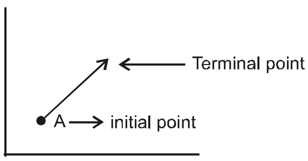
Vectors & Three Dimensional Geometry

What if angry vectors veer Round your sleeping head, and from. There's never need to fear Violence of the poor world's abstract storm. .... Warren, Robert PennNature is an infinite sphere of which the centre is everywhere and the circumference nowhere .... Pascal, Blaise

Vectors and their representation :



Vector quantities are specified by definite magnitude and definite direction. A vector is generally represented by a directed line segment, say $\overrightarrow{AB}$. A is called the **initial point** and B is called the **terminal point**. The magnitude of vector $\overrightarrow{AB}$ is expressed by $|\overrightarrow{AB}|$.

Zero vector :
A vector of zero magnitude i.e. which has the same initial and terminal point, is called a **zero vector.** It is denoted by **O**. **The direction of zero vector is indeterminate.**

Unit vector :
A vector of unit magnitude in the direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\vec{a}$, symbolically $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$.

Equal vectors :

Two vectors are said to be equal if they have the same magnitude, direction and represent the same physical quantity.

Collinear vectors :

Two vectors are said to be collinear if their directed line segments are parallel irrespective of their directions. Collinear vectors are also called **parallel vectors**. If they have the same direction($\rightrightarrows$) they are named as **like vectors** but if they have opposite direction ($\rightleftarrows$) then they are named as **unlike vectors**.

Symbolically, two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, $\vec{a} = \lambda\vec{b}$, where $\lambda \in R$

$$\vec{a} = \lambda\vec{b} \Leftrightarrow \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) = \lambda\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) \Leftrightarrow a_1 = \lambda b_1,\ a_2 = \lambda b_2,\ a_3 = \lambda b_3 \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}\ (= \lambda)$$

Vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are collinear if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$

Note : If $\vec{a}, \vec{b}$ are non zero, non–collinear vectors, such that $x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b} \Rightarrow x = x'$, $y = y'$, (where x, x', y, y' are scalars)

Example # 1 : Find unit vector of $\hat{i} - 2\hat{j} + 3\hat{k}$

Solution : $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$

if $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ then $|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$

$\therefore$ $|\vec{a}| = \sqrt{14}$ $\Rightarrow$ $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{1}{\sqrt{14}}\hat{i} - \dfrac{2}{\sqrt{14}}\hat{j} + \dfrac{3}{\sqrt{14}}\hat{k}$

Example # 2 : $\vec{a} = (x + 1)\hat{i} - (2x + y)\hat{j} + 3\hat{k}$ and $\vec{b} = (2x - 1)\hat{i} + (2 + 3y)\hat{j} + \hat{k}$

find $\vec{x}$ and $\vec{y}$ for which $\vec{a}$ and $\vec{b}$ are parallel.

Solution : $\vec{a}$ and $\vec{b}$ are parallel $\Rightarrow$ $\dfrac{x+1}{2x-1} = \dfrac{-(2x+y)}{2+3y} = \dfrac{3}{1}$ $\Rightarrow$ $x = 4/5$, $y = -19/25$

Coplanar vectors :

A given number vectors are called coplanar if their line segments are all parallel to the same plane. Note that **"two vectors are always coplanar"**.

Multiplication of a vector by a scalar :

If $\vec{a}$ is a vector and m is a scalar, then m is a vector parallel to $\vec{a}$ whose magnitude is $|m|$ times that of $\vec{a}$. This multiplication is called **scalar multiplication**. If $\vec{a}$ and $\vec{a}$ are vectors and m, n are scalars, then :

(i) $m\ (\vec{a}) = (\vec{a})\ m = m\vec{a}$ (ii) $m\ (n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$

(iii) $(m+n)\ \vec{a} = m\vec{a} + n\vec{a}$ (iv) $m\ (\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$

Self Practice Problems :

(1) Given a regular hexagon ABCDEF with centre O, show that
(i) $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OD} - \overrightarrow{OE}$ (ii) $\overrightarrow{EA} = 2\ \overrightarrow{OB} + \overrightarrow{OF}$ (iii) $\overrightarrow{AD} + \overrightarrow{EB} + \overrightarrow{FC} = 4\ \overrightarrow{AB}$

(2) Let ABCDEF be a regular hexagon. If $\overrightarrow{AD} = x\ \overrightarrow{BC}$ and $\overrightarrow{CF} = y\ \overrightarrow{AB}$ then find xy.

(3) The sum of the two unit vectors is a unit vector. Show that the magnitude of the their difference is $\sqrt{3}$.

Answers : (2) – 4

Addition of vectors :

(i) If two vectors $\vec{a}$ and $\vec{b}$ are represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$, then their sum $\vec{a}+\vec{b}$ is a vector represented by $\overrightarrow{OC}$, where OC is the diagonal of the parallelogram OACB.

(ii) $\vec{a}+\vec{b} = \vec{b}+\vec{a}$ **(commutative)** (iii) $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$ **(associative)**

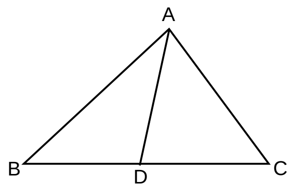
(iv) $\vec{a}+\vec{0} = \vec{a} = \vec{0}+\vec{a}$ (v) $\vec{a}+(-\vec{a}) = \vec{0} = (-\vec{a})+\vec{a}$

(vi) $|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$ (vii) $|\vec{a}-\vec{b}| \ge ||\vec{a}| - |\vec{b}||$

Example # 3 : The two sides of ΔABC are given by $\overrightarrow{AB} = 2\hat{i} + 4\hat{j} + 4\hat{k}$, $\overrightarrow{AC} = 2\hat{i} + 2\hat{j} + \hat{k}$. Then find the length of median through A.

Solution : Let D be mid point of BC

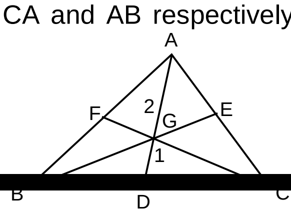
In ΔABC, $\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$ $\Rightarrow$ $\overrightarrow{AB} + \dfrac{1}{2}\overrightarrow{BC} = \overrightarrow{AD}$



$\Rightarrow$ $\dfrac{\overrightarrow{AB}+(\overrightarrow{AB}+\overrightarrow{BC})}{2} = \overrightarrow{AD}$

$\Rightarrow$ $\dfrac{\overrightarrow{AB}+\overrightarrow{AC}}{2} = \overrightarrow{AD}$ $\Rightarrow |\overrightarrow{AD}| = \left|\dfrac{4\hat{i}+6\hat{j}+5\hat{k}}{2}\right| = \dfrac{\sqrt{77}}{2}$

Example # 4 : In a triangle ABC, D, E, F are the mid-points of the sides BC, CA and AB respectively then prove that, $\overrightarrow{AD} = -(\overrightarrow{BE} + \overrightarrow{CF})$.



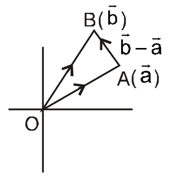
Solution : $\overrightarrow{AD} = 3\,\overrightarrow{GD} = 3.\dfrac{1}{2}(\overrightarrow{GB}+\overrightarrow{GC})$ where D is mid-point of BC

$$= \frac{3}{2}\left[\frac{2}{3}\overrightarrow{EB}+\frac{2}{3}\overrightarrow{FC}\right] = -(\overrightarrow{BE} + \overrightarrow{CF})$$

Position vector of a point:

Let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B, then

$\overrightarrow{AB} = \vec{b}-\vec{a}$ = position vector **(p.v.)** of B – position vector **(p.v.)** of A.

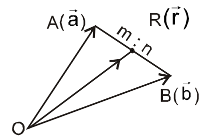


DISTANCE FORMULA

Distance between the two points $A(\vec{a})$ and $B(\vec{b})$ is $AB = \left|\vec{a}-\vec{b}\right|$

SECTION FORMULA

If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B, then the p.v. of a point which divides AB in the ratio m: n is given by $\vec{r} = \dfrac{n\vec{a}+m\vec{b}}{m+n}$.



Note : Position vector of mid point of AB = $\dfrac{\vec{a}+\vec{b}}{2}$.

Example # 5 : Let O be the centre of a regular pentagon ABCDE and $\overrightarrow{OA} = \vec{a}$.
Then $\overrightarrow{AB}+2\overrightarrow{BC}+3\overrightarrow{CD}+4\overrightarrow{DE}+5\overrightarrow{EA} =$

Solution : $\overrightarrow{OA}=\vec{a}, \overrightarrow{OB}=\vec{b}, \overrightarrow{OC}=\vec{c}, \overrightarrow{OD}=\vec{d}, \overrightarrow{OE}=\vec{e}$

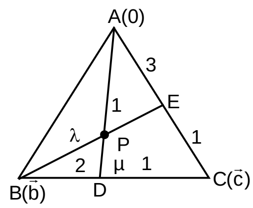
$$\overrightarrow{AB}+2\overrightarrow{BC}+3\overrightarrow{CD}+4\overrightarrow{DE}+5\overrightarrow{EA} = \left(\vec{b}-\vec{a}\right) + 2\left(\vec{c}-\vec{b}\right) + 3\left(\vec{d}-\vec{c}\right) + 4\left(\vec{e}-\vec{d}\right) + 5\left(\vec{a}-\vec{e}\right)$$

$$= 5\vec{a} - \left(\vec{a}+\vec{b}+\vec{c}+\vec{d}+\vec{e}\right) = 5\vec{a}, \quad (\text{since } \vec{a}+\vec{b}+\vec{c}+\vec{d}++\vec{e} = 0)$$

Example # 6 : In a triangle ABC, D and E are points on BC and AC respectively, such that BD = 2DC and AE = 3EC. Let P be the point of intersection of AD and BE. Find $\dfrac{BP}{PE}$ using vector method.

Solution : Let the position vectors of points B and C be respectively $\vec{b}$ and $\vec{c}$ referred to A as origin of reference.

Let $\dfrac{BP}{PE} = \lambda$ and $\dfrac{PD}{AP} = \mu$



$$\overrightarrow{AD} = \frac{2\vec{c}+\vec{b}}{3}, \quad \overrightarrow{AE} = \frac{3}{4}\vec{c} \Rightarrow \overrightarrow{AP} = \frac{\frac{3\lambda\vec{c}}{4}+\vec{b}}{\lambda+1} = \frac{\frac{2\vec{c}+\vec{b}}{3}}{\mu+1}$$

comparing the coefficient of $\vec{b}$ & $\vec{c}$

$$\frac{1}{\lambda+1} = \frac{1}{3(\mu+1)} \quad \text{and} \quad \frac{3\lambda}{4(\lambda+1)} = \frac{2}{3(\mu+1)}$$

solving above equations we get $\lambda = 8/3$

Self Practice Problems

(4) Express vectors $\overrightarrow{BC}$, $\overrightarrow{CA}$ and $\overrightarrow{AB}$ in terms of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$

(5) If $\vec{a}, \vec{b}$ are position vectors of the points(1,–1),(–2, m), find the value of m for which $\vec{a}$ and $\vec{b}$ are collinear.

(6) The vertices P, Q and S of a ΔPQS have position vectors $\vec{p}, \vec{q}$ and $\vec{s}$ respectively.

(i) Find the position vector of $\vec{t}$ of point T in terms of $\vec{p}, \vec{q}$ and $\vec{s}$, such that ST : TM = 2 : 1 and M is mid-point of PQ.

