Binomial Theorem

"Obvious" is the most dangerous word in mathematics......... Bell, Eric Temple

Binomial expression :

Any algebraic expression which contains two dissimilar terms is called binomial expression.

For example : $x + y$, $x^2y + \frac{1}{xy^2}$, $3 - x$, $\sqrt{x^2+1} + \frac{1}{(x^3+1)^{1/3}}$ etc.

Terminology used in binomial theorem :

Factorial notation : $\underline{|n}$ or n! is pronounced as factorial n and is defined as

$$n! = \begin{cases} n(n-1)(n-2)........3\ .\ 2\ .\ 1 & ; \text{ if } n \in N \\ 1 & ; \text{ if } n = 0 \end{cases}$$

Note : $n! = n\ .\ (n - 1)!$; $n \in N$

Mathematical meaning of ^nC_r : The term ^nC_r denotes number of combinations of r things choosen from n distinct things mathematically, $^nC_r = \frac{n!}{(n-r)!\ r!}$, $n, r \in W$, $0 \le r \le n$

Note : Other symbols of of ^nC_r are $\binom{n}{r}$ and C(n, r).

Properties related to ^nC_r :

(i) $^nC_r = {}^nC_{n-r}$

Note : If $^nC_x = {}^nC_y \Rightarrow$ Either $x = y$ or $x + y = n$

(ii) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iii) $\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$

(iv) $^nC_r = \frac{n}{r}\ {}^{n-1}C_{r-1} = \frac{n(n-1)}{r(r-1)}\ {}^{n-2}C_{r-2} = ... = \frac{n(n-1)(n-2).........(n-(r-1))}{r\ (r-1)(r-2).......2\ .1}$

(v) If n and r are relatively prime, then ^nC_r is divisible by n. But converse is not necessarily true.

Statement of binomial theorem :

$(a + b)^n = {}^nC_0\, a^nb^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + ... + {}^nC_r\, a^{n-r} b^r + ... + {}^nC_n\, a^0 b^n$

where $n \in N$

or $(a + b)^n = \sum_{r=0}^{n} {}^nC_r\ a^{n-r}b^r$

Note : If we put a = 1 and b = x in the above binomial expansion, then

or $(1 + x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + ... + {}^nC_r\, x^r + ... + {}^nC_n\, x^n$

or $(1 + x)^n = \sum_{r=0}^{n} {}^nC_r\ x^r$

Example # 1 : Expand the following binomials :

(i) $(x+\sqrt{2})^5$ (ii) $\left(1-\frac{3x^2}{2}\right)^4$

Solution : (i) $(x+\sqrt{2})^5 = {}^5C_0x^5 + {}^5C_1x^4\left(\sqrt{2}\right) + {}^5C_2x^3\left(\sqrt{2}\right)^2 + {}^5C_3\, x^2\left(\sqrt{2}\right)^3 + {}^5C_4x\left(\sqrt{2}\right)^4 + {}^5C_5\left(\sqrt{2}\right)^5$

$= x^5 + 5\sqrt{2}\, x^4 + 20x^3 + 20\sqrt{2}\, x^2 + 20x + 4\sqrt{2}$

(ii) $$\left(1-\frac{3x^2}{2}\right)^4 = {}^4C_0 + {}^4C_1\left(-\frac{3x^2}{2}\right) + {}^4C_2\left(-\frac{3x^2}{2}\right)^2 + {}^4C_3\left(-\frac{3x^2}{2}\right)^3 + {}^4C_4\left(-\frac{3x^2}{2}\right)^4$$

$$= 1 - 6x^2\,\frac{27}{2} + x^4 - \frac{27}{2}\,x^6 + \frac{81}{16}\,x^8$$

Example # 2 : Expand the binomial $\left(\frac{2}{x}+x\right)^{10}$ up to four terms

Solution : $$\left(\frac{2}{x}+x\right)^{10} = {}^{10}C_0\left(\frac{2}{x}\right)^{10} + {}^{10}C_1\left(\frac{2}{x}\right)^{9} x + {}^{10}C_2\left(\frac{2}{x}\right)^{8} x^2 + {}^{10}C_3\left(\frac{2}{x}\right)^{7} x^3 + ...$$

Self practice problems :

(1) Write the first three terms in the expansion of $\left(2-\frac{y}{3}\right)^6$.

(2) Expand the binomial $\left(\frac{x^2}{3}+\frac{3}{x}\right)^5$.

Ans. (1) $64 - 64y + \frac{80}{3}\,y^2$ (2) $\frac{x^{10}}{243} + \frac{5}{27}\,x^7 + \frac{10}{3}x^4 + 30x + \frac{135}{x^2} + \frac{243}{x^5}$.

Observations :

(i) The number of terms in the binomial expansion $(a + b)^n$ is $n + 1$.

(ii) The sum of the indices of a and b in each term is n.

