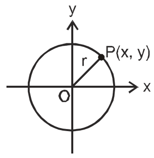
# CIRCLE

Four circles to the kissing come, The smaller are the benter. The bend is just the inverse of The distance from the centre. Through their intrigue left Euclid dumb There's now no need for rule of thumb. Since zero bend's a dead straight line And concave bends have minus sign, The sum of squares of all four bends Is half the square of their sum. ....... Soddy, Frederick

A circle is a locus of a point in a plane whose distance from a fixed point (called centre) is always constant (called radius).

## Equation of a circle in various forms :

(a) The circle with centre as origin & radius 'r' has the equation; $x^2 + y^2 = r^2$.



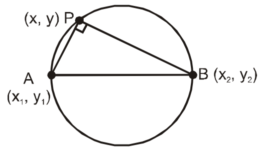

(b) The circle with centre (h, k) & radius 'r' has the equation; $(x – h)^2 + (y – k)^2 = r^2$.

(c) The general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$
with centre as (–g, –f) & radius = $\sqrt{g^2 + f^2 - c}$ .

This can be obtained from the equation $(x – h)^2 + (y – k)^2 = r^2$
$\Rightarrow$ $x^2 + y^2 – 2hx – 2ky + h^2 + k^2 – r^2 = 0$
Take $– h = g, – k = f$ and $h^2 + k^2 – r^2 = c$
Condition to define circle :-
$g^2 + f^2 – c > 0$ $\Rightarrow$ real circle.
$g^2 + f^2 – c = 0$ $\Rightarrow$ point circle.
$g^2 + f^2 – c < 0$ $\Rightarrow$ imaginary circle, with real centre, that is (– g, – f)

**Note :** That every second degree equation in x & y, in which coefficient of $x^2$ is equal to coefficient of $y^2$ & the coefficient of xy is zero, always represents a circle.

(d) The equation of circle with $(x_1, y_1)$ & $(x_2, y_2)$ as extremeties of its diameter is:
$(x – x_1) (x – x_2) + (y – y_1) (y – y_2) = 0$.




This is obtained by the fact that angle in a semicircle is a right angle.
$\therefore$ (Slope of PA) (Slope of PB) = – 1
$\Rightarrow \frac{y - y_1}{x - x_1} \cdot \frac{y - y_2}{x - x_2} = -1$ $\Rightarrow (x – x_1) (x – x_2) + (y – y_1) (y – y_2) = 0$
Note that this will be the circle of least radius passing through $(x_1, y_1)$ & $(x_2, y_2)$.

**Example # 1** Find the equation of the circle whose centre is (0, 3) and radius is 3.

**Solution.** The equation of the circle is $(x – 0)^2 + (y – 3)^2 = 3^2$
$\Rightarrow$ $x^2 + y^2 – 6y = 0$

**Example # 2** Find the equation of the circle which passes through(1, –1) and two of its dimeter are $x + 2y – 5 = 0$ and $x – y + 1 = 0$

**Solution.** Let P be the point of intersection of the lines
$x + 2y – 5 = 0$ ..........(i)
and $x – y + 1 = 0$ ..........(ii)
Solving (i) and (ii), we get x = 1, y = 2. So, coordinates of centre are (1, 2). since circle passes through (1,–1) so
radius = $\sqrt{(1-1)^2 + (2+1)^2}$ $\Rightarrow$ radius = 3
Hence the equation of the required circle is $(x – 1)^2 + (y + 2)^2 = 9$

**Example # 3** If the equation $ax^2 + (b – 3)xy + 3y^2 + 6ax + 2by – 3 = 0$ represents the equation of a circle then find a, b

**Solution.** $ax^2 + (b – 3)xy + 3y^2 + 6ax + 2by – 3 = 0$
above equation will represent a circle if
coefficient of $x^2$ = coefficient of $y^2$
$a = 3$
coefficient of $xy = 0$
$b = 3$

**Example # 4** Find the equation of a circle whose diametric end points are $(x_1, y_1)$ and $(x_2, y_2)$ where $x_1, x_2$ are the roots of $x^2 – ax + b = 0$ and $y_1, y_2$ are the roots of $y^2 – by + a = 0$.

**Solution.** We know that the equation of the circle described on the line segment joining $(x_1, y_1)$ and $(x_2, y_2)$ as a diameter is $(x – x_1)(x – x_2) + (y – y_1)(y – y_2) = 0$.
$x^2 + y^2 – (x_1 + x_2)x – (y_1 + y_2)y + x_1x_2 + y_1 y_2 = 0$
Here, $x_1 + x_2 = a$ , $x_1x_2 = b$
$y_1 + y_2 = b$ , $y_1y_2 = a$
So, the equation of the required circle is
$x^2 + y^2 – ax – by + a + b = 0$

**Self practice problems :**

(1) Find the equation of the circle passing through the point of intersection of the lines $x + 3y = 0$and $2x – 7y = 0$ and whose centre is the point of intersection of the lines $x + y + 1 = 0$ and $x – 2y + 4 = 0$.