(ii) If the parallelogram PQRS is now completed. Express , the position vector of the point R in terms of $\vec{p}, \vec{q}$ and $\vec{s}$

(7) In a quadrilateral ABCD, $\overrightarrow{AB} = \vec{p}$, $\overrightarrow{BC} = \vec{q}$, $\overrightarrow{DA} = \vec{p} - \vec{q}$. If E is the mid point of BC and F is the point on DE such that DF = $\frac{4}{5}$ DE. Show that the points. A,F,C are collinear.

(8) Point L, M, N divide the sides BC, CA, AB of ΔABC in the ratios 1 : 4, 3 : 2, 3 : 7 respectively. Prove that $\overrightarrow{AL} + \overrightarrow{BM} + \overrightarrow{CN}$ is a vector parallel to $\overrightarrow{CK}$, when K divides AB in the ratio 1 : 3.

Answers : (4) $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$, $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ (5) m = 2

(6) (i) $\vec{t} = \frac{1}{3}(\vec{p} + \vec{q} + \vec{s})$ (ii) $\vec{r} = (\vec{q} - \vec{p} + \vec{s})$

Distance formula

Distance between any two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given as $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

Distance of a point P from coordinate axes

Let PA, PB and PC are distances of the point P(x, y, z) from the coordinate axes OX, OY and OZ respectively then PA = $\sqrt{y^2 + z^2}$, PB = $\sqrt{z^2 + x^2}$, PC = $\sqrt{x^2 + y^2}$

Example # 7 : Find the locus of a point which is equidistance from A (0,2,3) and B (2, –2, 1).

Solution : let P (x, y, z) be any point which is equidistance from A (0,2,3) and B (2, – 2, 1)

PA = PB

$\Rightarrow \sqrt{(x-0)^2 + (y-2)^2 + (z-3)^2} = \sqrt{(x-2)^2 + (y+2)^2 + (z-1)^2} \Rightarrow x - 2y - z + 1 = 0$

Example # 8 : Find the locus of a point which moves such that the sum of its distances from points A(0, 0, – α) and B(0, 0, α) is constant.

Solution : Let the variable point whose locus is required be P(x, y, z)

Given PA + PB = constant = 2a (say)

$\therefore \quad \sqrt{(x-0)^2 + (y-0)^2 + (z+\alpha)^2} + \sqrt{(x-0)^2 + (y-0)^2 + (z-\alpha)^2} = 2a$

$\Rightarrow \quad \sqrt{x^2 + y^2 + (z+\alpha)^2} = 2a - \sqrt{x^2 + y^2 + (z-\alpha)^2}$

$\Rightarrow \quad x^2 + y^2 + z^2 + \alpha^2 + 2z\alpha = 4a^2 + x^2 + y^2 + z^2 + \alpha^2 - 2z\alpha - 4a\sqrt{x^2 + y^2 + (z-\alpha)^2}$

$\Rightarrow \quad 4z\alpha - 4a^2 = -4a\sqrt{x^2 + y^2 + (z-\alpha)^2} \quad \Rightarrow \quad \frac{z^2\alpha^2}{a^2} + a^2 - 2z\alpha = x^2 + y^2 + z^2 + \alpha^2 - 2z\alpha$

or, $\quad x^2 + y^2 + z^2\left(1 - \frac{\alpha^2}{a^2}\right) = a^2 - \alpha^2 \quad \Rightarrow \quad \frac{x^2}{a^2 - \alpha^2} + \frac{y^2}{a^2 - \alpha^2} + \frac{z^2}{a^2} = 1$

This is the required locus.

Self practice problems :

(9) One of the vertices of a cuboid is (0, 2, –1) and the edges from this vertex are along the positive x-axis, positive y-axis and positive z-axis respectively and are of lengths 2, 2, 3 respectively find out the vertices.

(10) Show that the points (0, 4, 1), (2, 3, –1), (4, 5, 0) and (2, 6, 2) are the vertices of a square.

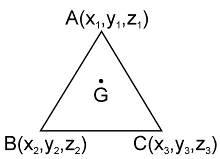
(11) Find the locus of point P if $AP^2 - BP^2 = 20$, where A $\equiv$ (2, –1, 3) and B $\equiv$ (–1, – 2, 1).

Answers : (9) (2,2, –1), (2, 4, –1), (2, 4, 2), (2, 2, 2), (0, 2, 2), (0, 4, 2), (0, 4, –1).

(11) x + y + 2z = 6

Centroid of a triangle

$$G \equiv \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

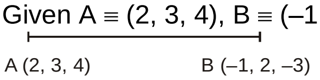


Incentre of triangle ABC

$$I \equiv \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}, \frac{az_1+bz_2+cz_3}{a+b+c}\right) \quad \text{Where AB = c, BC = a, CA = b}$$

Example # 9 : Show that the points A(2, 3, 4), B(–1, 2, –3) and C(–4, 1, –10) are collinear. Also find the ratio in which C divides AB.

Solution : Given A ≡ (2, 3, 4), B ≡ (–1, 2, –3), C ≡ (– 4, 1, –10).



Let C divide AB internally in the ratio k : 1, then

$$C \equiv \left(\frac{-k+2}{k+1}, \frac{2k+3}{k+1}, \frac{-3k+4}{k+1}\right) \therefore \frac{-k+2}{k+1} = -4 \quad \Rightarrow \quad 3k = -6 \Rightarrow \quad k = -2$$

For this value of k, $\frac{2k+3}{k+1} = 1$, and $\frac{-3k+4}{k+1} = -10$

Since k < 0, therefore C divides AB externally in the ratio 2 : 1 and points A, B, C are collinear.

Example # 10 : The vertices of a triangle are A(5, 4, 6), B(1, –1, 3) and C(4, 3, 2). The internal bisector of ∠BAC meets BC in D. Find AD.

Solution : $AB = \sqrt{4^2+5^2+3^2} = 5\sqrt{2}$

$AC = \sqrt{1^2+1^2+4^2} = 3\sqrt{2}$

Since AD is the internal bisector of BAC

$\therefore \quad \frac{BD}{DC} = \frac{AB}{AC} = \frac{5}{3} \quad \therefore$ D divides BC internally in the ratio 5 : 3

$$\therefore \quad D \equiv \left(\frac{5\times4+3\times1}{5+3}, \frac{5\times3+3(-1)}{5+3}, \frac{5\times2+3\times3}{5+3}\right) \quad \text{or,} \quad D = \left(\frac{23}{8}, \frac{12}{8}, \frac{19}{8}\right)$$

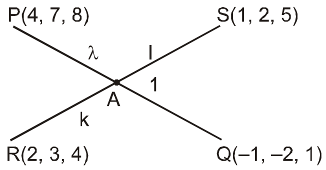
$$\therefore \quad AD = \sqrt{\left(5-\frac{23}{8}\right)^2+\left(4-\frac{12}{8}\right)^2+\left(6-\frac{19}{8}\right)^2} = \frac{\sqrt{1530}}{8} \text{ unit}$$

Example # 11 : If the points P, Q, R, S are (4, 7, 8), (– 1, – 2, 1), (2, 3, 4) and (1,2,5) respectively, show that PQ and RS intersect. Also find the point of intersection.

Solution : Let the lines PQ and RS intersect at point A.

Let A divide PQ in the ratio λ : 1, (λ ≠ –1) then $A \equiv \left(\frac{-\lambda+4}{\lambda+1}, \frac{-2\lambda+7}{\lambda+1}, \frac{\lambda+8}{\lambda+1}\right)$. .... (1)

Let A divide RS in the ratio k : 1, then $A \equiv \left(\frac{k+2}{k+1}, \frac{2k+3}{k+1}, \frac{5k+4}{k+1}\right)$ ..... (2)



From (1) and (2), we have,

$\frac{-\lambda+4}{\lambda+1} = \frac{k+2}{k+1} \Rightarrow -\lambda k - \lambda + 4k + 4 = \lambda k + 2\lambda + k + 2 \Rightarrow 2\lambda k + 3\lambda - 3k - 2 = 0$ .....(3)

$\frac{-2\lambda+7}{\lambda+1} = \frac{2k+3}{k+1} \Rightarrow -2\lambda k - 2\lambda + 7k + 7 = 2\lambda k + 3\lambda + 2k + 3 \Rightarrow 4\lambda k + 5\lambda - 5k - 4 = 0$ .....(4)

$\frac{\lambda+8}{\lambda+1} = \frac{5k+4}{k+1}$ ..... (5)

Multiplying equation (3) by 2, and subtracting from equation (4), we get $-\lambda + k = 0$ or, $\lambda = k$

Putting λ = k in equation (3), we get $2\lambda^2 + 3\lambda - 3\lambda - 2 = 0 \quad \Rightarrow \quad \lambda = 1 = k$

Clearly λ = k = 1 satisfies eqn. (5), hence our assumption is correct.

$$\therefore \quad A \equiv \left(\frac{-1+4}{2}, \frac{-2+7}{2}, \frac{1+8}{2}\right) \quad \text{or,} \quad A \equiv \left(\frac{3}{2}, \frac{5}{2}, \frac{9}{2}\right).$$

Self practice problems :

(12) Find the ratio in which yz plane divides the line joining the points A (4, 3, 5) and B (7, 4, 5).

(13) Find the co-ordinates of the foot of perpendicular drawn from the point A(1, 2, 1) to the line joining the point B(1, 4, 6) and C(5, 4, 4).

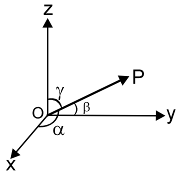
(14) Two vertices of a triangle are (4, –6, 3) and (2, –2, 1) and its centroid is $\left(\frac{8}{3}, -1, 2\right)$. Find the third vertex.

(15) Show that $\left(\frac{3}{2},\frac{7}{2},\frac{1}{2}\right)$ is the circumcentre of the triangle whose vertices are A (2, 3, 2), B (0, 4, 1) and C (3, 3, 0) and hence find its orthocentre.

Answers : (12) 4 : 7 Externally (13) (3, 4, 5) (14) (2, 5, 2) (15) (2, 3, 2)

Direction cosines and direction ratios

(i) **Direction cosines :** Let α, β, γ be the angles which a directed line makes with the positive directions of the axes of x, y and z respectively, then cos α, cosβ, cos γ are called the direction cosines of the line. The direction cosines are usually denoted by ℓ, m, n.



Thus $\ell = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

(ii) If ℓ, m, n be the direction cosines of a line, then $\ell^2 + m^2 + n^2 = 1$

(iii) **Direction ratios :** Let a, b, c be proportional to the direction cosines ℓ, m, n then a, b, c are called the direction ratios.

If a, b, c, are the direction ratios of any line L, then $a\hat{i} + b\hat{j} + c\hat{k}$ will be a vector parallel to the line L.

If ℓ, m, n are direction cosines of line L, then $\ell\,\hat{i} + m\,\hat{j} + n\,\hat{k}$ is a unit vector parallel to the line L.

(iv) If ℓ, m, n be the direction cosines and a, b, c be the direction ratios of a vector, then (ℓ, m, n)

$= \left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$ or $\left(\frac{-a}{\sqrt{a^2+b^2+c^2}}, \frac{-b}{\sqrt{a^2+b^2+c^2}}, \frac{-c}{\sqrt{a^2+b^2+c^2}}\right)$

(v) If the coordinates P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2), then the direction ratios of line PQ are, $a = x_2 - x_1$, $b = y_2 - y_1$ & $c = z_2 - z_1$ and the direction cosines of line PQ are $\ell = \frac{x_2 - x_1}{|PQ|}$, $m = \frac{y_2 - y_1}{|PQ|}$ and $n = \frac{z_2 - z_1}{|PQ|}$.