(iii) The binomial coefficients (${}^nC_0$, ${}^nC_1$ ....${}^nC_n$) of the terms equidistant from the beginning and the end are equal, i.e. ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$ etc. $\{\because\ {}^nC_r = {}^nC_{n-r}\}$

(iv) The binomial coefficient can be remembered with the help of the following pascal's Triangle (also known as Meru Prastra provided by Pingla)

Index of the binomial	The binomial coefficient
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

Regarding Pascal's Triangle, we note the following :

(a) Each row of the triangle begins with 1 and ends with 1.

(b) Any entry in a row is the sum of two entries in the preceding row, one on the immediate left and the other on the immediate right.

Example # 3 : The number of dissimilar terms in the expansion of $(1 + x^4 - 2x^2)^{15}$ is

(A) 21 (B) 31 (C) 41 (D) 61

Solution : $(1 - x^2)^{30}$

Therefore number of dissimilar terms = **31.**

General term :

$(x + y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + ...+ {}^nC_r\, x^{n-r} y^r + ...+ {}^nC_n\, x^0 y^n$

$(r + 1)^{th}$ term is called general term and denoted by T_{r+1}.

$T_{r+1} = {}^nC_r\, x^{n-r} y^r$

Note : The r^{th} term from the end is equal to the $(n - r + 2)^{th}$ term from the begining, i.e. ${}^nC_{n-r+1}\, x^{r-1} y^{n-r+1}$

Example # 4 : Find (i) 15^{th} term of $(2x - 3y)^{20}$ (ii) 4^{th} term of $\left(\frac{3x}{5} - y\right)^7$

Solution : (i) $T_{14+1} = {}^{20}C_{14}\,(2x)^6\,(-3y)^{14} = {}^{20}C_{14}\,2^6\,3^{14}\,x^6.y^{14}$

(ii) $T_{3+1} = {}^{7}C_3\left(\frac{3x}{5}\right)^4 (-y)^3 = {}^{-7}C_3\left(\frac{3}{5}\right)^4 x^4y^3$

Example # 5 : Find the number of rational terms in the expansion of $\left(2^{\frac{1}{3}} + 3^{\frac{1}{5}}\right)^{600}$

Solution : The general term in the expansion of $\left(2^{\frac{1}{3}} + 3^{\frac{1}{5}}\right)^{600}$ is

$$T_{r+1} = {}^{600}C_r\left(2^{\frac{1}{3}}\right)^{600-r}\left(3^{\frac{1}{5}}\right)^r = {}^{600}C_r\ 2^{\frac{600-r}{3}}\,3^{\frac{r}{5}}$$

The above term will be rational if exponent of 3 and 2 are integers

It means $\frac{600-r}{3}$ and $\frac{r}{5}$ must be integers.

The possible set of values of r is {0, 15, 30,45............,600}

Hence, number of rational terms is 41

Middle term(s) :

(a) If n is even, there is only one middle term, which is $\left(\frac{n+2}{2}\right)^{th}$ term.

(b) If n is odd, there are two middle terms, which are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+1}{2}+1\right)^{th}$ terms.

Example # 6 : Find the middle term(s) in the expansion of

(i) $(1 + 2x)^{12}$ (ii) $\left(2y - \frac{y^2}{2}\right)^{11}$

Solution : (i) $(1 + 2x)^{12}$

Here, n is even, therefore middle term is $\left(\frac{12+2}{2}\right)^{th}$ term.

It means T_7 is middle term $T_7 = {}^{12}C_6\,(2x)^6$

(ii) $\left(2y - \frac{y^2}{2}\right)^{11}$

Here, n is odd therefore, middle terms are $\left(\frac{11+1}{2}\right)^{th}$ & $\left(\frac{11+1}{2}+1\right)^{th}$.

It means T_6 & T_7 are middle terms

$$T_6 = {}^{11}C_5\,(2y)^6\left(-\frac{y^2}{2}\right)^5 = -2\ {}^{11}C_5\,y^{16} \Rightarrow T_7 = {}^{11}C_6\,(2y)^5\left(-\frac{y^2}{2}\right)^6 = \frac{{}^{11}C_6}{2}y^{17}$$

Example # 7 : Find term which is independent of x in $\left(x^2 - \frac{1}{x^6}\right)^{16}$

Solution : $T_{r+1} = {}^{16}C_r\,(x^2)^{16-r}\left(-\frac{1}{x^6}\right)^r$

For term to be independent of x, exponent of x should be 0

$32 - 2r = 6r \quad \Rightarrow \quad r = 4 \quad \therefore\ T_5$ is independent of x.

Numerically greatest term in the expansion of $(a + b)^n$, $n \in N$

Binomial expansion of $(a + b)^n$ is as follows **: –**

$(a + b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + ... + {}^nC_r\, a^{n-r} b^r + ... + {}^nC_n\, a^0 b^n$

If we put certain values of a and b in RHS, then each term of binomial expansion will have certain value. The term having numerically greatest value is said to be numerically greatest term.