(2) Find the equation of the circle whose centre is (1, 2) and which passes through the point (4, 6)

(3) Find the equation of a circle whose radius is 6 and the centre is at the origin.

**Answers :**
(1) $x^2 + y^2 + 4x – 2y = 0$ (2) $x^2 + y^2 – 2x – 4y – 20 = 0$ (3) $x^2 + y^2 = 36$.

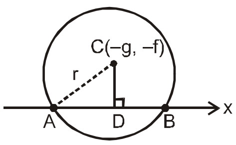
## Intercepts made by a circle on the axes:

The intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on the co–ordinate axes are $2\sqrt{g^2 - c}$ (on x-axis) & $2\sqrt{f^2 - c}$ (on y-axis) respectively.

If $g^2 > c$ $\Rightarrow$ circle cuts the x axis at two distinct points.

$g^2 = c$ $\Rightarrow$ circle touches the x–axis.

$g^2 < c$ $\Rightarrow$ circle lies completely above or below the x–axis.




$AB = 2AD = 2\sqrt{r^2 - CD^2} = 2\sqrt{r^2 - f^2} = 2\sqrt{g^2 + f^2 - c - f^2} = 2\sqrt{g^2 - c}$

**Example # 5** Find the locus of the centre of the circle whose x and y intercepts are a and b respectively.

**Solution.** Equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$
x intercept = a
$2\sqrt{g^2 - c} = a$ $g^2 – c = \frac{a^2}{4}$ ...... (i)
y intercept = b
$2\sqrt{f^2 - c} = b$ $f^2 – c = \frac{b^2}{4}$ ...... (ii)
subtracting equation (i) and (ii)

$g^2 - f^2 = \frac{a^2 - b^2}{4}$

hence locus of centre is $x^2 - y^2 = \frac{a^2 - b^2}{4}$

**Self practice problems :**

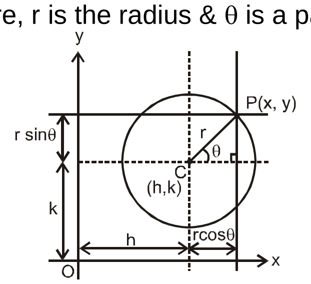
(4) Find the equation of a circle which touches the positive axis of y at a distance 3 from the origin and intercepts a distance 6 on the axis of x.

(5) Find the equation of a circle which touches positive y-axis at a distance of 2 units from the origin and cuts an intercept of 3 units with the positive direction of x-axis.

**Answers :** (4) $x^2 + y^2 \pm 6\sqrt{2}\,x - 6y + 9 = 0$ (5) $x^2 + y^2 - 5x - 4y + 4 = 0$

## Parametric equations of a circle:

The parametric equations of $(x - h)^2 + (y - k)^2 = r^2$ are: $x = h + r\cos\theta$ ; $y = k + r\sin\theta$ ; $-\pi < \theta \le \pi$ where (h, k) is the centre, r is the radius & $\theta$ is a parameter.




**Example # 6** Find the parametric equations of the circle $x^2 + y^2 + 4x + 6y + 9 = 0$

**Solution.** We have : $x^2 + y^2 + 4x + 6y + 9 = 0$

$\Rightarrow \quad (x + 2)^2 + (y + 3)^2 = 2^2$

So, the parametric equations of this circle are

$x = -2 + 2\cos\theta$ , $y = -3 + 2\sin\theta$.

**Example # 7** Find the equation of the following curve in cartesian form

$x + y = \cos\theta$ , $x - y = \sin\theta$ where $\theta$ is the parameter.

**Solution.** We have : $x + y = \cos\theta$ ....... (i)

$x - y = \sin\theta$ ....... (ii)

$(i)^2 + (ii)^2$

$\Rightarrow \quad (x + y)^2 + (x - y)^2 = 1$

$x^2 + y^2 = \frac{1}{2}$

Clearly, it is a circle with centre at (0, 0) and radius $\frac{1}{\sqrt{2}}$.

**Self practice problems :**

(6) Find the parametric equations of circle $x^2 + y^2 - 6x + 4y - 12 = 0$

(7) Find the cartesian equations of the curve $x = 1 + \sqrt{2}\cos\theta$, $y = 2 - \sqrt{2}\sin\theta$

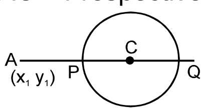
**Answers :** (6) $x = 3 + 5\cos\theta$, $y = -2 + 5\sin\theta$ (7) $(x - 1)^2 + (y - 2)^2 = 2$

## Position of a point with respect to a circle:

The point $(x_1, y_1)$ is inside, on or outside the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$.

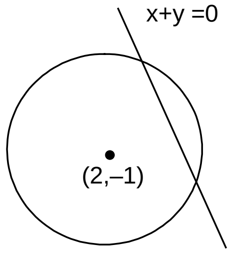
according as $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c <, = \text{ or } > 0$.

**Note :** The greatest & the least distance of a point A (lies outside the circle) from a circle with centre C & radius r is AC + r & AC – r respectively.