Example # 12 : If a line makes angle α, β, γ with the co-ordinate axes. Then find the value of $\sum\frac{\cos 3\alpha}{\cos\alpha}$.

Solution : $\sum\frac{\cos 3\alpha}{\cos\alpha} = \sum\frac{4\cos^3\alpha - 3\cos\alpha}{\cos\alpha} = \Sigma(4\cos^2\alpha - 3)$

$= 4(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 3 - 3 - 3 = 4 - 9 = -5$ **Ans.** –5

Example # 13 : If the direction ratios of two lines are given by $mn - 4n\ell + 3\ell m = 0$ and $\ell + 2m + 3n = 0$ then find the direction ratios of the lines.

Solution : Eliminating ℓ we have $m = \pm\sqrt{2}\,n$ $\therefore$ $\frac{\ell}{-2\sqrt{2}-3} = \frac{m}{\sqrt{2}} = \frac{n}{1}$ & $\frac{\ell}{2\sqrt{2}-3} = \frac{m}{-\sqrt{2}} = \frac{n}{1}$

Ans. $((-2\sqrt{2}-3)\lambda, \sqrt{2}\lambda, \lambda), (2\sqrt{2}-3)\lambda,\ \sqrt{2}\lambda - \lambda$ where $\lambda \in R - \{0\}$

Self practice problems:

(16) Find the direction cosines of a line lying in the xy plane and making angle 30° with x-axis.

(17) A line makes an angle of 60° with each of x and y axes, find the angle which this line makes with z-axis.

(18) A plane intersects the co-ordinates axes at point A(2, 0, 0), B(0, 4, 0), C(0, 0, 6) ; O is origin. Find the direction ratio of the line joining the vertex B to the centroid of face ABC.

Answers : (16) $\ell = \frac{\sqrt{3}}{2}$, m = $\pm \frac{1}{2}$, n = 0 (17) 45° (18) $\frac{2}{3}, -\frac{8}{3}, 2$

Scalar product (Dot Product) of two vectors :

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta, \ (0 \le \theta \le \pi)$$

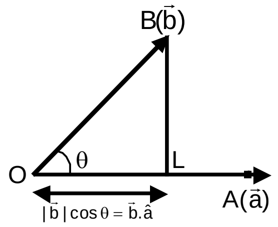
Note: (a) If θ is acute, then $\vec{a} \cdot \vec{b} > 0$ and if θ is obtuse, then $\vec{a} \cdot \vec{b} < 0$.

(b) $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ $(\vec{a} \ne 0, \ \vec{b} \ne 0)$

(c) Maximum value of $\vec{a} \cdot \vec{b}$ is $|\vec{a}|\,|\vec{b}|$ (d) Minimum value of $\vec{a} \cdot \vec{b}$ is $-|\vec{a}|\,|\vec{b}|$

Geometrical interpretation of scalar product :

As shown in Figure, projection of vector $\overrightarrow{OB}$ (or $\vec{b}$) along vector $\overrightarrow{OA}$ (or $\vec{a}$) is OL = $|\vec{b}| \cos\theta = \frac{\vec{b}.\vec{a}}{|\vec{a}|} = \vec{b}.\hat{a}$



Properties of Dot Product

(i) Projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

(ii) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ **(commutative)**

(iii) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ **(distributive)**

(iv) $(m\vec{a}) \cdot \vec{b} = \vec{a} \cdot (m\vec{b}) = m(\vec{a} \cdot \vec{b})$, where m is a scalar.

(v) $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$; $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

(vi) $\vec{a} \cdot \vec{a} = |\vec{a}|^2 = \vec{a}^2$

(vii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

(viii) $\left|\vec{a} \pm \vec{b}\right| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 \pm 2|\vec{a}||\vec{b}|\cos\theta}$, where θ is the angle between the vectors

Example # 14 : Find the value of p for which the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are

(i) perpendicular (ii) parallel

Solution : (i) $\vec{a} \perp \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = 0 \Rightarrow \left(3\hat{i} + 2\hat{j} + 9\hat{k}\right) \cdot \left(\hat{i} + p\hat{j} + 3\hat{k}\right) = 0$

$\Rightarrow$ 3 + 2p + 27 = 0 $\Rightarrow$ p = – 15

(ii) vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are parallel iff

$$\frac{3}{1} = \frac{2}{p} = \frac{9}{3} \quad \Rightarrow \quad 3 = \frac{2}{p} \quad \Rightarrow \quad p = \frac{2}{3}$$

Example # 15 : If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vectors such that each is inclined at an angle $\pi/3$ with the other two and $|\vec{a}| = 1$, $|\vec{b}| = 2$, $|\vec{c}| = 3$, then find the scalar product of the vectors $2\vec{a} + 3\vec{b} - 5\vec{c}$ and $4\vec{a} - 6\vec{b} + 10\vec{c}$.

Solution : Dot products is $8^2\vec{a} - 18^2\vec{b} - 50^2\ \vec{c} + \vec{a}.\vec{b}\ (-12 + 12) + \vec{b}.\vec{c}\ (30 + 30) + \vec{c}.\ \vec{a}\ (20 - 20)$

$= 8 - 18\ (4) - 50(9) + 60\ \left(2.3\cos\frac{\pi}{3}\right) = 188 - 522 = -334$

Example # 16 : Find the values of x for which the angle between the vectors
$\vec{a} = 2x^2\,\hat{i} + 4x\,\hat{j} + \hat{k}$ and $\vec{b} = 7\,\hat{i} - 2\,\hat{j} + x\,\hat{k}$ is obtuse.

Solution : The angle θ between vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a}\,.\,\vec{b}}{|\vec{a}||\vec{b}|}$

Now, θ is obtuse $\Rightarrow$ $\cos\theta < 0$ $\Rightarrow$ $\dfrac{\vec{a}\,.\,\vec{b}}{|\vec{a}||\vec{b}|} < 0 \Rightarrow \vec{a}\,.\,\vec{b} < 0$ $\quad [\because |\vec{a}|, |\vec{b}| > 0]$

$\Rightarrow\ 14x^2 - 8x + x < 0 \Rightarrow 7x\ (2x - 1) < 0 \Rightarrow x(2x - 1) < 0 \Rightarrow 0 < x < \dfrac{1}{2}$

Hence, the angle between the given vectors is obtuse if $x \in (0, 1/2)$

Example # 17 : If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $= 2\vec{a} - \hat{j} + 3\hat{k}$, then find

(i) Component of $\vec{b}$ along $\vec{a}$. (ii) Component of $\vec{b}$ in plane of $\vec{a}$ & $\vec{b}$ but $\perp$ to $\vec{a}$.

Solution : (i) Component of $\vec{b}$ along $\vec{a}$ is $\left(\dfrac{\vec{a}\,.\,\vec{b}}{|\vec{a}|^2}\right)\vec{a}$; Here $\vec{a}.\vec{b} = 2 - 1 + 3 = 4$ and $|\vec{a}|^2 = 3$

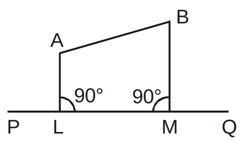
Hence $\left(\dfrac{\vec{a}\,.\,\vec{b}}{|\vec{a}|^2}\right)\vec{a} = \dfrac{4}{3}\vec{a} = \dfrac{4}{3}\ (\hat{i} + \hat{j} + \hat{k})$

(ii) Component of $\vec{b}$ in plane of $\vec{a}$ & $\vec{b}$ but $\perp$ to $\vec{a}$ is $\vec{b} - \left(\dfrac{\vec{a}\,.\,\vec{b}}{|\vec{a}|^2}\right)\vec{a}. = \dfrac{1}{3}\left(2\hat{i} - 7\hat{j} + 5\hat{k}\right)$

Example # 18 : Find the projection of the line joining A(1, 2, 3) and B(–1, 4, 2) on the line having direction ratios 2, 3, –6.

Solution : $\overrightarrow{AB} = -2\hat{i} + 2\hat{j} - \hat{k}$

Projection of $\overrightarrow{AB} = -2\hat{i} + 2\hat{j} - \hat{k}$ on $2\hat{i} + 3\hat{j} - 6\hat{k}$ is $\dfrac{-4 + 6 + 6}{\sqrt{4 + 9 + 36}} = \dfrac{8}{7}$



Self Practice Problems :

(19) If $\vec{a}$ and $\vec{b}$ are unit vectors and θ is angle between them, prove that $\tan\dfrac{\theta}{2} = \dfrac{|\vec{a} - \vec{b}|}{|\vec{a} + \vec{b}|}$.

(20) Find the values of x for which the angle between the vectors $\vec{a} = 2x\,\hat{i} + 4\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 3\hat{j} + x\hat{k}$ is 90º

(21) if a, b, c are the pth, qth, rth terms of a HP then find the angle between the vectors $= (q - r)\,\hat{i} + (r - q)\,\hat{j} + (p - q)\,\hat{k}$ and $\vec{v} = \dfrac{1}{a}\,\hat{i} + \dfrac{1}{b}\,\hat{j} + \dfrac{1}{c}\,\hat{k}$.

(22) The points O, A, B, C, D are such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = 2\vec{a} + 3\vec{b}$, $\overrightarrow{OD} = \vec{a} + 2\vec{b}$
Given that the length of $\overrightarrow{OA}$ is three times the length of $\overrightarrow{OB}$. Show that $\overrightarrow{BD}$ and $\overrightarrow{AC}$ are perpendicular.

(23) ABCD is a tetrahedron and G is the centroid of the base BCD. Prove that
$AB^2 + AC^2 + AD^2 = GB^2 + GC^2 + GD^2 + 3GA^2$

(24) A (2, 3, –2), B (1, 5, 4,), C(0, –1, 2) D (4, 0, 3). Find the projection of line segment AB on CD line.

(25) The projections of a directed line segment on co-ordinate axes are 3, 4, –12. Find its length and direction cosines.

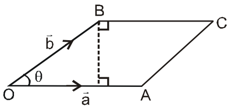
Answers : (20) x = 12/7 (21) $\pi/2$

(24) $\frac{2\sqrt{2}}{3}$ (25) 13, $\frac{3}{13}, \frac{4}{13}, \frac{-12}{13}$

Vector product (Cross Product) of two vectors:

(i) If $\vec{a}$, $\vec{b}$ are two vectors and θ is the angle between them, then $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ forms a right handed screw system.

(ii) Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ and $\vec{b}$.



(iii) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ **(not commutative)**

(iv) $(m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b}) = m(\vec{a} \times \vec{b})$, where m is a scalar.

(v) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ **(distributive)**

(vi) $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a}$ and $\vec{b}$ are parallel (collinear) $(\vec{a} \neq 0,\ \vec{b} \neq 0)$ i.e. $\vec{a} = K\vec{b}$, where K is a scalar.

(vii) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$; $\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$

(viii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(x) A vector of magnitude 'r' and perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\pm\frac{r(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$

(xii) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vectors of 3 points A, B and C respectively, then the vector area of $\Delta ABC = \frac{1}{2}(\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a})$. The points A, B and C are collinear if $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$

(xiii) Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ and $\vec{d}_2$ is given by $\frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$

Example # 19 : Given $\vec{a} = \hat{i} + \hat{j} - \hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = -\hat{i} + 2\hat{j} - \hat{k}$, then find a unit vector perpendicular to both $\vec{a} + \vec{b}$ and $\vec{b} + \vec{c}$.