Let T_r and T_{r+1} be the r^{th} and $(r + 1)^{th}$ terms respectively

$T_r = {}^nC_{r-1}\ a^{n-(r-1)}\, b^{r-1}$

$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Now, $\left|\dfrac{T_{r+1}}{T_r}\right| = \left|\dfrac{{}^nC_r}{{}^nC_{r-1}} \dfrac{a^{n-r}\ b^r}{a^{n-r+1} b^{r-1}}\right| = \dfrac{n-r+1}{r} \cdot \left|\dfrac{b}{a}\right|$

Consider $\left|\dfrac{T_{r+1}}{T_r}\right| \ge 1$

$\left(\dfrac{n-r+1}{r}\right)\left|\dfrac{b}{a}\right| \ge 1 \quad \Rightarrow \quad \dfrac{n+1}{r} - 1 \ge \left|\dfrac{a}{b}\right| \quad \Rightarrow r \le \dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$

Case - I When $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ is an integer (say m), then

(i) $T_{r+1} > T_r$ when $r < m$ $(r = 1, 2, 3 ...., m - 1)$
i.e. $T_2 > T_1, T_3 > T_2, ..., T_m > T_{m-1}$

(ii) $T_{r+1} = T_r$ when $r = m$
i.e. $T_{m+1} = T_m$

(iii) $T_{r+1} < T_r$ when $r > m$ $(r = m + 1, m + 2, ...n)$
i.e. $T_{m+2} < T_{m+1}, T_{m+3} < T_{m+2}, ...T_{n+1} < T_n$

Conclusion :

When $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ is an integer, say m, then T_m and T_{m+1} will be numerically greatest terms (both terms are equal in magnitude)

Case - II

When is not an integer (Let its integral part be m), then

(i) $T_{r+1} > T_r$ when r < (r = 1, 2, 3,........, m–1, m)
i.e. $T_2 > T_1, T_3 > T_2, ..., T_{m+1} > T_m$

(ii) $T_{r+1} < T_r$ when r > (r = m + 1, m + 2, ...n)
i.e. $T_{m+2} < T_{m+1}, T_{m+3} < T_{m+2}, ..., T_{n+1} < T_n$

Conclusion :

When is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

Note : (i) In any binomial expansion, the middle term(s) has greatest binomial coefficient.
In the expansion of $(a + b)^n$

If n	No. of greatest binomial coefficient	Greatest binomial coefficient
Even	1	${}^nC_{n/2}$
Odd	2	${}^nC_{(n-1)/2}$ and ${}^nC_{(n+1)/2}$

(Values of both these coefficients are equal)

(ii) In order to obtain the term having numerically greatest coefficient, put a = b = 1, and proceed as discussed above.

Example # 8 : Find the numerically greatest term in the expansion of $(7 - 3x)^{25}$ when $x = \frac{1}{3}$.

Solution : $m = \dfrac{n+1}{1+\left|\frac{a}{b}\right|} = \dfrac{25+1}{1+\left|\frac{7}{-1}\right|} = \dfrac{26}{8}$

[m] = 3 ([m] denotes GIF)

$\therefore$ T_4 is numerically greatest term

Self practice problems :

(3) Find the term independent of x in $\left(x^2 - \frac{3}{x}\right)^9$

(4) The sum of all rational terms in the expansion of $(3^{1/7} + 5^{1/2})^{14}$ is

(A) 3^2 (B) $3^2 + 5^7$ (C) $3^7 + 5^2$ (D) 5^7

(5) Find the coefficient of x^{-2} in $(1 + x^2 + x^4)\left(1 - \frac{1}{x^2}\right)^{18}$

(6) Find the middle term(s) in the expansion of $(1 + 3x + 3x^2 + x^3)^{2n}$

(7) Find the numerically greatest term in the expansion of $(2 + 5x)^{21}$ when $x = \frac{2}{5}$.

Ans. (3) 28.3^7 (4) B (5) –681

(6) ${}^{6n}C_{3n} \cdot x^{3n}$ (7) $T_{11} = T_{12} = {}^{21}C_{10}\ 2^{21}$

Example # 9 : Show that $7^n + 5$ is divisible by 6, where n is a positive integer.

Solution : $7^n + 5 = (1 + 6)^n + 5 = {}^nC_0 + {}^nC_1 . 6 + {}^nC_2 . 6^2 + ... + {}^nC_n 6^n + 5.$

$= 6. C_1 + 6^2. C_2 + ... + C_n . 6^n + 6.$

$= 6\lambda$, where λ is a positive integer

Hence, $7^n + 5$ is divisible by 6.

Example # 10 : What is the remainder when 7^{81} is divided by 5.

