pi}{10}}, 2e^{i\frac{11\pi}{10}}$.

Example # 21 : Find the value $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - \cos\frac{2\pi k}{7}\right)$

Solution : $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}\right) - \sum_{k=1}^{6}\left(\cos\frac{2\pi k}{7}\right) = \sum_{k=0}^{6}\sin\frac{2\pi k}{7} - \sum_{k=0}^{6}\cos\frac{2\pi k}{7} + 1$

$= \sum_{k=0}^{6}$ (Sum of imaginary part of seven seventh roots of unity)

$- \sum_{k=0}^{6}$ (Sum of real part of seven seventh roots of unity) + 1 = 0 – 0 + 1 = 1

Self Practice Problems

(24) If 1, α_1, α_2 , α_3 , α_4 are the fifth roots of unity then find $\sum_{i=1}^{4}\frac{1}{2 - \alpha_i}$

(25) If α, β, γ are the roots of $x^3 - 3x^2 + 3x + 7 = 0$ and ω is a complex cube root of unity then prove that $\frac{\alpha-1}{\beta-1}+\frac{\beta-1}{\gamma-1}+\frac{\gamma-1}{\alpha-1}=3\omega^2$

(26) Find all values of $(-256)^{1/4}$. Interpret the result geometrically.

Answers : (24) $\frac{49}{31}$

(26) $4\left[\cos\left(\frac{2r+1}{4}\right)\pi+i\sin\left(\frac{2r+1}{4}\right)\pi\right]$, r = 0, 1, 2, 3; vertices of a square in a circle of radius 4 & centre (0, 0)

The Sum Of The Following Series Should Be Remembered :

(i) $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \cos\left(\frac{n+1}{2}\right)\theta.$

(ii) $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \sin\left(\frac{n+1}{2}\right)\theta.$

Note : If $\theta = (2\pi/n)$ then the sum of the above series vanishes.

Geometrical Properties :

Section formula

If z_1 and z_2 are affixes of the two points P and Q respectively and point C divides the line segment joining P and Q internally in the ratio m : n then affix z of C is given by

$z = \frac{mz_2+nz_1}{m+n}$ where m, n > 0

If C divides PQ in the ratio m : n externally then $z = \frac{mz_2-nz_1}{m-n}$

Note : If a, b, c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where a + b + c = 0 and a,b,c are not all simultaneously zero, then the complex numbers z_1, z_2 & z_3 are collinear.

(1) If the vertices A, B, C of a Δ are represented by complex numbers z_1, z_2, z_3 respectively and a, b, c are the length of sides then,

(i) Centroid of the Δ ABC = $\frac{z_1+z_2+z_3}{3}$:

(ii) Orthocentre of the Δ ABC =

$$\frac{(a\sec A)z_1+(b\sec B)z_2+(c\sec C)z_3}{a\sec A+b\sec B+c\sec C} \text{ or } \frac{z_1\tan A+z_2\tan B+z_3\tan C}{\tan A+\tan B+\tan C}$$

(iii) Incentre of the Δ ABC = $(az_1 + bz_2 + cz_3) \div (a + b + c)$.

(iv) Circumcentre of the Δ ABC = :
$(Z_1 \sin 2A + Z_2 \sin 2B + Z_3 \sin 2C) \div (\sin 2A + \sin 2B + \sin 2C)$.

(2) amp(z) = θ is a ray emanating from the origin inclined at an angle θ to the positive x– axis.

(3) $|z-a| = |z-b|$ is the perpendicular bisector of the line joining a to b.

(4) The equation of a line joining z_1 & z_2 is given by, $z = z_1 + t(z_1 - z_2)$ where t is a real parameter.

(5) $z = z_1 (1 + it)$ where t is a real parameter is a line through the point z_1 & perpendicular to the line joining z_1 to the origin.

(6) The equation of a line passing through z_1 & z_2 can be expressed in the determinant form as $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$. This is also the condition for three complex numbers z, z_1, z_2 to be collinear. The above equation on manipulating, takes the form $\bar{\alpha} z + \alpha \bar{z} + r = 0$ where r is real and α is a non zero complex constant.