Solution : $\vec{\alpha} \times \vec{\beta}$ is $\perp$ to both $\vec{\alpha}$ and $\vec{\beta}$

$\vec{a} + \vec{b} = 3\hat{j}$, $\vec{b} + \vec{c} = -2\hat{i} + 4\hat{j}$

$\therefore$ $3\hat{j} \times (-2\hat{i} + 4\hat{j})$ is $\perp$ to both or $-6\hat{j} \times \hat{i} = 6\hat{k}$ $\therefore$ hence a unit vector is $\hat{k}$.

Example # 20 : If $\vec{\alpha} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\vec{\beta} = -\hat{i} + 2\hat{j} - 4\hat{k}$, $\vec{\gamma} = \hat{i} + \hat{j} + \hat{k}$, then find value $(\vec{\alpha} \times \vec{\beta}).(\vec{\alpha} \times \vec{\gamma})$.

Solution : $\vec{\alpha} \times \vec{\beta} = -10\hat{i} + 9\hat{j} + 7\hat{k}$ and $\vec{\alpha} \times \vec{\gamma} = 4\hat{i} - 3\hat{j} - \hat{k}$ their dot product = –40 – 27 – 7 = –74

Example # 21 : Let $\overrightarrow{OA} = \vec{a} + 3\vec{b}$, $\overrightarrow{OB} = 5\vec{a} + 4\vec{b}$ and $\overrightarrow{OC} = 2\vec{a} - \vec{b}$ where O is origin. Let p denote the area of the quadrilateral OABC and q denote the area of the parallelogram with OA and OC as adjacent sides. Find $\frac{p}{q}$.

Solution : We have, p = Area of the quadrilateral OABC

$\Rightarrow \quad p = \frac{1}{2}|\overrightarrow{OB} \times \overrightarrow{AC}| = \frac{1}{2}|\overrightarrow{OB} \times (\overrightarrow{OC} - \overrightarrow{OA})| \quad \Rightarrow \quad p = \frac{1}{2}|(5\vec{a} + 4\vec{b}) \times (4\vec{b} - \vec{a})|$

$\Rightarrow \quad p = \frac{1}{2}\ |20(\vec{a} \times \vec{b}) - 4(\vec{b} \times \vec{a})| = 12\ |\vec{a} \times \vec{b}|$ .....(i)

and q = Area of the parallelogram with OA and OC as adjacent sides

$\Rightarrow \quad q = |\overrightarrow{OA} \times \overrightarrow{OC}| = |(\vec{a} \times 3\vec{b}) \times (2\vec{a} - \vec{b})| = 7|\vec{a} \times \vec{b}|$ .....(ii)

From (i) and (ii), we get $\frac{p}{q} = \frac{12}{7}$

Self Practice Problems :

(26) If $\vec{p}$ and $\vec{q}$ are unit vectors forming an angle of 30º. Find the area of the parallelogram having $\vec{a} = \vec{p} + 2\vec{q}$ and $\vec{b} = 2\vec{p} + \vec{q}$ as its diagonals.

(27) ABC is a triangle and EF is any straight line parallel to BC meeting AC, AB in E, F respectively. If BR and CQ be drawn parallel to AC, AB respectively to meet EF in R and Q respectively, prove that Δ ARB = ΔACQ.

Answers : (26) 3/4 sq. units

A LINE

Equation of a line

(i) Vector equation: Vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$ where λ is a scalar.

(ii) Vector equation of a straight line passing through two points with position vectors $\vec{a}$ & $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

(iii) The equation of a line passing through the point (x_1, y_1, z_1) and having direction ratios a, b, c is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ = r. This form is called symmetric form. A general point on the line is given by (x_1 + ar, y_1 + br, z_1 + cr).

(iv) The equation of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

(v) Reduction of cartesian form of equation of a line to vector form & vice versa

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \Leftrightarrow = \vec{r}\ (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda\,(a\hat{i} + b\hat{j} + c\hat{k}).$$

(vi) The equations of the bisectors of the angles between the lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{a} + \mu\vec{c}$ are : $\vec{r} = \vec{a} + t\left(\hat{b} + \hat{c}\right)$ and $\vec{r} = \vec{a} + p\left(\hat{c} - \hat{b}\right)$.

Example # 22 : Find the equation of the line through the points (4, –5, 8) and (–1, 2, 7) in vector form as well as in cartesian form.

Solution : Let A $\equiv$ (4, –5, 8) , B $\equiv$ (–1, 2, 7)

Now $\vec{a} = \overrightarrow{OA} = 4\hat{i} - 5\hat{j} + 8\hat{k}$ and $\vec{b} = \overrightarrow{OB} = -\hat{i} + 2\hat{j} + 7\hat{k}$

Equation of the line through A($\vec{a}$) and B($\vec{b}$) is $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

or $\vec{r} = 4\hat{i} - 5\hat{j} + 8\hat{k} + t(-5\hat{i} + 7\hat{j} - \hat{k})$ ..... (1)

Equation of AB in cartesian form is $\frac{x-4}{5} = \frac{y+5}{-7} = \frac{z-8}{1}$

Example # 23 : Find the equation of the line passing through point (1, 0, 2) having direction ratio 3, –1, 5. Prove that this line passes through (4, – 1, 7).

Solution. equation of line is $\frac{x-1}{3} = \frac{y-0}{-1} = \frac{z-2}{5}$,

Now $\frac{4-1}{3} = \frac{-1-0}{-1} = \frac{7-2}{5} = 1$, so line passes through point (4, –1,7)

Example # 24 : Find the equation of the line drawn through point (–1, 7, 0) to meet at right angles the line $\frac{x-2}{2} = \frac{y+3}{-1} = \frac{z-1}{2}$

Solution : Given line is $\frac{x-2}{2} = \frac{y+3}{-1} = \frac{z-1}{2}$ ..... (1)

Let $P \equiv (-1, 7, 0)$

Co-ordinates of any point on line (1) may be taken as $Q \equiv (2r + 2, - r - 3, 2r + 1)$

Direction ratios of PQ are 2r + 3, – r – 10, 2r + 1

Direction ratios of line AB are 2, – 1, 2

Since $PQ \perp AB$

$\therefore$ $2(2r + 3) + (- r - 10)(-1) + 2(2r + 1) = 0 \Rightarrow r = -2$

Therefore, direction ratios of PQ are – 1 , – 8, – 3

Equation of line PQ is $\frac{x+1}{1} = \frac{y-7}{8} = \frac{z}{3}$

Example # 25 : A line passes through the point $3\hat{i}$ and is parallel to the vector $-\hat{i}+\hat{j}+\hat{k}$ and another line passes through the point $\hat{i}+\hat{j}$ and is parallel to the vector $\hat{i}+\hat{k}$, then find the point of intersection of lines.

Solution : A point on the first line is $3\hat{i}+s(-\hat{i}+\hat{j}+\hat{k})$ ....(i)

A point on the second line is $\hat{i}+\hat{j}+t(\hat{i}+\hat{k})$ ....(ii)

At the point of intersection (i) and (ii) are same.

$\therefore$ $3 - s = 1 + t$, $s = 1$, $s = t$ $\therefore$ $s = t = 1$

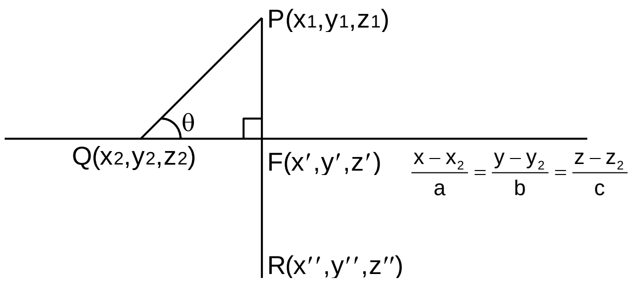
hence the point is $3\hat{i}+(-\hat{i}+\hat{j}+\hat{k}) = 2\hat{i}+\hat{j}+\hat{k}$ Ans. (2,1,1)

Self practice problems:

(28) Find the equation of the line parallel to line $\frac{x-3}{4} = \frac{y+1}{-1} = \frac{z-7}{5}$ and passing through the point (2, 3, –2).

Answers : (28) $\frac{x-2}{4} = \frac{y-3}{1} = \frac{z+2}{5}$

Foot, Reflection, length of perpendicular from a point to a line :



Let $L \equiv \frac{x-x_2}{a} = \frac{y-y_2}{b} = \frac{z-z_2}{c}$ is a given line and $P(x_1, y_1, z_1)$ is given point as shown in figure.

Let $F(x', y', z') = (ar+x_2, br+ y_2, cr+ z_2)$ ….(1) be the foot of the point P (x_1, y_1, z_1) with respect to the line L. Apply $\overrightarrow{PF}.(a\hat{i} + b\hat{j} + c\hat{k}) = 0$ we get 'r'. Now put this value of 'r' in (1) we get F

Now for calculating the reflection $R(x'', y'', z'')$ of the point P (x_1, y_1, z_1) with respect to the line L, apply midpoint formula (midpoint of P & R is F)

$$PF = PQ \sin\theta = \left|\frac{\overrightarrow{PF}.(a\hat{i} + b\hat{j} + c\hat{k})}{\sqrt{a^2+b^2+c^2}}\right|$$

Example # 26 : Find the length of the perpendicular from P (2, – 3, 1) to the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$.

Solution : Co-ordinates of any point on given line may be taken as $Q \equiv (2r - 1, 3r + 3, - r - 2)$

Direction ratios of PQ are $2r - 3, 3r + 6, - r - 3$

Direction ratios of AB are 2, 3, – 1

Since PQ $\perp$ AB

$\therefore$ $2(2r - 3) + 3(3r + 6) - 1(- r - 3) = 0 \Rightarrow 14r + 15 = 0 \Rightarrow r = \frac{-15}{14}$

$\therefore$ $Q \equiv \left(\frac{-22}{7}, \frac{-3}{14}, \frac{-13}{14}\right)$ $\therefore$ $PQ = \sqrt{\frac{531}{14}}$ units.

Self practice problems :

(29) Find the length and foot of perpendicular drawn from point (2,3,4) to the line $\frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3}$. Also find the image of the point in the line.

Answers : (29) $3\sqrt{5}$, $N \equiv (2, 6, -2)$, $I \equiv (2, 9, -8)$

Angle between two line :

If two lines have direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 respectively, then we can consider two vectors parallel to the lines as $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ and angle between them can be given as.