Solution : $7^{81} = 7.7^{80} = 7. (49)^{40} = 7 (50 - 1)^{40}$

$= 7 [{}^{40}C_0 (50)^{40} - {}^{40}C_1 (50)^{39} + ... - {}^{40}C_{39} (50)^1 + {}^{40}C_{40} (50)^0]$

= 5(k) + 7(where k is a positive integer) = 5 (k + 1) + 2

Hence, remainder is 2.

Example # 11 : Find the last digit of the number $(13)^{12}$.

Solution : $(13)^{12} = (169)^6 = (170 - 1)^6$

$= {}^6C_0 (170)^6 - {}^6C_1 (170)^5 + ... - {}^6C_5 (170)^1 + {}^6C_6 (170)^0$

Hence, last digit is 1

Note : We can also conclude that last three digits are 481.

Example-12 : Which number is larger $(1.1)^{100000}$ or 10,000 ?

Solution : By Binomial Theorem

$(1.1)^{100000} = (1 + 0.1)^{100000} = 1 + {}^{100000}C_1 (0.1)$ + other positive terms

= 1 + 100000 × 0.1 + other positive terms

= 1 + 10000 + other positive terms

Hence $(1.1)^{100000} > 10,000$

Self practice problems :

(8) If n is a positive integer, then show that $6^n - 5n - 1$ is divisible by 25.

(9) What is the remainder when 3^{257} is divided by 80 .

(10) Find the last digit, last two digits and last three digits of the number $(81)^{25}$.

(11) Which number is larger $(1.3)^{2000}$ or 600

Ans. (9) 3 (10) 1, 01, 001 (11) $(1.3)^{2000}$.

Some standard expansions :

(i) Consider the expansion

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r\ x^{n-r} y^r = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + \ldots\ldots\ldots + {}^nC_r x^{n-r} y^r + \ldots\ldots\ldots + {}^nC_n x^0 y^n \quad \ldots(i)$$

(ii) Now replace $y \to -y$ we get

$$(x-y)^n = \sum_{r=0}^{n} {}^nC_r\ (-1)^r x^{n-r} y^r = {}^nC_0 x^n y^0 - {}^nC_1 x^{n-1} y^1 + \ldots + {}^nC_r (-1)^r x^{n-r} y^r + \ldots + {}^nC_n (-1)^n x^0 y^n \quad \ldots(ii)$$

(iii) Adding (i) & (ii), we get

$(x+y)^n + (x-y)^n = 2[{}^nC_0 x^n y^0 + {}^nC_2 x^{n-2} y^2 + \ldots\ldots]$

(iv) Subtracting (ii) from (i), we get

$(x+y)^n - (x-y)^n = 2[{}^nC_1 x^{n-1} y^1 + {}^nC_3 x^{n-3} y^3 + \ldots\ldots]$

Properties of binomial coefficients :

$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_rx^r + \ldots\ldots + C_nx^n \quad \ldots\ldots(1)$

where C_r denotes ${}^nC_r$

(1) The sum of the binomial coefficients in the expansion of $(1+x)^n$ is 2^n

Putting $x = 1$ in (1)

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_n = 2^n \quad \ldots\ldots(2)$$

or $$\sum_{r=0}^{n} {}^nC_r = 2^n$$

(2) Again putting $x = -1$ in (1), we get

$${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots\ldots\ldots + (-1)^n\, {}^nC_n = 0 \quad \ldots\ldots(3)$$

or $$\sum_{r=0}^{n} (-1)^r\ {}^nC_r = 0$$

(3) The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to 2^{n-1}.

from (2) and (3)

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots\ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots\ldots\ldots = 2^{n-1}$$

(4) Sum of two consecutive binomial coefficients

${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

$$\text{L.H.S.} = {}^nC_r + {}^nC_{r-1} = \frac{n!}{(n-r)!\ r!} + \frac{n!}{(n-r+1)!\ (r-1)!}$$

$$= \frac{n!}{(n-r)!\ (r-1)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(n-r)!\ (r-1)!}\ \frac{(n+1)}{r(n-r+1)}$$

$$= \frac{(n+1)!}{(n-r+1)!\ r!} = {}^{n+1}C_r = \text{R.H.S.}$$

(5) Ratio of two consecutive binomial coefficients

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$$

(6) $${}^nC_r = \frac{n}{r}\ {}^{n-1}C_{r-1} = \frac{n(n-1)}{r(r-1)}\ {}^{n-2}C_{r-2} = \ldots\ldots\ldots = \frac{n(n-1)(n-2)\ldots\ldots(n-(r-1))}{r\ (r-1)(r-2)\ldots\ldots 2\ .1}$$

Example # 13 : If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + c_nx^n$, then show that

(i) $C_0 + 4C_1 + 4^2C_2 + ... + 4^n C_n = 5^n$. (ii) $3C_0 + 5C_1 + 7.\ C_2 + ... + (2n + 3)\ C_n = 2^n (n + 3)$.