**Example # 8** Check whether the point (1, 2) lies in smaller or larger region made by circle $x^2 + y^2 - 4x + 2y - 11 = 0$ and the line $x + y = 0$

**Solution :** We have $x^2 + y^2 - 4x + 2y - 11 = 0$ or $S = 0$,




where $S = x^2 + y^2 - 4x + 2y - 11$.

For the point (1, 2), we have $S_1 = 1^2 + 2^2 - 4 \times 1 + 2 \times 2 - 11 < 0$

Hence, the point (1, 2) lies inside the circle

Points (1,2) and (2, –1) lie on same side of the line $x + y = 0$

Hence the point (1,2) lies in the larger region.

**Self practice problem :**

(8) How are the points (0, 1) (3, 1) and (1, 3) situated with respect to the circle $x^2 + y^2 - 2x - 4y + 3 = 0$?

**Answer :** (8) (0, 1) lies on the circle ; (3, 1) lies outside the circle ; (1, 3) lies inside the circle.

## Line and a circle:

Let L = 0 be a line & S = 0 be a circle. If r is the radius of the circle & p is the length of the perpendicular from the centre on the line, then:

(i) $p > r$ $\Leftrightarrow$ the line does not meet the circle i. e. passes out side the circle.

(ii) $p = r$ $\Leftrightarrow$ the line touches the circle. (It is tangent to the circle)

(iii) $p < r$ $\Leftrightarrow$ the line is a secant of the circle.

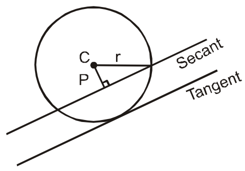
(iv) $p = 0$ $\Rightarrow$ the line is a diameter of the circle.

Also, if $y = mx + c$ is line and $x^2 + y^2 = a^2$ is circle then

(i) $c^2 < a^2 (1 + m^2)$ $\Leftrightarrow$ the line is a secant of the circle.

(ii) $c^2 = a^2 (1 + m^2)$ $\Leftrightarrow$ the line touches the circle. (It is tangent to the circle)

(iii) $c^2 > a^2 (1 + m^2)$ $\Leftrightarrow$ the line does not meet the circle i. e. passes out side the circle.




These conditions can also be obtained by solving $y = mx + c$ with $x^2 + y^2 = a^2$ and making the discriminant of the quadratic greater than zero for secant, equal to zero for tangent and less the zero for the last case.

**Example # 9** For what value of $\lambda$, does the line $x + y = \lambda$ touch the circle $x^2 + y^2 - 2x - 2y = 0$

**Solution.** We have : $x + y = \lambda$ ......(i) and $x^2 + y^2 - 2x - 2y = 0$....... (ii)

If the line (i) touches the circle (ii), then

length of the $\perp$ from the centre (1, 1) = radius of circle (ii)

$$\Rightarrow \left| \frac{1+1-\lambda}{\sqrt{1^2+1^2}} \right| = \sqrt{2} \quad \Rightarrow \quad |2-\lambda| = 2 \quad \Rightarrow \quad \lambda = 0 \text{ or } 4$$

Hence, the line (i) touches the circle (ii) for $\lambda = 0$ or 4

**Self practice problem :**

(9) Find the range of values of m for which the line $y = mx + 2$ cuts the circle $x^2 + y^2 = 1$ at distinct points

**Answers :** (9) $m \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

## Slope form of tangent :

$y = mx + c$ is always a tangent to the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$. Hence, of tangent is $y = mx \pm a\sqrt{1+m^2}$ and the point of contact is $\left(-\frac{a^2m}{c}, \frac{a^2}{c}\right)$.

## Point form of tangent :

(i) The equation of the tangent to the circle $x^2 + y^2 = a^2$ at its point $(x_1, y_1)$ is, $x x_1 + y y_1 = a^2$.

(ii) The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at its point $(x_1, y_1)$ is : $xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$.

**Note :** In general the equation of tangent to any second degree curve at point $(x_1, y_1)$ on it can be obtained by replacing $x^2$ by $x x_1$, $y^2$ by $yy_1$, $x$ by $\frac{x + x_1}{2}$, $y$ by $\frac{y + y_1}{2}$, $xy$ by $\frac{x_1y + xy_1}{2}$ and c remains as c.

## Parametric form of tangent :

The equation of a tangent to circle $x^2 + y^2 = a^2$ at $(a\cos\alpha, a\sin\alpha)$ is $x\cos\alpha + y\sin\alpha = a$.