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}.$$

(i) The lines will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(ii) The lines will be parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Example # 27 : What is the angle between the lines whose direction cosines are

$-\frac{\sqrt{3}}{4}, \frac{1}{4}, -\frac{\sqrt{3}}{2}$ and $-\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}$

Solution : Let θ be the required angle, then $\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$

$$= \left(-\frac{\sqrt{3}}{4}\right)\left(-\frac{\sqrt{3}}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(-\frac{\sqrt{3}}{2}\right) . \left(\frac{\sqrt{3}}{2}\right) = \frac{3}{16} + \frac{1}{16} - \frac{3}{4} = -\frac{1}{2} \Rightarrow \theta = 120°,$$

Example # 28 : P is a point on line $\vec{r} = 5\hat{i} + 7\hat{j} - 2\hat{k} + s(3\hat{i} - \hat{j} + \hat{k})$ and Q is a point on the line $\vec{r} = -3\hat{i} + 3\hat{j} + 6\hat{k} + t\left(-3\hat{i} + 2\hat{j} + 4\hat{k}\right)$. If $\overrightarrow{PQ}$ is parallel to the vector, $2\hat{i} + 7\hat{j} - 5\hat{k}$, find P and Q

Solution : $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$ is parallel to $2\hat{i} + 7\hat{j} - 5\hat{k}$

$\therefore \quad -8\hat{i}-4\hat{j}+8\hat{k}+t(-3\hat{i}+2\hat{j}+4\hat{k})-s\left(3\hat{i}-\hat{j}+\hat{k}\right) = 2\alpha = -8-3t-3s$

$7\alpha = -4+2t+s \quad \Rightarrow \quad -5\alpha = 8+4t-s$

solving, $\alpha = t = s = -1$

$\therefore \quad P = (5, 7, -2) - (3, -1, 1) = (2, 8, -3) \quad \Rightarrow \quad Q = (-3, 3, 6) - (-3, 2, 4) = (0, 1, 2)$

Self practice problems :

(30) Find the angle between the lines whose direction cosines are given by $\ell + m + n = 0$ and $\ell^2 + m^2 - n^2 = 0$

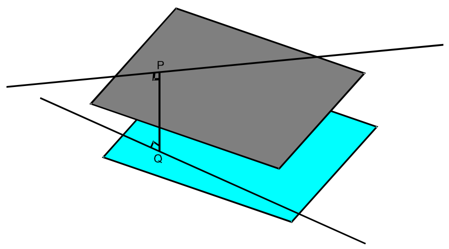
(31) Let P (6, 3, 2), Q (5, 1, 4), R (3, 3, 5) are vertices of a Δ find $\angle Q$.

(32) Show that the direction cosines of a line which is perpendicular to the lines having directions cosines $\ell_1\ m_1\ n_1$ and $\ell_2\ m_2\ n_2$ respectively are proportional to $m_1n_2 - m_2n_1,\ n_1\ell_2 - n_2\ell_1,\ \ell_1m_2 - \ell_2m_1$

Answers : (30) 60° (31) 90°

Skew Lines :

Lines in space which do not intersect and are also not parallel are called **skew line**.



If lines $\vec{r} = \vec{a} + \lambda_1\vec{p}$ & $\vec{r} = \vec{b} + \lambda_2\vec{q}$ are skew lines then $(\vec{b} - \vec{a}) \cdot (\vec{p} \times \vec{q}) \neq 0$

If lines are not skew lines then they are coplanar which means if $(\vec{b} - \vec{a}) \cdot (\vec{p} \times \vec{q}) = 0$, then lines are coplanar.

Shortest distance between two lines

(i) **Shortest distance (d) between lines** $\vec{r} = \vec{a} + \lambda_1\vec{p}$ & $\vec{r} = \vec{b} + \lambda_2\vec{q}$

is $d = \left|\dfrac{(\vec{b} - \vec{a}) \cdot (\vec{p} \times \vec{q})}{|\vec{p} \times \vec{q}|}\right|$

(ii) Shortest distance (d) between two skew lines $\dfrac{x-\alpha}{\ell} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$ and $\dfrac{x-\alpha'}{\ell'} = \dfrac{y-\beta'}{m'} = \dfrac{z-\gamma'}{n'}$

is $d = \begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} \div \sqrt{\sum(mn' - m'n)^2}$

For Skew lines the direction of the shortest distance would be perpendicular to both the lines.
If d = 0, the lines are coplanar

(iii) Shortest distance between two parallel lines $\vec{r}_1 = \vec{a}_1 + K\vec{b}$ and $\vec{r}_2 = \vec{a}_2 + K\vec{b}$, is given by

$d = \left|\dfrac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$

Example # 29 : Find the shortest distance and the vector equation of the line of shortest distance between the lines given by $\vec{r} = 3\hat{i}+8\hat{j}+3\hat{k}+\lambda\left(3\hat{i}-\hat{j}+\hat{k}\right)$ and $\vec{r} = -3\hat{i}-7\hat{j}+6\hat{k}+\mu\left(-3\hat{i}+2\hat{j}+4\hat{k}\right)$

Solution : Equation of given lines in cartesian form is $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} = \lambda$ (say L_1)

and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} = \mu$ (say L_2)

Let L on L_1 is $(3\lambda + 3, -\lambda + 8, \lambda + 3)$ and M on L_2 is $(-3\mu - 3, 2\mu - 7, 4\mu + 6)$

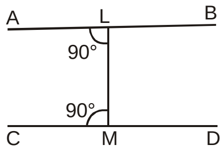
Direction ratios of LM are $3\lambda + 3\mu + 6, -\lambda - 2\mu + 15, \lambda - 4\mu - 3$.

Since LM $\perp$ L_1

$\therefore$ $3(3\lambda + 3\mu + 6) - 1(-\lambda - 2\mu + 15) + 1(\lambda - 4\mu - 3) = 0$ or, $11\lambda + 7\mu = 0$ ..... (1)

Again LM $\perp$ CD

$\therefore$ $-3(3\lambda + 3\mu + 6) + 2(-\lambda - 2\mu + 15) + 4(\lambda - 4\mu - 3) = 0$ or, $-7\lambda - 29\mu = 0$ ..... (2)



Solving (1) and (2), we get $\lambda = 0, \mu = 0 \Rightarrow$ $L \equiv (3, 8, 3), M \equiv (-3, -7, 6)$

Hence shortest distance LM = $\sqrt{(3+3)^2 + (8+7)^2 + (3-6)^2} = \sqrt{270} = 3\sqrt{30}$ units

Vector equation of LM is $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + t\left(6\hat{i} + 15\hat{j} - 3\hat{k}\right)$

Note : Cartesian equation of LM is $\frac{x-3}{6} = \frac{y-8}{15} = \frac{z-3}{-3}$.

Self practice problems:

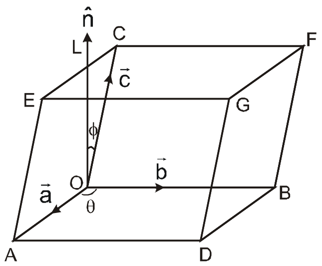
(33) Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$. Find also its equation.

Answers : (33) $\frac{1}{\sqrt{6}}$, $6x - y = 10 - 3y = 6z - 25$

Scalar triple product (Box Product) (S.T.P.) :

(i) The scalar triple product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is defined as: $\vec{a} \times \vec{b} \cdot \vec{c} = |\vec{a}|\ |\vec{b}|\ |\vec{c}|$, $\sin\theta \cdot \cos\phi$ where θ is the angle between $\vec{a}$, $\vec{b}$ **(i.e. $\vec{a} \wedge \vec{b} = \theta$)** and ϕ is the angle between $\vec{a} \times \vec{b}$ and $\vec{c}$ **($(\vec{a} \times \vec{b}) \wedge \vec{c} = \phi$)** . It is (i.e. $\vec{a} \times \vec{b} \cdot \vec{c}$) also written as $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$ and spelled as box product.

(ii) Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are



represented by $\vec{a}, \vec{b}$ and $\vec{c}$ i.e. $V = |[\vec{a}\ \vec{b}\ \vec{c}]|$

(iii) In a scalar triple product the position of dot and cross can be interchanged i.e.

$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ $\Rightarrow$ $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

(iv) $\vec{a} \cdot (\vec{b} \times \vec{c}) = -\vec{a} \cdot (\vec{c} \times \vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$

(v) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

In general, if $\vec{a} = a_1\vec{\ell} + a_2\vec{m} + a_3\vec{n}$; $\vec{b} = b_1\vec{\ell} + b_2\vec{m} + b_3\vec{n}$ and $\vec{c} = c_1\vec{\ell} + c_2\vec{m} + c_3\vec{n}$

then $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \left[\vec{\ell}\ \vec{m}\ \vec{n}\right]$, where $\vec{\ell}$, $\vec{m}$ and $\vec{n}$ are non-coplanar vectors.

(vi) If $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar, then $\Rightarrow$ $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

(vii) If $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-coplanar, then $[\vec{a}\ \vec{b}\ \vec{c}] > 0$ for right handed system and $[\vec{a}\ \vec{b}\ \vec{c}] < 0$ for left handed system.

(viii) $[\hat{i}\ \hat{j}\ \hat{k}] = 1$

(ix) $[K\vec{a}\ \vec{b}\ \vec{c}] = K\ [\vec{a}\ \vec{b}\ \vec{c}]$

(x) $[(\vec{a}+\vec{b})\vec{c}\,\vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$

(xi) $\left[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}\right] = 0$ and $\left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\right] = 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

(xii) $\left[\vec{a}\,\vec{b}\,\vec{c}\right]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$

Volume of Parallelopiped/ Tetrahedron and their properties :

(a) The volume of the parallelopiped whose three coterminous edges are $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $V = [\vec{a}\ \vec{b}\ \vec{c}]$

(b) The volume of the tetrahedron OABC with O as origin and the position vectors of A, B and C being $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively is given by $V = \frac{1}{6}\left|\left[\vec{a}\ \vec{b}\ \vec{c}\right]\right|$

(c) If the position vectors of the vertices of tetrahedron are $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$, then the position vector of its centroid is given by $\frac{1}{4}\ (\vec{a}+\vec{b}+\vec{c}+\vec{d})$.

Note : that this is also the point of concurrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron.

Example # 30 : The volume of the paralleopiped whose edges are represented by $-12\hat{i} + \lambda\hat{k}$, $3\hat{j} - k$, $2\hat{i} + \hat{j} - 15\hat{k}$ is 546, then find λ.

Solution : $V = \left|\begin{vmatrix} -12 & 0 & \lambda \\ 0 & 3 & -1 \\ 2 & 1 & -15 \end{vmatrix}\right|$ $\quad\therefore\quad 546 = |12 \times 44 - 6\lambda| \quad\therefore\quad \lambda = -3\ ,\ 179$

Example # 31 : Find the volume of the tetrahedron whose four vertices have position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$.

Solution : Let four vertices be A, B, C, D with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ respectively.

$\therefore\quad \overrightarrow{DA} = (\vec{a}-\vec{d}) \Rightarrow \overrightarrow{DB} = (\vec{b}-\vec{d}) \Rightarrow \overrightarrow{DC} = (\vec{c}-\vec{d})$

Hence volume $V = \frac{1}{6}\ [\vec{a}-\vec{d}\ \ \vec{b}-\vec{d}\ \ \vec{c}-\vec{d}]$

$= \frac{1}{6}\ (\vec{a}-\vec{d}).\,[(\vec{b}-\vec{d}) \times (\vec{c}-\vec{d})] = \frac{1}{6}\ (\vec{a}-\vec{d})\,.\,[\vec{b}\times\vec{c} - \vec{b}\times\vec{b} + \vec{c}\times\vec{d}\ ]$

$= \frac{1}{6}\{[\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{d}] + [\vec{a}\ \vec{c}\ \vec{d}] - [\vec{d}\ \vec{b}\ \vec{c}]\} = \frac{1}{6}\{[\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{d}] + [\vec{a}\ \vec{c}\ \vec{d}] - [\vec{b}\ \vec{c}\ \vec{d}]\}$

Example # 32 : Prove that vectors $\vec{r}_1 = (\sec^2 A, 1, 1)$; $\vec{r}_2 = (1, \sec^2 B, 1)$; $\vec{r}_3 = (1, 1, \sec^2 C)$ are always non-coplanar vectors if A, B, C $\in (0, \pi)$.