(iii) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

Solution : (i) $(1 + x)^n = C_0 + C_1 x + C_2x^2 + ... + C_nx^n$

put x = 4

$C_0 + 4C_1 + 4^2C_2 + ... + 4^n C_n = 5^n$.

(ii) L.H.S. $= 3C_0 + 5C_1 + 7.\ C_2 + ... + (2n + 3)\ C_n$

$= \sum_{r=0}^{n}(2r+3)\ .\ {}^nC_r = 2\sum_{r=0}^{n} r.\ {}^nC_r + 3\sum_{r=0}^{n}{}^nC_r$

$= 2n \sum_{r=1}^{n} {}^{n-1}C_{r-1} + 3 \sum_{r=0}^{n} {}^nC_r = 2n\ .\ 2^{n-1} + 3.\ 2^n = 2^n(n + 3)$ RHS

(iii) I **<u>Method : By Summation</u>**

L.H.S. $= C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + ... + \frac{C_n}{n+1}$

$= \sum_{r=0}^{n} .\ \frac{{}^nC_r}{r+1} = \frac{1}{n+1}\sum_{r=0}^{n} .\ {}^{n+1}C_{r+1}$ $\left\{\frac{n+1}{r+1}\ .\ {}^nC_r = {}^{n+1}C_{r+1}\right\} = \frac{2^{n+1}-1}{n+1}$ R.H.S.

II **<u>Method : By Integration</u>**

$(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_n x^n$. Integrating both sides, within the limits 0 to 1.

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + ... + C_n\frac{x^{n+1}}{n+1}\right]_0^1$$

$$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = \left(C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1}\right) - 0$$

$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$ Proved

Example # 14 : If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ...+ C_nx^n$, then prove that

(i) $C_0C_1 + C_1C_2 + C_2C_3 + ... + C_{n-1} C_n = {}^{2n}C_{n-1}$ or ${}^{2n}C_{n+1}$

(ii) $1^2.\ C_1^2 + 2^2\ .\ C_2^2 + 3^2.\ C_3^2 + ... + n^2 C_n^2\ . = n^2.\ {}^{2n-2}C_{n-1}$

Solution : (i) $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_n x^n$. ....(i)

$(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_n x^0$ ....(ii)

Multiplying (i) and (ii)

$(C_0 + C_1x + C_2x^2 + ... + C_nx^n)\ (C_0x^n + C_1x^{n-1} + ... + C_nx^0) = (1 + x)^{2n}$

Comparing coefficient of x^{n-1},

$C_0C_1 + C_1C_2 + C_2C_3 + ... + C_{n-1} C_n = {}^{2n}C_{n-1}$ or ${}^{2n}C_{n+1}$

(ii) $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_n x^n$. ....(i)

differentiating w.r.t x......

$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + ... + nC_n x^{n-1}$.

multiplying by x.......

$n\ x(1 + x)^{n-1} = C_1x + 2C_2x^2 + 3C_3x^3 + ... + nC_n x^n$

Now differentiate w.r.t. x.....

$n(1 + x)^{n-1} + n\ (n-1)x.(1+x)^{n-2} = 1^2C_1 + 2^2C_2x + 3^2C_3x^2 + ... + n^2C_nx^{n-1}$ ....(ii)

$(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_n x^0$ ....(iii)

multiplying (ii) & (iii) and comparing the cofficient of x^{n-1}

$1^2.\ C_1^2 + 2^2\ .\ C_2^2 + 3^2.\ C_3^2 + ... + n^2 C_n^2. = n\left({}^{2n-1}C_{n-1} - {}^{2n-2}C_{n-2}\right) + n^2\ {}^{2n-2}C_{n-2}$

$= n^2\ {}^{2n-2}C_{n-1}$ = R.H.S.

Example # 15 : Find the summation of the following series –

(i) ${}^{m}C_0 + {}^{m+1}C_1 + {}^{m+2}C_2 + \ldots\ldots\ldots\ldots + {}^{n}C_m$ (ii) ${}^{n}C_3 + 2\,.\,{}^{n+1}C_3 + 3.\,{}^{n+2}C_3 + \ldots\ldots\ldots + n\,.\,{}^{2n-1}C_3$

Solution : (i) I **Method :** Using property, ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$

${}^{m}C_0 + {}^{m+1}C_1 + {}^{m+2}C_2 + \ldots\ldots\ldots\ldots + {}^{n}C_m$

${}^{m}C_m + {}^{m+1}C_m + {}^{m+2}C_m + \ldots\ldots\ldots\ldots + {}^{n}C_m$

$= \underbrace{{}^{m+1}C_{m+1} + {}^{m+1}C_m} + {}^{m+2}C_m + \ldots\ldots\ldots\ldots + {}^{n}C_m \quad \{\because {}^{m}C_m = {}^{m+1}C_{m+1}\}$