**NOTE :** The point of intersection of the tangents at the points $P(\alpha)$ & $Q(\beta)$ is $\left(\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \frac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

**Example # 10** Find the equation of the tangent to the circle $x^2 + y^2 - 2x - 2y - 11 = 0$ at (3, 4).

**Solution.** Equation of tangent is

$$3x + 4y - 2\left(\frac{x+3}{2}\right) - 2\left(\frac{y+4}{2}\right) - 11 = 0$$

or $2x + 3y - 18 = 0$

Hence, the required equation of the tangent is $2x + 3y - 18 = 0$

**Example # 11** Find the equation of tangents to the circle $x^2 + y^2 - 4x + 2y = 0$ which are perpendicular to the line $x + 2y + 4 = 0$

**Solution.** Given circle is $x^2 + y^2 - 4x + 2y = 0$ .......(i)

and given line is $x + 2y + 4 = 0$ .......(ii)

Centre of circle (i) is (2,–1)) and its radius $\sqrt{5}$ is Equation of any line

$2x - y + k = 0$ perpendicular to the line (ii) .......(iii)

If line (iii) is tangent to circle (i) then

$\frac{|4+1+k|}{\sqrt{5}} = \sqrt{5}$ or $|k + 5| = 5$ or $k = 0, -10$

Hence equation of required tangents are $2x - y = 0$ and $2x - y - 10 = 0$

**Self practice problem :**

(10) Find the equation of the tangents to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ which are

(i) parallel,

(ii) perpendicular to the line $3x - 4y - 1 = 0$

**Answer.**

(10) (i) $3x - 4y + 20 = 0$ and $3x - 4y - 10 = 0$ (ii) $4x + 3y + 5 = 0$ and $4x + 3y - 25 = 0$

## Normal :

If a line is normal / orthogonal to a circle, then it must pass through the centre of the circle. Using this fact normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at $(x_1, y_1)$ is; $y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$.

**Example # 12** Two normals of a circle are $2x + 3y = 5$ and $3x - 4y + 1 = 0$. Find its equation having radius 2

**Solution.** Since point of intersection of normals is the centre of the circle

point of intersection of lines $2x + 3y = 5$ and $3x - 4y + 1 = 0$ is (1,1)

equation of circle having centre (1,1) and radius 2 is

$(x - 1)^2 + (y - 1)^2 = 4$

**Self practice problem :**

(11) Find the equation of the normal to the circle $x^2 + y^2 - 2x - 4y + 3 = 0$ at the point (2, 3).

**Answer :** (11) $x - y + 1 = 0$

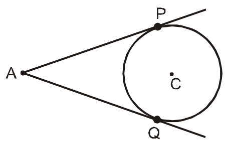
## Pair of tangents from a point :

The equation of a pair of tangents drawn from the point A $(x_1, y_1)$ to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is : $SS_1 = T^2$.

Where $S \equiv x^2 + y^2 + 2gx + 2fy + c$ ; $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$.




**Example # 13** Find the equation of the pair of tangents drawn to the circle $x^2 + y^2 + 4x - 6y + 9 = 0$ from thepoint (2, 1)

**Solution.** Given circle is $S = x^2 + y^2 + 4x - 6y + 9 = 0$

Let $P \equiv (2, 1)$

For point P, $S_1 = 16$

Clearly P lies outside the circle

and $T \equiv 2x + y + 2(x + 2) - 3(y + 1) + 9 = 0$

i.e $T \equiv 2(2x - y + 5)$

Now equation of pair of tangents from P(2, 1) to circle (1) is $SS_1 = T^2$

or $16(x^2 + y^2 + 4x - 6y + 9) = 4(2x - y + 5)^2$ or $12y^2 - 16x - 56y + 16xy + 44 = 0$

or $3y^2 - 4x - 14y + 4xy + 11 = 0$

**Self practice problems :**

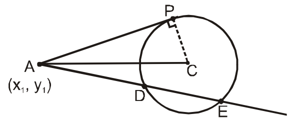
(12) Find the joint equation of the tangents through (7, 1) to the circle $x^2 + y^2 = 25$.

**Answer :** (12) $12x^2 - 12y^2 + 7xy - 175x - 25y + 625 = 0$

## Length of a tangent and power of a point :

The length of a tangent from an external point $(x_1, y_1)$ to the circle

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is given by $L = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2f_1y + c} = \sqrt{S_1}$ .




AP = length of tangent

$AP^2 = AD . AE$

Square of length of the tangent from the point A is also called the power of point w.r.t. a circle.

Power of a point w.r.t. a circle remains constant.

Power of a point P is positive, negative or zero according as the point 'A' is outside, inside or on the circle respectively.

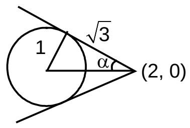
**Example # 14** Find the angle between the tangents drawn from the point (2, 0) to the circle $x^2 + y^2 = 1$

**Solution.** Given circle is $x^2 + y^2 = 1$ .........(i)

Given point is (2, 0).

Now length of the tangent from (2, 0) to circle (i) = $\sqrt{2^2 + 0^2 - 1} = \sqrt{3}$

$\tan\alpha = \dfrac{1}{\sqrt{3}}$




$\alpha = \dfrac{\pi}{6}$

so angle between tangents = $2\alpha = \dfrac{\pi}{3}$

**Self practice problems :**

(13) The lenght of tangents from P(1, –1) & Q(3, 3) to a circle are $\sqrt{2}$ and $\sqrt{6}$ respectively. Then find the lenght of tangent from R (– 1, – 5) to the same circle.