Solution : Condition of coplanarity gives D = 0 $\Rightarrow \begin{vmatrix} \sec^2 A & 1 & 1 \\ 1 & \sec^2 B & 1 \\ 1 & 1 & \sec^2 C \end{vmatrix} = 0$

$\Rightarrow$ $\sec^2 A\,[\sec^2 B \sec^2 C - 1] - 1(\sec^2 c - 1) + 1(1 - \sec^2 B) = 0$

$\Rightarrow$ $(1 + \tan^2 A)(\tan^2 B + \tan^2 C + \tan^2 B \tan^2 C) - \tan^2 C - \tan^2 B = 0$

$\Rightarrow$ $\tan^2 B \tan^2 C + \tan^2 A \tan^2 B + \tan^2 C \tan^2 A + \tan^2 A \tan^2 B \tan^2 C = 0$

divide by $\tan^2 A \tan^2 B \tan^2 C$

$\cot^2 A + \cot^2 B + \cot^2 C = -1$ it is a not possible

Example # 33 : If two pairs of opposite edges of a tetrahedron are mutually perpendicular, show that the third pair will also be mutually perpendicular.

Solution : Let OABC be the tetrahedron, where O is the origin and co-ordinates of A, B, C are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, x_3) respectively.

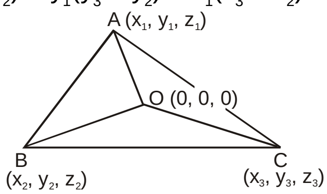
Let $OA \perp BC$ and $OB \perp CA$.

We have to prove that $OC \perp BA$.

Now, direction ratios of OA are x_1, y_1, z_1 and of BC are $(x_3 - x_2), (y_3 - y_2), (z_3 - z_2)$.

$\because$ $OA \perp BC$ and $OB \perp CA$

$\Rightarrow$ $x_1(x_3 - x_2) + y_1(y_3 - y_2) + z_1(z_3 - z_2) = 0$ and $x_2(x_1 - x_3) + y_2(y_1 - y_3) + z_2(z_1 - z_3) = 0$



Adding above two equations we get $x_3(x_1 - x_2) + y_3(y_1 - y_2) + z_3(z_1 - z_2) = 0$

$\therefore$ $OC \perp BA$ ($\because$ direction ratios of OC are x_3, y_3, z_3 and that of BA are $(x_1 - x_2), (y_1 - y_2), (z_1 - z_2)$)

Self practice problems :

(34) Show that $\vec{a}\ .\ (\vec{b} + \vec{c}) \times (\vec{a} + \vec{b} + \vec{c}) = 0$

(35) One vertex of a parallelopiped is at the point A (1, –1, –2) in the rectangular cartesian co- ordinate. If three adjacent vertices are at B(–1, 0, 2), C(2, –2, 3) and D(4, 2, 1), then find the volume of the parallelopiped.

(36) Show that the vector $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if $\vec{b} + \vec{c}$, $\vec{c} + \vec{a}$, $\vec{a} + \vec{b}$ are coplanar.

(37) Show that $\{(\vec{a} + \vec{b} + \vec{c}) \times (\vec{c} - \vec{b})\}\ .\ \vec{a} = 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

(38) Find the value of m such that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} + m\hat{j} + 5\hat{k}$ are coplanar.

(39) Find the value of λ for which the four points with position vectors $-\hat{j} - \hat{k}$, $4\hat{i} + 5\hat{j} + \lambda\hat{k}$, $3\hat{i} + 9\hat{j} + 4\hat{k}$, and $-4\hat{i} + 4\hat{j} + 4\hat{k}$ are coplanar.

Answer : (35) 72 (38) – 4 (39) $\lambda = 1$

Vector triple product :

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors, then the expression $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector & is called a vector triple product. This vector is perpendicular to $\vec{a}$ and lies in plane containing vectors $\vec{b}$ and $\vec{c}$

- $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$
- $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$
- In general $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

Example # 34 : $[\vec{a} \times (3\vec{b} + 2\vec{c}) \;\; \vec{b} \times (\vec{c} - 2\vec{a}) \;\;\; 2\vec{c} \times (\vec{a} - 3\vec{b})] =$

Solution : Let $\vec{b} \times \vec{c} = \vec{p}$, $\vec{c} \times \vec{a} = \vec{q}$, $\vec{a} \times \vec{b} = \vec{r}$

$\therefore \quad [\vec{p}\ \vec{q}\ \vec{r}\,] = [\vec{a}\ \vec{b}\ \vec{c}\,]^2$ ....(i)

$\vec{a} \times (3\vec{b} + 2\vec{c}) = 3\vec{r} - 2q$ etc.

$\therefore \quad E = [3\vec{r} - 2\vec{q}\ \vec{p} + 2\vec{r}\ , 2\vec{q} + 6\vec{p}\,]$

$$= [0\vec{p} - 2\vec{q} + 3\vec{r}\ ,\ \vec{p} + 0\vec{q} + 2\vec{r}\ ,\ 6\vec{p} + 2\vec{q} + 0\vec{r}\,] = \begin{vmatrix} 0 & -2 & 3 \\ 1 & 0 & 2 \\ 6 & 2 & 0 \end{vmatrix} [\vec{a}\ \vec{b}\ \vec{c}\,]^2 = -18\,[\vec{a}\ \vec{b}\ \vec{c}\,]^2$$

Example # 35 : If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar unit vectors such that $(\vec{a} \times \vec{b}) \times \vec{c} = \left(\dfrac{\sqrt{3}\vec{b} + \vec{a}}{2}\right)$. Then find angles which makes $\vec{c}$ with $\vec{a}$ & $\vec{b}$ ($\vec{a}$ and $\vec{b}$ are non-collinear)

Solution : $(\vec{a} \times \vec{b}) \times \vec{c} = \left(\dfrac{\sqrt{3}\vec{b} + \vec{a}}{2}\right) \Rightarrow (\vec{a} \cdot \vec{c}.),\vec{b} - (\vec{b} \cdot \vec{c})\vec{a} = \dfrac{\sqrt{3}\vec{b} + \vec{a}}{2}$

$\vec{a} \cdot \vec{c} = \dfrac{\sqrt{3}}{2}$, and $\vec{b} \cdot \vec{c} = -\dfrac{1}{2}$.

$\cos\theta = \dfrac{\sqrt{3}}{2}$ and $\cos\phi = -\dfrac{1}{2}$.

$\theta = \dfrac{\pi}{6}$ and $\phi = \dfrac{2\pi}{3}$

Example # 36 : Prove that $\vec{a} \times \{\vec{b} \times (\vec{c} \times \vec{d})\} = (\vec{b} \cdot \vec{d})(\vec{a} \times \vec{c}) - (\vec{b} \cdot \vec{c})(\vec{a} \times \vec{d})$

Solution : We have, $\vec{a} \times \{\vec{b} \times (\vec{c} \times \vec{d})\} = \vec{a} \times \{(\vec{b} \cdot \vec{d})\,\vec{c} - (\vec{b} \cdot \vec{c})\,\vec{d}\}$

$= \vec{a} \times \{(\vec{b} \cdot \vec{d})\,\vec{c}\} - \vec{a} \times \{(\vec{b} \cdot \vec{c})\,\vec{d}\}$ [by dist. law]

$= (\vec{b} \cdot \vec{d})\,(\vec{a} \times \vec{c}) - (\vec{b} \cdot \vec{c})\,(\vec{a} \times \vec{d})$.

Self Practice Problems :

(40) Prove that $\vec{a} \times \{\vec{a} \times (\vec{a} \times \vec{b})\} = (\vec{a} \cdot \vec{a})\,(\vec{b} \times \vec{a})$.

(41) Let $\vec{b}$ and $\vec{c}$ be noncollinear vectors. If $\vec{a}$ is a vector such that $\vec{a} \cdot (\vec{b} + \vec{c}) = 4$ and $\vec{a} \times (\vec{b} \times \vec{c}) = (x^2 - 2x + 6)\,\vec{b} + \vec{c}\,\sin y$, then find x and y.

(42) Find a unit vector coplanar with $\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$ and perpendicular to $\hat{i} + \hat{j} + \hat{k}$ is

Answer : (41) $x = 1$ & $y = (4n + 1)\,\pi/2$, $n \in I$ (42) $\pm\left(\dfrac{\hat{j} - \hat{k}}{\sqrt{2}}\right)$

Linear combinations :

Given a finite set of vectors $\vec{a}, \vec{b}, \vec{c}, \ldots\ldots$, then the vector $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \ldots\ldots$ is called a linear combination of $\vec{a}, \vec{b}, \vec{c}, \ldots\ldots$ for any x, y, z..... $\in$ R. We have the following results :

(a) If $\vec{a},\vec{b}$ are non zero, non–collinear vectors, then $x\vec{a}+y\vec{b}=x'\vec{a}+y'\vec{b} \Rightarrow x = x'$, $y = y'$

(b) **Fundamental Theorem in plane :** Let $\vec{a},\vec{b}$ be non zero, non collinear vectors, then any vector $\vec{r}$ coplanar with $\vec{a},\vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}$ and $\vec{b}$
i.e. there exist some unique x, y $\in$ R such that $x\vec{a}+y\vec{b}=\vec{r}$.

(c) If $\vec{a},\vec{b},\vec{c}$ are non–zero, non–coplanar vectors, then
$x\vec{a}+y\vec{b}+z\vec{c}=x'\vec{a}+y'\vec{b}+z'\vec{c} \Rightarrow x = x'$, $y = y'$, $z = z'$

(d) **Fundamental theorem in space:** Let $\vec{a},\vec{b},\vec{c}$ be non–zero, non–coplanar vectors in space. Then any vector $\vec{r}$ can be uniquely expressed as a linear combination of $\vec{a},\vec{b},\vec{c}$ i.e. there exist some unique x,y, z $\in$ R such that $x\vec{a}+y\vec{b}+z\vec{c}=\vec{r}$.

(e) If $\vec{x}_1,\vec{x}_2,......,\ \vec{x}_n$ are n non zero vectors and $k_1,k_2,.....,k_n$ are n scalars and if the linear combination $k_1\vec{x}_1+k_2\vec{x}_2+.......+k_n\vec{x}_n=\vec{0} \ \Rightarrow\ k_1=0,\ k_2=0\ ,\ .....,\ k_n=0$, then we say that vectors $\vec{x}_1,\ \vec{x}_2,\ ......,\vec{x}_n$ are **linearly independent vectors**.

(f) If $k_1\vec{x}_1+k_2\vec{x}_2+k_3\vec{x}_3......+k_r\vec{x}_r+......+k_n\vec{x}_n=\vec{0}$ and if there exists at least one $k_r \neq 0$, then $\vec{x}_1,\ \vec{x}_2,\ ......,\ \vec{x}_n$ are said to be **linearly dependent vectors.**
If $k_r \neq 0$ then $\vec{x}_r$ is expressed as a linear combination of vectors $\vec{x}_1,\ \vec{x}_2,........,\ \vec{x}_{r-1},\ \vec{x}_{r+1},\ ....,\ \vec{x}_n$

Note :

☞ In general, in 3 dimensional space every set of four vectors is a linearly dependent system.