$= \underbrace{{}^{m+2}C_{m+1} + {}^{m+2}C_m} + \ldots\ldots\ldots\ldots\ldots\ldots + {}^{n}C_m = {}^{m+3}C_{m+1} + \ldots\ldots\ldots\ldots + {}^{n}C_m = {}^{n}C_{m+1} + {}^{n}C_m = \mathbf{{}^{n+1}C_{m+1}}$

II **Method**

${}^{m}C_m + {}^{m+1}C_m + {}^{m+2}C_m + \ldots\ldots\ldots + {}^{n}C_m$

The above series can be obtained by writing the coefficient of x^m in

$(1 + x)^m + (1 + x)^{m+1} + \ldots\ldots\ldots + (1 + x)^n$

Let $S = (1 + x)^m + (1 + x)^{m+1} + \ldots\ldots\ldots\ldots + (1 + x)^n$

$$= \frac{(1+x)^m\left[(1+x)^{n-m+1} - 1\right]}{x} = \frac{(1+x)^{n+1} - (1+x)^m}{x}$$

$= \text{coefficient of } x^m \text{ in } \dfrac{(1+x)^{n+1}}{x} - \dfrac{(1+x)^m}{x} = {}^{n+1}C_{m+1} + 0 = {}^{n+1}C_{m+1}$

(ii) ${}^{n}C_3 + 2\,.\,{}^{n+1}C_3 + 3\,.\,{}^{n+2}C_3 + \ldots\ldots\ldots + n\,.\,{}^{2n-1}C_3$

The above series can be obatined by writing the coefficient of x^3 in

$(1 + x)^n + 2\,.\,(1 + x)^{n+1} + 3\,.\,(1 + x)^{n+2} + \ldots\ldots\ldots\ldots + n\,.\,(1 + x)^{2n-1}$

Let $S = (1 + x)^n + 2\,.\,(1 + x)^{n+1} + 3.\,(1 + x)^{n+2} + \ldots\ldots\ldots\ldots + n\,(1 + x)^{2n-1}$ .....(i)

$(1 + x)S = (1 + x)^{n+1} + 2\,(1 + x)^{n+2} + \ldots\ldots\ldots\ldots + (n - 1)\,(1 + x)^{2n-1} + n(1 + x)^{2n}$ ....(ii)

Subtracting (ii) from (i)

$-xS = (1 + x)^n + (1 + x)^{n+1} + (1 + x)^{n+2} + \ldots\ldots\ldots\ldots + (1 + x)^{2n-1} - n(1 + x)^{2n}$

$$= \frac{(1+x)^n\left[(1+x)^n - 1\right]}{x} - n\,(1 + x)^{2n}$$

$$S = \frac{-(1+x)^{2n} + (1+x)^n}{x^2} + \frac{n(1+x)^{2n}}{x}$$

$x^3 : S$ (coefficient of x^3 in S)

$$x^3 : \frac{-(1+x)^{2n} + (1+x)^n}{x^2} + \frac{n(1+x)^{2n}}{x}$$

Hence, required summation of the series is $-{}^{2n}C_5 + {}^{n}C_5 + n\,.\,{}^{2n}C_4$

Example # 16 : Prove that $C_1 - C_3 + C_5 - \ldots\ldots\ldots = 2^{n/2} \sin\dfrac{n\pi}{4}$.

Solution : Consider the expansion $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots\ldots + C_n x^n$ ....(i)

putting $x = -i$ in (i) we get

$(1 - i)^n = C_0 - C_1 i - C_2 + C_3 i + C_4 + \ldots\ldots\ (-1)^n C_n i^n$

or $2^{n/2}\left[\cos\left(-\dfrac{n\pi}{4}\right) + i \sin\left(-\dfrac{n\pi}{4}\right)\right] = (C_0 - C_2 + C_4 - \ldots.) - i\,(C_1 - C_3 + C_5 - \ldots..)$ ....(ii)

Equating the imaginary part in (ii) we get $C_1 - C_3 + C_5 - \ldots\ldots\ldots = 2^{n/2} \sin\dfrac{n\pi}{4}$.

Self practice problems :

(12) Prove the following

(i) $5C_0 + 7C_1 + 9C_2 + \ldots\ldots\ldots\ldots + (2n + 5)\,C_n = 2^n\,(n + 5)$

(ii) $4C_0 + \dfrac{4^2}{2}\,.\,C_1 + \dfrac{4^3}{3}\,C_2 + \ldots\ldots\ldots\ldots + \dfrac{4^{n+1}}{n+1}\,C_n = \dfrac{5^{n+1} - 1}{n+1}$

(iii) ${}^{n}C_0\,.\,{}^{n+1}C_n + {}^{n}C_1\,.\,{}^{n}C_{n-1} + {}^{n}C_2\,.\,{}^{n-1}C_{n-2} + \ldots\ldots\ldots\ldots + {}^{n}C_n\,.\,{}^{1}C_0 = 2^{n-1}\,(n + 2)$

(iv) ${}^{2}C_2 + {}^{3}C_2 + \ldots\ldots\ldots + {}^{n}C_2 = {}^{n+1}C_3$

Binomial theorem for negative and fractional indices :

If $n \in R$, then $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2\,!} x^2 + \frac{n(n-1)(n-2)}{3\,!} x^3 + ...$

$................. + \frac{n(n-1)(n-2).......(n-r+1)}{r\,!} x^r + ... \infty.$

Remarks

(i) The above expansion is valid for any rational number other than a whole number if $|x| < 1$.