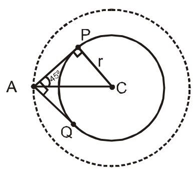
(14) Find the lenght of tangent drawn from any point on circle $x^2 + y^2 + 4x + 6y - 3 = 0$ to the circle $x^2 + y^2 + 4x + 6y + 4 = 0$.

**Answer.** (13) $\sqrt{38}$ (14) $\sqrt{7}$

## Director circle :

The locus of the point of intersection of two perpendicular tangents is called the director circle of the given circle. The director circle of a circle is the concentric circle having radius equal to  times the original circle.

Proof :




AC = r cosec 45º = $r\sqrt{2}$

**Example # 15** Find the equation of director circle of the circle $x^2 + y^2 + 6x + 8y - 2 = 0$

**Solution :** Centre & radius of given circle are (–3, –4) & $\sqrt{27}$ respectively.

Centre and radius of the director circle will be (–3, –4) & $\sqrt{27}.\sqrt{2} = \sqrt{54}$ respectively.

$\therefore$ equation of director circle is $(x + 3)^2 + (y + 4)^2 = 54$

$\Rightarrow$ $x^2 + y^2 + 6x + 8y - 29 = 0$

**Self practice problems :**

(15) Find the angle between the tangents drawn from $(5, \sqrt{7})$ to the circle $x^2+y^2 = 16$

**Answer** (15) $\dfrac{\pi}{2}$

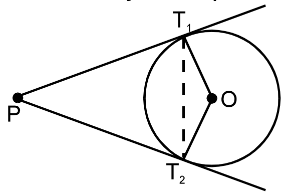
## Chord of contact :

If two tangents $PT_1$ & $PT_2$ are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact $T_1T_2$ is:

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

**Note :** Here R = radius; L = length of tangent.

(a) Chord of contact exists only if the point 'P' is not inside.




(b) Length of chord of contact $T_1T_2 = \dfrac{2LR}{\sqrt{R^2 + L^2}}$.

(c) Area of the triangle formed by the pair of the tangents & its chord of contact = $\dfrac{RL^3}{R^2 + L^2}$

(d) Tangent of the angle between the pair of tangents from $(x_1, y_1) = \left(\dfrac{2RL}{L^2 - R^2}\right)$

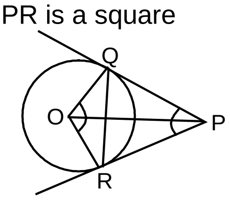
(e) Equation of the circle circumscribing the triangle $PT_1T_2$ is:
$(x - x_1)(x + g) + (y - y_1)(y + f) = 0$.

**Example # 16** Find the equation of the chord of contact of the tangents drawn from (0, 1) to the circle $x^2 + y^2 - 2x + 4y = 0$

**Solution.** Given circle is $x^2 + y^2 - 2x + 4y = 0$ .......(i)
Let P = (0, 1)
For point P (0, 1), $T = x \cdot 0 + y \cdot 1 - (x + 0) + 2(y + 1)$
i.e. $T = x - 3y - 2$
Now equation of the chord of contact of point P(0, 1) w.r.t. circle (i) will be
$x - 3y - 2 = 0$

**Example # 17** If the chord of contact of the tangents drawn from $(\alpha, \beta)$ to the circle $x^2 + y^2 = a^2$ subtends right angle at the centre then prove that $\alpha^2 + \beta^2 = 2a^2$.

**Solution.** $\angle QOR = \angle QPR = \dfrac{\pi}{2}$
so OQPR is a square




$OQ^2 = PQ^2$
$a^2 = \alpha^2 + \beta^2 - a^2$
$\alpha^2 + \beta^2 = 2a^2$

**Self practice problems :**

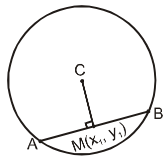
(16) Find the co-ordinates of the point of intersection of tangents at the points where the line $x - 2y + 1 = 0$ meets the circle $x^2 + y^2 = 25$

(17) If the chord of contact of the tangetns drawn from a point on circle $x^2 + y^2 = a^2$ to another circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$ then prove that a,b,c are in G.P.

**Answers :** (16) (–25, 50) (17) $\dfrac{405\sqrt{3}}{52}$ sq. unit ; $4x + 6y - 25 = 0$

## Equation of the chord with a given middle point:

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $T = S_1$.




**Notes : (i)** The shortest chord of a circle passing through a point 'M' inside the circle is one chord whose middle point is M.

**(ii)** The chord passing through a point ' M ' inside the circle and which is at a maximum distance from the centre is a chord with middle point M.