☞ $\hat{i},\hat{j},\hat{k}$ are **Linearly Independent** set of vectors. For $K_1\hat{i}+K_2\hat{j}+K_3\hat{k}=\vec{0} \ \Rightarrow\ K_1=K_2=K_3=0$

☞ Two vectors $\vec{a}$ and $\vec{b}$ are linearly dependent $\Rightarrow$ $\vec{a}$ is parallel to $\vec{b}$ i.e. $\vec{a}\times\vec{b}=\vec{0} \Rightarrow$ linear dependence of $\vec{a}$ and $\vec{b}$. Conversely if $\vec{a}\times\vec{b}\neq\vec{0}$ then $\vec{a}$ and $\vec{b}$ are linearly independent.

☞ If three vectors $\vec{a},\vec{b},\vec{c}$ are linearly dependent, then they are coplanar i.e. $[\vec{a}\ \ \vec{b}\ \ \vec{c}]=0$. Conversely if $[\vec{a}\ \ \vec{b}\ \ \vec{c}]\neq 0$ then the vectors are linearly independent.

Example # 37 : If $\vec{a},\ \vec{b},\ \vec{c}$ are three non-coplanar vectors, solve the vector equation $\vec{r}\cdot\vec{a}=\vec{r}\cdot\vec{b}=\vec{r}\cdot\vec{c}=1$

Solution : since $\vec{a},\ \vec{b},\ \vec{c}$ are three non-coplanar vectors therefore $\vec{a}\times\vec{b},\ \vec{b}\times\vec{c}$ & $\vec{c}\times\vec{a}$ are also non-coplanar vectors
Let $\vec{r}=x(\vec{a}\times\vec{b})+y(\vec{b}\times\vec{c})+z(\vec{c}\times\vec{a})$.
Then, $\vec{r}\cdot\vec{a}=1 \ \Rightarrow\ 1=y[(\vec{b}\times\vec{c})\ \vec{a}.]$
$y=\dfrac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}$, similarly $x = z = \dfrac{1}{[\vec{a}\,\vec{b}\,\vec{c}]} \Rightarrow \vec{r}=\dfrac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}\ ((\vec{a}\times\vec{b})+(\vec{b}\times\vec{c})+(\vec{c}\times\vec{a}))$

Example # 38 : Given that position vectors of points A, B, C are respectively
$\vec{a}-2\vec{b}+3\vec{c},\ 2\vec{a}+3\vec{b}-4\vec{c},\ -7\vec{b}+10\vec{c}$ then prove that vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly dependent.

Solution : Let A, B, C be the given points and O be the point of reference then
$\overrightarrow{OA}=\vec{a}-2\vec{b}+3\vec{c},\ \overrightarrow{OB}=2\vec{a}+3\vec{b}-4\vec{c}$ and $\overrightarrow{OC}=-7\vec{b}+10\vec{c}$
Now $\overrightarrow{AB}$ = p.v. of B – p.v. of A
$=\overrightarrow{OB}-\overrightarrow{OA}=(\vec{a}+5\vec{b}-7\vec{c})$ and $\overrightarrow{AC}$ = p.v. of C – p.v of A
$=\overrightarrow{OC}-\overrightarrow{OA}=-(\vec{a}+5\vec{b}-7\vec{c})=-\overrightarrow{AB}$
$\therefore\ \overrightarrow{AC}=\lambda\ \overrightarrow{AB}$
where $\lambda = -1$. Hence $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly dependent.

Example # 39 : Prove that the vectors $5\vec{a}+6\vec{b}+7\vec{c},\ 7\vec{a}-8\vec{b}+9\vec{c}$ and $3\vec{a}+20\vec{b}+5\vec{c}$ are linearly dependent, where $\vec{a},\vec{b},\vec{c}$ being linearly independent vectors.

Solution : We know that if these vectors are linearly dependent , then we can express one of them as a linear combination of the other two.

Now let us assume that the given vector are coplanar, then we can write

$5\,\vec{a} + 6\vec{b} + 7\vec{c} = \ell(7\vec{a} - 8\vec{b} + 9\vec{c}) + m\,(3\vec{a} + 20\vec{b} + 5\vec{c})$ where ℓ, m are scalars

Comparing the coefficients of $\vec{a}$, $\vec{b}$ and $\vec{c}$ on both sides of the equation

$5 = 7\ell + 3m$, $6 = -8\ell + 20\,m$, $7 = 9\ell + 5m$

$\Rightarrow \ \ell = \dfrac{1}{2} = m$. Hence the given vectors are linearly dependent.

Self Practice Problems :

(43) Given that $\vec{x} + \dfrac{1}{\vec{p}^2}(\vec{p}\ .\ \vec{x})\,\vec{p} = \vec{q}$, show that $\vec{p}\ .\ \vec{x} = \dfrac{1}{2}\vec{p}\ .\ \vec{q}$ and find $\vec{x}$ in terms of $\vec{p}$ and $\vec{q}$.

(44) If $\vec{x}\ .\ \vec{a} = 0$, $\vec{x}\ .\ \vec{b} = 0$ and $\vec{x}\ .\ \vec{c} = 0$ for some non-zero vector $\vec{x}$, then show that $[\vec{a}\ \ \vec{b}\ \ \vec{c}] = 0$

(45) Prove that $\vec{r} = \dfrac{(\vec{r}\ .\ \vec{a})\ (\vec{b}\times\vec{c})}{[\ \vec{a}\ \ \vec{b}\ \ \vec{c}\ ]} + \dfrac{(\vec{r}\ .\ \vec{b})\ (\vec{c}\times\vec{a})}{[\ \vec{a}\ \ \vec{b}\ \ \vec{c}\ ]} + \dfrac{(\vec{r}\ .\ \vec{c})\ (\vec{a}\times\vec{b})}{[\ \vec{a}\ \ \vec{b}\ \ \vec{c}\ ]}$

where $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors

(46) Does there exist scalars u, v, w such that $u\vec{e}_1 + v\vec{e}_2 + w\vec{e}_3 = \hat{i}$ where $\vec{e}_1 = \hat{k}$, $\vec{e}_2 = \hat{j} + \hat{k}$, $\vec{e}_3 = -\hat{j} + 2\hat{k}$?

(47) If $\vec{a}$ and $\vec{b}$ are non-collinear vectors and $\vec{A} = (x + 4y)\ \vec{a} + (2x + y + 1)\ \vec{b}$ and $\vec{B} = (y - 2x + 2)\,\vec{a} + (2x - 3y - 1)\ \vec{b}$, find x and y such that $3\vec{A} = 2\vec{B}$.

(48) If vectors $\vec{a}, \vec{b}, \vec{c}$ be linearly independent, then show that

(i) $\vec{a} - 2\vec{b} + 3\vec{c}$, $-2\vec{a} + 3\vec{b} - 4\vec{c}$, $-\vec{b} + 2\vec{c}$ are linearly dependent

(ii) $\vec{a} - 3\vec{b} + 2\vec{c}$, $-2\vec{a} - 4\vec{b} - \vec{c}$, $3a + 2\vec{b} - \vec{c}$ are linearly independent.

(49) Prove that a vector $\vec{r}$ in space can be expressed linearly in terms of three non-coplanar, non-zero vectors $\vec{a}, \vec{b}, \vec{c}$ in the form

$$\vec{r} = \frac{[\vec{r}\ \ \vec{b}\ \ \vec{c}]\ \vec{a} + [\ \vec{r}\ \ \vec{c}\ \ \vec{a}\ ]\ \vec{b} + [\ \vec{r}\ \ \vec{a}\ \ \vec{b}\ ]\ \vec{c}}{[\ \vec{a}\ \ \vec{b}\ \ \vec{c}]}$$

Answers : (43) $\vec{x} = \vec{q} - \left(\dfrac{\vec{p}\ .\ \vec{q}}{2\,|\vec{p}|^2}\right)\vec{p}$ (46) No (47) x = 2, y = –1

Test of collinearity :

Three points A,B,C with position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively are collinear, if & only if there exist scalars x, y, z not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} = 0 = \vec{0}$, where x + y + z = 0.

Test of coplanarity :

Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{0}$, where x + y + z + w = 0.

Example # 40 : Prove that four points $2\vec{a} + 3\vec{b} - \vec{c}$, $\vec{a} - 2\vec{b} + 3\vec{c}$, $3\vec{a} + 4\vec{b} - 2\vec{c}$ and $\vec{a} - 6\vec{b} + 6\vec{c}$ are coplanar.

Solution : Let the given four points be P, Q, R and S respectively. These points are coplanar if the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$ are coplanar. These vectors are coplanar iff one of them can be expressed as a linear combination of other two. So let $\overrightarrow{PQ} = x\overrightarrow{PR} + y\ \overrightarrow{PS}$

$\Rightarrow \quad -\vec{a} - 5\vec{b} + 4\vec{c} = x\left(\vec{a} + \vec{b} - \vec{c}\right) + y\left(-\vec{a} - 9\vec{b} + 7\vec{c}\right)$

$\Rightarrow \quad -\vec{a} - 5\vec{b} + 4\vec{c} = (x - y)\ \vec{a} + (x - 9y)\ \vec{b} + (-x + 7y)\ \vec{c}$

$\Rightarrow \quad x - y = -1,\ x - 9y = -5,\ -x + 7y = 4$ [Equating coeff. of $\vec{a}, \vec{b}, \vec{c}$ on both sides]

Solving the first two equations of these three equations, we get $x = -\frac{1}{2}$, $y = \frac{1}{2}$.

These values also satisfy the third equation. Hence the given four points are coplanar.

Self Practice Problems :

(50) If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are any four vectors in 3-dimensional space with the same initial point and such that $3\vec{a} - 2\vec{b} + \vec{c} - 2\vec{d} = \vec{0}$, show that the terminal A, B, C, D of these vectors are coplanar. Find the point (P) at which AC and BD meet. Also find the ratio in which P divides AC and BD.

Answers : (50) $\vec{p} = \dfrac{3\vec{a} + \vec{c}}{4}$ divides AC in 1 : 3 and BD in 1 : 1 ratio

A PLANE

If line joining any two points on a surface lies completely on it then the surface is a plane.

OR

If line joining any two points on a surface is perpendicular to some fixed straight line. Then this surface is called a plane. This fixed line is called the normal to the plane.

Equation of a plane :

(i) **Vector form :** The equation $(\vec{r} - \vec{r}_0)\ .\ \vec{n} = 0$ represents a plane containing the point with position vector is a vector normal to the plane.
The above equation can also be written as $\vec{r}\ .\ \vec{n} = d$, where d = $\vec{r}_0\ .\ \vec{n}$

(ii) **Cartesian form :** The equation of a plane passing through the point (x_1, y_1, z_1) is given by $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ where a, b, c are the direction ratios of the normal to the plane.

(iii) **Normal form :** Vector equation of a plane normal to unit vector and at a distance d from the origin is $\vec{r}\ .\vec{n} = d$. Normal form of the equation of a plane is $\ell x + my + nz = p$, where, ℓ,m, n are the direction cosines of the normal to the plane and p is the distance of the plane from the origin.

(iv) **General form :** ax + by + cz + d = 0 is the equation of a plane, where a, b, c are the direction ratios of the normal to the plane.