(ii) When the index is a negative integer or a fraction then number of terms in the expansion of $(1 + x)^n$ is infinite, and the symbol ^nC_r cannot be used to denote the coefficient of the general term.

(iii) The first term must be unity in the expansion, when index 'n' is a negative integer or fraction

$$(x+y)^n = \begin{cases} x^n\left(1+\frac{y}{x}\right)^n = x^n\left\{1+n\,.\,\frac{y}{x}+\frac{n\,(n-1)}{2\,!}\left(\frac{y}{x}\right)^2+.....\right\} & \text{if } \left|\frac{y}{x}\right|<1 \\ y^n\left(1+\frac{x}{y}\right)^n = y^n\left\{1+n\,.\,\frac{x}{y}+\frac{n\,(n-1)}{2\,!}\left(\frac{x}{y}\right)^2+.....\right\} & \text{if } \left|\frac{x}{y}\right|<1 \end{cases}$$

(iv) The general term in the expansion of $(1 + x)^n$ is $T_{r+1} = \frac{n(n-1)(n-2).........(n-r+1)}{r\,!} x^r$

(v) When 'n' is any rational number other than whole number then approximate value of $(1 + x)^n$ is $1 + nx$ (x^2 and higher powers of x can be neglected)

(vi) Expansions to be remembered ($|x| < 1$)

(a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + ... + (-1)^r x^r + ...\infty$

(b) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + ... + x^r + ...\infty$

(c) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... + (-1)^r (r + 1) x^r + ...\infty$

(d) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ... + (r + 1)x^r + ... \infty$

Example # 17 : Prove that the coefficient of x^r in $(1 - x)^{-n}$ is $^{n+r-1}C_r$

Solution: $(r + 1)^{th}$ term in the expansion of $(1 - x)^{-n}$ can be written as

$$T_{r+1} = \frac{-n(-n-1)(-n-2)......(-n-r+1)}{r\,!} (-x)^r$$

$$= (-1)^r \frac{n(n+1)(n+2)......(n+r-1)}{r\,!} (-x)^r = \frac{n(n+1)(n+2)......(n+r-1)}{r\,!} x^r$$

$$= \frac{(n-1)!\ n(n+1)......(n+r-1)}{(n-1)\,!\ r\,!} x^r$$ Hence, coefficient of x^r is $\frac{(n+r-1)!}{(n-1)!\ r!} = {}^{n+r-1}C_r$ Proved

Example-18 : If x is so small such that its square and higher powers may be neglected, then find the value of $\frac{(1-2x)^{1/3}+(1+5x)^{-3/2}}{(9+x)^{1/2}}$

Solution : $$\frac{(1-2x)^{1/3}+(1+5x)^{-3/2}}{(9+x)^{1/2}} = \frac{1-\frac{2}{3}x+1-\frac{15x}{2}}{3\left(1+\frac{x}{9}\right)^{1/2}} = \frac{1}{3}\left(2-\frac{49}{6}x\right)\left(1+\frac{x}{9}\right)^{-1/2}$$

$$= \frac{1}{3}\left(2-\frac{49}{6}x\right)\left(1-\frac{x}{18}\right) = \frac{1}{2}\left(2-\frac{x}{9}-\frac{49}{6}x\right) = 1 - \frac{x}{18} - \frac{49}{12}x = 1 - \frac{149}{36}x$$

Self practice problems :

(13) Find the possible set of values of x for which expansion of $(3 - 2x)^{1/2}$ is valid in ascending powers of x.

(14) If $y = \frac{2}{5} + \frac{1.3}{2\,!}\left(\frac{2}{5}\right)^2 + \frac{1.3.5}{3\,!}\left(\frac{2}{5}\right)^3 + ...$, then find the value of $y^2 + 2y$

(15) The coefficient of x^{50} in $\dfrac{2-3x}{(1-x)^3}$ is

(A) 500 (B) 1000 (C) –1173 (D) 1173

Ans. (13) $x \in \left(-\dfrac{3}{2}, \dfrac{3}{2}\right)$ (14) 4 (15) C

Multinomial theorem : As we know the Binomial Theorem $(x + y)^n = \sum_{r=0}^{n} {}^nC_r \; x^{n-r} y^r = \sum_{r=0}^{n} \dfrac{n!}{(n-r)!\; r!} x^{n-r} y^r$

putting $n - r = r_1$, $r = r_2$ therefore, $(x + y)^n = \sum_{r_1+r_2=n} \dfrac{n!}{r_1!\; r_2!} x^{r_1} \cdot y^{r_2}$