**Example # 18** Find the equation of the chord of the circle $x^2 + y^2 + 2x - 2y - 4 = 0$, whose middle point is (0, 0)

**Solution.** Equation of given circle is $S \equiv x^2 + y^2 + 2x - 2y - 4 = 0$

Let $L \equiv (0, 0)$

For point L(0, 0), $S_1 = -4$ and

$T \equiv x.0 + y(0) + (x + 0) - (y + 0) - 4$ i.e. $T \equiv x - y - 4$

Now equation of the chord of circle (i) whose middle point is L(0, 0) is

$T = S_1$ or $x - y = 0$

Second Method : Let C be the centre of the given circle, then $C \equiv (-1, 1)$. $L \equiv (0, 0)$ slope of CL = –1

$\therefore$ Equation of chord of circle whose middle point is L, is $y - 0 = 1(x - 0)$

($\because$ chord is perpendicular to CL) or $x - y = 0$

**Self practice problems :**

(18) Find the equation of that chord of the circle $x^2 + y^2 = 15$, which is bisected at (3, 2)

(19) A variable chord is drawn through the origin to the circle $x^2 + y^2 - 2ax = 0$. Find the locus of the centre of the circle drawn on this chord as diameter.

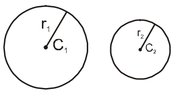
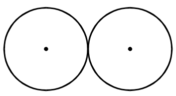
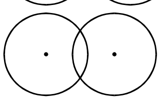
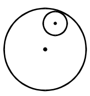
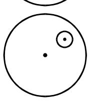
**Answers :** (18) $3x + 2y - 13 = 0$ (19) $x^2 + y^2 - ax = 0$

## Equation of the chord joining two points of circle :

The equation of chord PQ to the circle $x^2 + y^2 = a^2$ joining two points P($\alpha$) and Q($\beta$) on it is given by the equation of a straight line joining two point $\alpha$ & $\beta$ on the circle $x^2 + y^2 = a^2$ is

$$x \cos \frac{\alpha+\beta}{2} + y \sin \frac{\alpha+\beta}{2} = a \cos \frac{\alpha-\beta}{2}.$$

## Common tangents to two circles:

| | Case | Number of Tangents | Condition |
|---|---|---|---|
| (i) |  | 4 common tangents (2 direct and 2 transverse) | $r_1 + r_2 < c_1 c_2$. |
| (ii) |  | 3 common tangents. | $r_1 + r_2 = c_1 c_2$. |
| (iii) |  | 2 common tangents. | $\lvert r_1 - r_2 \rvert < c_1 c_2 < r_1 + r_2$ |
| (iv) |  | 1 common tangent. | $\lvert r_1 - r_2 \rvert = c_1 c_2$. |
| (v) |  | No common tangent. | $c_1 c_2 < \lvert r_1 - r_2 \rvert$. |

**(Here $C_1C_2$ is distance between centres of two circles.)**

**Notes : (i)** The direct common tangents meet at a point which divides the line joining centre of circles externally in the ratio of their radii.

Transverse common tangents meet at a point which divides the line joining centre of circles internally in the ratio of their radii.

**(ii)** Length of an external (or direct) common tangent & internal (or transverse) common tangent to the two circles are given by: $L_{ext} = \sqrt{d^2 - (r_1 - r_2)^2}$ & $L_{int} = \sqrt{d^2 - (r_1 + r_2)^2}$, where d = distance between the centres of the two circles and $r_1$, $r_2$ are the radii of the two circles. Note that length of internal common tangent is always less than the length of the external or direct common tangent.

**Example # 19** Examine if the two circles $x^2 + y^2 - 4x - 6y + 9 = 0$ and $x^2 + y^2 - 10x - 6y + 18 = 0$ intersect or not

**Solution.** Given circles are $x^2 + y^2 - 4x - 6y + 9 = 0$ ...........(i)

and $x^2 + y^2 - 10x - 6y + 18 = 0$ ...........(ii)

Let A and B be the centres and $r_1$ and $r_2$ the radii of circles (i) and (ii) respectively, then

$A \equiv (2, 3)$, $B \equiv (5, 3)$, $r_1 = 2$, $r_2 = 4$

Now AB = 3 and $r_1 + r_2 = 6$, $|r_1 - r_2| = 2$

Thus $|r_1 - r_2| < AB < r_1 + r_2$, hence the two circles intersect.

**Self practice problems :**

(20) Find the position of the circles $x^2 + y^2 - 10x + 4y - 20 = 0$ and $x^2 + y^2 + 14x - 6y + 22 = 0$ with respect to each other.

**Answer :** (20) touch externally

## Orthogonality of two circles:

Two circles $S_1 = 0$ & $S_2 = 0$ are said to be orthogonal or said to intersect orthogonally if the tangents at their point of intersection include a right angle. The condition for two circles to be orthogonal is:

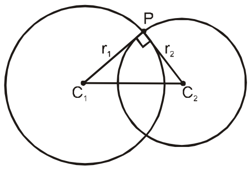
$2g_1g_2 + 2f_1f_2 = c_1 + c_2$.

Proof :

$(C_1C_2)^2 = (C_1P)^2 + (C_2P)^2$

$\Rightarrow \quad (g_1 - g_2)^2 + (f_1 - f_2)^2 = g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2$

$\Rightarrow \quad 2g_1g_2 + 2f_1f_2 = c_1 + c_2$




**Notes :**

(a) The centre of a variable circle orthogonal to two fixed circles lies on the radical axis of two circles.