(7) The equation of the circle described on the line segment joining z_1 & z_2 as diameter is $\arg \dfrac{z - z_2}{z - z_1} = \pm \dfrac{\pi}{2}$

or $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$.

(8) Condition for four given points z_1, z_2, z_3 & z_4 to be concyclic is the number $\dfrac{z_3 - z_1}{z_3 - z_2} \cdot \dfrac{z_4 - z_2}{z_4 - z_1}$ should be real. Hence the equation of a circle through 3 non collinear points z_1, z_2 & z_3 can be taken as $\dfrac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}$ is real $\Rightarrow$ $\dfrac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)} = \dfrac{(\bar{z} - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}{(\bar{z} - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}$.

(9) $\operatorname{Arg}\left(\dfrac{z - z_1}{z - z_2}\right) = \theta$ represent (i) a line segment if $\theta = \pi$

(ii) Pair of ray if $\theta = 0$ (iii) a part of circle, if $0 < \theta < \pi$.

(10) If $|z - z_1| + |z - z_2| = K > |z_1 - z_2|$ then locus of z is an ellipse whose focii are z_1 & z_2

(11) If $\left|\dfrac{z - z_1}{z - z_2}\right| = k$ where $k \in (0, 1) \cup (1, \infty)$, then locus of z is circle.

(12) If $\big|\,|z - z_1| - |z - z_2|\,\big| = K < |z_1 - z_2|$ then locus of z is a hyperbola, whose focii are z_1 & z_2.

Match the following columns :

Column - I		**Column - II**	
(i)	If $\mid z - 3+2i \mid - \mid z + i \mid = 0$, then locus of z represents	(i)	circle
(ii)	If arg $\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$, then locus of z represents...	(ii)	Straight line
(iii)	if $\mid z - 8 - 2i \mid + \mid z - 5 - 6i \mid = 5$ then locus of z represents	(iii)	Ellipse
(iv)	If arg $\left(\frac{z-3+4i}{z+2-5i}\right) = \frac{5\pi}{6}$, then locus of z represents	(iv)	Hyperbola
(v)	If $\mid z - 1 \mid + \mid z + i \mid = 10$ then locus of z represents	(v)	Major Arc
(vi)	$\mid z - 3 + i \mid - \mid z + 2 - i \mid = 1$ then locus of z represents	(vi)	Minor arc
(vii)	$\mid z - 3i \mid = 25$	(vii)	Perpendicular bisector of a line segment
(viii)	arg $\left(\frac{z-3+5i}{z+i}\right) = \pi$	(viii)	Line segment

Ans.

I	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)
II	(ii),(vii)	(v)	(viii)	(vi)	(iii)	(iv)	(i)	(viii)

Example # 22 : If z_1, z_2 & z_3 are the affixes of three points A, B & C respectively and satisfy the condition $|z_1 - z_2| = |z_1| + |z_2|$ and $|(2 - i) z_1 + iz_3| = |z_1| + |(1 - i) z_1 + iz_3|$ then prove that Δ ABC in a right angled.

Solution : $|z_1 - z_2| = |z_1| + |z_2|$

$\Rightarrow$ z_1, z_2 and origin will be collinear and z_1, z_2 will be opposite side of origin

Similarly $|(2 - i) z_1 + iz_3| = |z_1| + |(1 - i) z_1 + iz_3|$

$\Rightarrow$ z_1 and $(1 - i) z_1 + iz_3 = z_4$ say, are collinear with origin and lies on same side of origin. Let $z_4 = \lambda z_1$, λ real

then $(1 - i) z_1 + iz_3 = \lambda z_1$

$\Rightarrow$ $i (z_3 - z_1) = (\lambda - 1) z_1 \Rightarrow \frac{(z_3 - z_1)}{-z_1} = (\lambda - 1)\, I \Rightarrow \frac{z_3 - z_1}{0 - z_1} = me^{i\pi/2}$, $m = \lambda - 1$

$\Rightarrow$ $z_3 - z_1$ is perpendicular to the vector $0 - z_1$.

i.e. also z_2 is on line joining origin and z_1

so we can say the triangle formed by z_1 , z_2 and z_3 is right angled.