Total number of terms in the expansion of $(x + y)^n$ is equal to number of non-negative integral solution of $r_1 + r_2 = n$ i.e. ${}^{n+2-1}C_{2-1} = {}^{n+1}C_1 = n + 1$

In the same fashion we can write the multinomial theorem

$(x_1 + x_2 + x_3 + \ldots\ldots\ldots\ldots x_k)^n = \sum_{r_1+r_2+\ldots+r_k=n} \dfrac{n!}{r_1!\; r_2! \ldots\; r_k!} x_1^{r_1} \cdot x_2^{r_2} \ldots x_k^{r_k}$

Here total number of terms in the expansion of $(x_1 + x_2 + \ldots\ldots\ldots + x_k)^n$ is equal to number of non-negative integral solution of $r_1 + r_2 + \ldots\ldots\ldots + r_k = n$ i.e. ${}^{n+k-1}C_{k-1}$

Example # 19 : Find the coefficient of $a^2 b^3 c^4 d$ in the expansion of $(a - b - c + d)^{10}$

Solution : $(a - b - c + d)^{10} = \sum_{r_1+r_2+r_3+r_4=10} \dfrac{(10)!}{r_1!\; r_2! r_3!\; r_4!} (a)^{r_1} \; (-b)^{r_2} \; (-c)^{r_3} \; (d)^{r_4}$

we want to get $a^2 b^3 c^4 d$ this implies that $r_1 = 2, r_2 = 3, r_3 = 4, r_4 = 1$

$\therefore$ coeff. of $a^2 b^3 c^4 d$ is $\dfrac{(10)!}{2!\; 3!\; 4!\; 1!} (-1)^3 (-1)^4 = -12600$

Example # 20 : In the expansion of $\left(1 + x + \dfrac{7}{x}\right)^{11}$, find the term independent of x.

Solution : $\left(1 + x + \dfrac{7}{x}\right)^{11} = \sum_{r_1+r_2+r_3=11} \dfrac{(11)!}{r_1!\; r_2!\; r_3!} (1)^{r_1} \; (x)^{r_2} \; \left(\dfrac{7}{x}\right)^{r_3}$

The exponent 11 is to be divided among the base variables 1, x and $\dfrac{7}{x}$ in such a way so that we get x^0. Therefore, possible set of values of (r_1, r_2, r_3) are (11, 0, 0), (9, 1, 1), (7, 2, 2), (5, 3, 3), (3, 4, 4), (1, 5, 5)

Hence the required term is

$$\frac{(11)!}{(11)!}(7^0) + \frac{(11)!}{9!\;1\;!1\;!} 7^1 + \frac{(11)!}{7!\;2\;!\;2\;!} 7^2 + \frac{(11)!}{5!\;3\;!\;3\;!} 7^3 + \frac{(11)!}{3!\;4\;!\;4\;!} 7^4 + \frac{(11)!}{1\;!\;5\;!\;5\;!} 7^5$$

$$= 1 + \frac{(11)!}{9\;!\;2\;!} \cdot \frac{2\;!}{1\;!\;1\;!} 7^1 + \frac{(11)!}{7\;!\;4\;!} \cdot \frac{4\;!}{2\;!\;2\;!} 7^2 + \frac{(11)\;!}{5\;!\;6\;!} \cdot \frac{6\;!}{3\;!\;3\;!} 7^3$$

$$+ \frac{(11)\;!}{3\;!\;8\;!} \cdot \frac{8\;!}{4\;!\;4\;!} 7^4 + \frac{(11)\;!}{1\;!\;10\;!} \cdot \frac{(10)\;!}{5\;!\;5\;!} 7^5$$

$$= 1 + {}^{11}C_2 \cdot {}^2C_1 \cdot 7^1 + {}^{11}C_4 \cdot {}^4C_2 \cdot 7^2 + {}^{11}C_6 \cdot {}^6C_3 \cdot 7^3 + {}^{11}C_8 \cdot {}^8C_4 \cdot 7^4 + {}^{11}C_{10} \cdot {}^{10}C_5 \cdot 7^5 = 1 + \sum_{r=1}^{5} {}^{11}C_{2r} \cdot {}^{2r}C_r \cdot 7^r$$

Self practice problems :

(16) The number of terms in the expansion of $(a + b + c + d + e)^n$ is

(A) ${}^{n+4}C_4$ (B) ${}^{n+3}C_n$ (C) ${}^{n+5}C_n$ (D) n + 1

(17) Find the coefficient of $x^2 y^3 z^1$ in the expansion of $(x - 2y - 3z)^7$

(18) Find the coefficient of x^{17} in $(2x^2 - x - 3)^9$

Ans. (16) A (17) $\dfrac{7!}{2!\; 3!\; 1!}$ 24 (18) 2304