(b) If two circles are orthogonal, then the polar of a point 'P' on first circle w.r.t. the second circle passes through the point Q which is the other end of the diameter through P. Hence locus of a point which moves such that its polars w.r.t. the circles $S_1 = 0$, $S_2 = 0$ & $S_3 = 0$ are concurrent in a circle which is orthogonal to all the three circles.

(c) The centre of a circle which is orthogonal to three given circles is the radical centre provided the radical centre lies outside all the three circles.

**Example # 20** If the circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $2x^2 + 2y^2 + 2g_2x + 2f_2y + c_2 = 0$ are orthogonal to each other then prove that $g_1g_2 + f_1f_2 = c_1 + \frac{c_2}{2}$

**Solution.** Given circles are $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ ...........(i)

and $2x^2 + 2y^2 + 2g_2x + 2f_2y + c_2 = 0$

or $x^2 + y^2 + g_2x + f_2y + \frac{c_2}{2} = 0$ ..........(ii)

Since circles (i) and (ii) cut orthogonally

$\therefore \quad 2g_1\left(\frac{g_2}{2}\right) + 2f_1\left(\frac{f_2}{2}\right) = c_1 + \frac{c_2}{2}$

$g_1g_2 + f_1f_2 = c_1 + \frac{c_2}{2}$

**Self practice problems :**

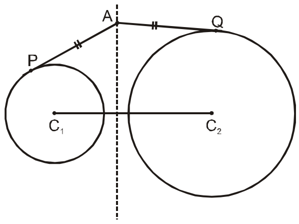
(21) For what value of $\lambda$ the circles $x^2 + y^2 + 8x + 3y + 9 = 0$ and $x^2 + y^2 + 2x - y - \lambda = 0$ cut orthogonally.

(22) Find the equation to the circle which passes through the origin and has its centre on the line $x - y = 0$ and cuts the circle $x^2 + y^2 - 4x - 6y + 10 = 0$ orthogonally.

**Answer :** (21) $\frac{5}{2}$ (22) $x^2 + y^2 - 2x - 2y = 0$

## Radical axis and radical centre:

The radical axis of two circles is the locus of points whose powers w.r.t. the two circles are equal. The equation of radical axis of the two circles $S_1 = 0$ & $S_2 = 0$ is given by
$S_1 - S_2 = 0$ i.e. $2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0$.




The common point of intersection of the radical axes of three circles taken two at a time is called the radical centre of three circles. Note that the length of tangents from radical centre to the three circles are equal.

**Notes :**

(a) If two circles intersect, then the radical axis is the common chord of the two circles.

(b) If two circles touch each other, then the radical axis is the common tangent of the two circlesat the common point of contact.

(c) Radical axis is always perpendicular to the line joining the centres of the two circles.

(d) Radical axis will pass through the mid point of the line joining the centres of the two circles only if the two circles have equal radii.

(e) Radical axis bisects a common tangent between the two circles.
(f) A system of circles, every two which have the same radical axis, is called a coaxial system.

(g) Pairs of circles which do not have radical axis are concentric.

**Example # 21** Find the co-ordinates of the point from which the lengths of the tangents to the following three circles be equal.

$x^2 + y^2 = 1$
$x^2 + y^2 - 8x + 15 = 0$
$x^2 + y^2 + 10y + 24 = 0$

**Solution :** Here we have to find the radical centre of the three circles. First reduce them to standard form in which coefficients of $x^2$ and $y^2$ be each unity. Subtracting in pairs the three radical axes are

$x = 2$ ; $8x + 10y + 9 = 0$
$10y + 25 = 0$

solving any two, we get the point $\left(2, -\frac{5}{2}\right)$ which satisfies the third also. This point is called the radical centre and by definition the length of the tangents from it to the three circles are equal.

**Self practice problem :**

(23) Find the point from which the tangents to the three circles $x^2 + y^2 - 4x + 7 = 0$, $2x^2 + 2y^2 - 3x + 5y + 9 = 0$ and $x^2 + y^2 + y = 0$ are equal in length. Find also this length.

**Answer :** (23) (2, – 1) ; 2.

## Family of Circles:

This article is aimed at obtaining the equation of a group of circles having a specific characteristic. For example, the equation $x^2 + y^2 + 4x + 2y + \lambda = 0$ where $\lambda$ is arbitrary, represents a family of circles with fixed centre (–2, –1) but variable radius. We have the following results for some other families of circles.

(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is : $S_1 + K S_2 = 0$
($K \neq -1$, provided the co–efficient of $x^2$ & $y^2$ in $S_1$ & $S_2$ are same)

(b) The equation of the family of circles passing through the point of intersection of a circle S = 0 & a line L = 0 is given by S + KL = 0.

(c) The equation of a family of circles passing through two given points $(x_1, y_1)$ & $(x_2, y_2)$ can be written in the form:

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + K \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$ where K is a parameter.

(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point $(x_1, y_1)$ is $(x - x_1)^2 + (y - y_1)^2 + K(y - y_1 - m(x - x_1)) = 0$, where K is a parameter.

(e) Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$, $L_2 = 0$ and $L_3 = 0$ is given by; $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided co–efficient of xy = 0 and co–efficient of $x^2$ = co–efficient of $y^2$.

(f) Equation of circle circumscribing a quadrilateral whose side in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ are $u L_1L_3 + \lambda L_2L_4 = 0$ where values of u & $\lambda$ can be found out by using condition that co–efficient of $x^2$ = co–efficient of $y^2$ and co–efficient of xy = 0.

**Example # 22** Find the equation of the circle passing through the point (1, 1) and points of intersection of the circles $x^2 + y^2 + 13x - 3y = 0$ and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$.

**Solution.** Any circle through the intersection of given circles is $S_1 + \lambda S_2 = 0$

or $x^2 + y^2 + 13x - 3y + \lambda(x^2 + y^2 + 2x - 7y/2 - 25/2) = 0$

This circle passes through (1, 1)

$1 + 1 + 13 - 3 + \lambda(1 + 1 + 2 - 7/2 - 25/2) = 0$

$\lambda = 1$

Putting the value of $\lambda$ in (i) the required circle is $4x^2 + 4y^2 + 30x - 13y - 25 = 0$

**Example # 23** Find the equations of smallest circle which passes through the points of intersection of the line x + y = 1 and the circle $x^2 + y^2 = 9$.

**Solution.** The required circle by $S + \lambda L = 0$ is

$x^2 + y^2 - 9 + \lambda(x + y - 1) = 0$ ....(i)

centre $(-g, -f) = \left(-\frac{\lambda}{2}, -\frac{\lambda}{2}\right)$

centre lies on the line x + y = 1

$-\frac{\lambda}{2} - \frac{\lambda}{2} = 1$

$\lambda = -1$

Putting the value of $\lambda$ in (i) the required circle is

$x^2 + y^2 - x - y - 8 = 0$

**Example # 24** Find the equation of circle passing through the points A(1, 1) & B(0, 3) and whose radius is $\sqrt{\frac{5}{2}}$.

**Solution.** Equation of AB is 2x + y – 3 = 0

$\therefore$ equation of circle is

$(x - 1)(x) + (y - 1)(y - 3) + \lambda(2x + y - 3)$

$= 0$ or $x^2 + y^2 + (2\lambda - 1)x + (\lambda - 4)y + 3 - 3\lambda = 0$

$$\sqrt{\left(\frac{2\lambda - 1}{2}\right)^2 + \left(\frac{\lambda - 4}{2}\right) + 3\lambda - 3} = \sqrt{\frac{5}{2}}$$

$\lambda = 1$

$\therefore$ equation of circle is $x^2 + y^2 + x - 3y = 0$

**Example # 25** A variable circle always touhces x + y = 2 at (1, 1), cuts the circle $x^2 + y^2 + 4x + 5y - 6 = 0$. Prove that all common chords pass through a fixed point. Also find the point.

**Solution :** Equation of circle is $(x - 1)^2 + (y - 1)^2 + \lambda(x + y - 2) = 0$

$x^2 + y^2 + x(\lambda - 2) + y(\lambda - 2) + 2 - 2\lambda = 0$

common chord of this circle with $x^2 + y^2 + 4x + 5y - 6 = 0$ is

$(\lambda - 6)x + (\lambda - 7)y + 8 - 2\lambda = 0$

$\lambda(x + y - 2) + (-6x - 7y + 8) = 0$

this chord passes through the point of intersection of the lines $x + y - 2 = 0$ and $-6x - 7y + 8 = 0$ which is (6, –4)

**Example # 26** Find the equation of circle circumcscribing the triangle whose sides are $3x - y - 12 = 0$, $5x - 3y - 28 = 0$ & $x + y - 4 = 0$.

**Solution :** $L_1L_2 + \lambda L_2L_3 + \mu L_1L_3 = 0$

$(3x - y - 12)(5x - 3y - 28) + \lambda(5x - 3y - 28)(x + y - 4) + \mu(3x - y - 12)(x + y - 4) = 0$

coefficient of $x^2$ = coefficient of $y^2$

$\Rightarrow \quad 5\lambda + 3\mu + 15 = 3 - 3\lambda - \mu$

$2\lambda + \mu + 3 = 0$ ...........(ii)

coefficient of xy = 0

$\Rightarrow \quad \lambda + \mu - 7 = 0$ ..........(iii)

Solving (ii) and (iii), we have

$\lambda = -10, \mu = 17$

Puting these values of $\lambda$ & $\mu$ in equation (i), we get $2x^2 + 2y^2 - 9x + 11y + 4 = 0$

**Self practice problems :**

(24) Find the equation of the circle passing through the points of intersection of the circles $x^2 + y^2 - 6x + 2y + 4 = 0$ and $x^2 + y^2 + 2x - 4y - 6 = 0$ and with its centre on the line y = x.

(25) Find the equation of circle circumscribing the quadrilateral whose sides are x + y = 10, x – 7y + 50 = 0, 22x – 4y + 125 = 0 and 2x – 4y – 5

**Answers :** (24) $7x^2 + 7y^2 - 10x - 10y - 12 = 0$

(25) $x^2 + y^2 = \frac{125}{2}$