(v) **Plane through three points :** The equation of the plane through three non–collinear points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is $\begin{vmatrix} x - x_3 & y - y_3 & z - z_3 \\ x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \\ x_2 - x_3 & y_2 - y_3 & z_2 - z_3 \end{vmatrix} = 0$

(vi) **Intercept Form :** The equation of a plane cutting intercept a, b, c on the axes is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Note :

☞ Equation of yz–plane, xz–plane and xy–plane is x = 0, y = 0 and z = 0

☞ **Transformation of the equation of a plane to the normal form:** To reduce any equation ax + by + cz – d = 0 to the normal form, first write the constant term on the right hand side and make it positive, then divide each term by $\sqrt{a^2 + b^2 + c^2}$, where a, b, c are coefficients of x, y and z respectively e.g.

$$\frac{ax}{\pm\sqrt{a^2 + b^2 + c^2}} + \frac{by}{\pm\sqrt{a^2 + b^2 + c^2}} + \frac{cz}{\pm\sqrt{a^2 + b^2 + c^2}} = \frac{d}{\pm\sqrt{a^2 + b^2 + c^2}}$$

Where (+) sign is to be taken if d > 0 and (–) sign is to be taken if d < 0.

☞ A plane ax + by + cz + d = 0 divides the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2). in the ratio $\left(-\dfrac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}\right)$

☞ Coplanarity of four points

The points $A(x_1\, y_1\, z_1)$, $B(x_2\, y_2\, z_2)$ $C(x_3\, y_3\, z_3)$ and $D(x_4\, y_4\, z_4)$ are coplanar then

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix} = 0$$

Example # 41 : Find the equation of the plane upon which the length of normal from origin is 10 and direction ratios of this normal are 3, 2, 6.

Solution : If p be the length of perpendicular from origin to the plane and ℓ, m, n be the direction cosines of this normal, then its equation is

$\ell x + my + nz = 10$ ..... (1)

Direction ratios of normal to the plane are 3, 2, 6

$\therefore$ Direction cosines of normal to the required plane are $\ell = \dfrac{3}{7}$, $m = \dfrac{2}{7}$, $n = \dfrac{6}{7}$

Equation of required plane is $\dfrac{3}{7}x + \dfrac{2}{7}y + \dfrac{6}{7}z = 10$ or, $3x + 2y + 6z = 70$

Example # 42 : Find the plane through the points (2,–3,3), (–5, 2, 0), (1, – 7, 1)

Solution : $\begin{vmatrix} x-2 & y+3 & z-3 \\ -5-2 & 2+3 & 0-3 \\ 1-2 & -7+3 & 1-3 \end{vmatrix} = 0$ or $\begin{vmatrix} x-2 & y+3 & z-3 \\ -7 & 5 & -3 \\ -1 & -4 & -2 \end{vmatrix} = 0 \Rightarrow 2x + y - 3z + 8 = 0$

Example # 43 : If P be any point on the plane $\ell x + my + nz = p$ and Q be a point on the line OP such that $OP \,.\, OQ = p^2$, show that the locus of the point Q is $p(\ell x + my + nz) = x^2 + y^2 + z^2$.

Solution : Let $P \equiv (\alpha, \beta, \gamma)$, $Q \equiv (x_1, y_1, z_1)$

Direction ratios of OP are α, β, γ and direction ratios of OQ are x_1, y_1, z_1.

Since O, Q, P are collinear, we have $\dfrac{\alpha}{x_1} = \dfrac{\beta}{y_1} = \dfrac{\gamma}{z_1}$ = k (say) ..... (1)

As P (α, β, γ) lies on the plane $\ell x + my + nz = p$,

$\ell\alpha + m\beta + n\gamma = p$ or $k(\ell x_1 + my_1 + nz_1) = p$ ..... (2)

Given $OP \,.\, OQ = p^2$ $\Rightarrow$ $\sqrt{\alpha^2+\beta^2+\gamma^2}\ \sqrt{x_1^2+y_1^2+z_1^2} = p^2$

or, $\sqrt{k^2(x_1^2+y_1^2+z_1^2)}\ \sqrt{x_1^2+y_1^2+z_1^2} = p^2$ or, $k\,(x_1^2+y_1^2+z_1^2) = p^2$ ....(3)

On dividing (2) by (3), we get $\dfrac{\ell x_1 + my_1 + nz_1}{x_1^2+y_1^2+z_1^2} = \dfrac{1}{p}$ or, $p(\ell x_1 + my_1 + nz_1) = x_1^2+y_1^2+z_1^2$

Hence the locus of point Q is $p(\ell x + my + nz) = x^2 + y^2 + z^2$.

Example # 44 : A moving plane passes through a fixed point (α,β,γ) and cuts the coordinate axes A, B, C . Find the locus of the centroid of the tetrahedron OABC.

Solution : Let the plane be $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c} = 1$, 0 (0,0,0) , A (a, 0,0), B (0, b, 0)

C (0,0,c) . Centroid of OABC is $\left(\dfrac{a}{4},\dfrac{b}{4},\dfrac{c}{4}\right)$

The plane passes through (α,β,γ) $\therefore$ $\dfrac{\alpha}{a}+\dfrac{\beta}{b}+\dfrac{\gamma}{c} = 1$ ......(i)

Centroid , $x = \dfrac{a}{4}$, $y = \dfrac{b}{4}$, $z = \dfrac{c}{4}$ or $a = 4x$, $b = 4y$, $c = 4z$

Now (1) gives the locus of G as $\frac{\alpha}{x}+\frac{\beta}{y}+\frac{\gamma}{z} = 4$

Self practice problems :

(51) Check whether given points are coplanar if yes find the equation of plane containing them
$A \equiv (0, -1, -1)$, $B \equiv (4, 5, 1)$, $C \equiv (3, 9, 4)$, $D \equiv (-4, 4, 4)$

(52) Find the plane passing through point (3, 2, 1) and perpendicular to the line joining the points (2, 4, 3) and (3, –1, 5).

(53) Find the equation of plane passing through the point (2, 4, 6) and making equal intercepts on the coordinate axes.

(54) Find the equation of plane passing through (1, 2, –3) and (2, 3, 3) and perpendicular to the plane $2x + y - 3z + 4 = 0$.

(55) Find the equation of the plane parallel to $2\hat{i}+\hat{j}-\hat{k}$ and $\hat{i}-2\hat{j}-3\hat{k}$ and passing through (2, 1, 3).

(56) Find the equation of the plane passing through the point (1, 1, – 1) and perpendicular to the planes $x + 2y + 3z - 7 = 0$ and $2x - 3y + 4z = 0$.

Answers :

(51)	yes, $5x - 7y + 11z + 4 = 0$	(52)	$x - 5y + 2z + 5 = 0$
(53)	$x + y + z = 12$	(54)	$9x - 15y + z + 24 = 0$
(55)	$x - y + z = 4$	(56)	$17x + 2y - 7z = 26$

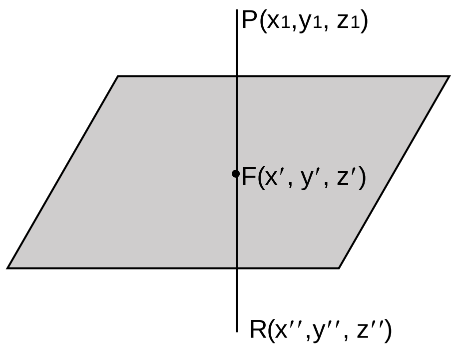
Position of point with respect to plane :

A plane divides the three dimensional space in two equal parts. Two points A $(x_1\ y_1\ z_1)$ and B $(x_2\ y_2\ z_2)$ are on the same side of the plane $ax + by + cz + d = 0$ if $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are both positive or both negative and are opposite side of plane if both of these values are in opposite sign.

Example # 45 : Show that the points (1, 2, 3) and (2, – 1, 4) lie on opposite sides of the plane $x + 4y + z - 3 = 0$.

Solution : Since the numbers $1 + 4 \times 2 + 3 - 3 = 9$ and $2 - 4 + 4 - 3 = -1$ are of opposite sign, then points are on opposite sides of the plane.

A plane & a point



Let $P \equiv ax + by + cz + d = 0$ is a given plane and $P(x_1, y_1, z_1)$ is given point as shown in figure.
Let $F(x', y', z')$ be the foot of the point P (x_1, y_1, z_1) with respect to the plane P.
And $R(x'', y'', z'')$ be the reflection of point P (x_1, y_1, z_1) with respect to the plane P.

(i) Distance of the point (x_1, y_1, z_1) from the plane $ax + by + cz + d = 0$ is given by $\left|\dfrac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$.

(ii) The length of the perpendicular from a point having position vector $\vec{a}$ to plane $\vec{r}.\vec{n} = d$ is $p = \dfrac{|\vec{a}\ .\ \vec{n} - d|}{|\vec{n}|}$.

(iii) The coordinates of the foot (F) of perpendicular from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ are $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c} = -\dfrac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$

(iv) The coordinates of the Image (R) of point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ are $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c} = -\dfrac{2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$

Example # 46 : Find the image of the point P (3, 5, 7) in the plane $2x + y + z = 0$.

Solution : Given plane is $2x + y + z = 0$ ..... (1)

Direction ratios of normal to plane (1) are 2, 1, 1

Let Q be the image of point P in plane (1). Let PQ meet plane (1) in R then PQ $\perp$ plane (1)

Let $R \equiv (2r + 3, r + 5, r + 7)$

Since R lies on plane (1)

$\therefore$ $2(2r + 3) + r + 5 + r + 7 = 0$ or, $6r + 18 = 0$ $\therefore$ $r = -3$

$\therefore$ $R \equiv (-3, 2, 4)$

Let $Q \equiv (\alpha, \beta, \gamma)$

Since R is the middle point of PQ

$\therefore$ $-3 = \dfrac{\alpha + 3}{2}$ $\Rightarrow$ $\alpha = -9$ and $2 = \dfrac{\beta + 5}{2}$ $\Rightarrow$ $\beta = -1$ and $4 = \dfrac{\gamma + 7}{2}$ $\Rightarrow$ $\gamma = 1$

$\therefore$ $Q = (-9, -1, 1)$.

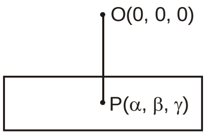
Example # 47 : A plane passes through a fixed point (a, b, c). Show that the locus of the foot of perpendicular to it from the origin is the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$

Solution : Let the equation of the variable plane be $\ell x + my + nz + d = 0$ ..... (1)

Plane passes through the fixed point (a, b, c) $\therefore$ $\ell a + mb + nc + d = 0$ ..... (2)

Let P (α, β, γ) be the foot of perpendicular from origin to plane (1).

Direction ratios of OP are



$\alpha - 0, \beta - 0, \gamma - 0$ i.e. α, β, γ

From equation (1), it is clear that the direction ratios of normal to the plane i.e. OP are ℓ, m, n;

α, β, γ and ℓ, m, n are the direction ratios of the same line OP

$\therefore$ $\dfrac{\alpha}{\ell} = \dfrac{\beta}{m} = \dfrac{\gamma}{n} = \dfrac{1}{k}$ (say) $\therefore$ $\ell = k\alpha, m = k\beta, n = k\gamma$ ..... (3)

Putting the values of ℓ, m, n in equation (2), we get $ka\alpha + kb\beta + kc\gamma + d = 0$ ..... (4)

Since α, β, γ lies in plane (1) $\therefore$ $\ell\alpha + m\beta + n\gamma + d = 0$ ..... (5)

Putting the values of ℓ, m, n from (3) in (5), we get $k\alpha^2 + k\beta^2 + k\gamma^2 + d = 0$ ..... (6)

or $k\alpha^2 + k\beta^2 + k\gamma^2 - ka\alpha - kb\beta - kc\gamma = 0$ [putting the value of d from (4) in (6)]

or $\alpha^2 + \beta^2 + \gamma^2 - a\alpha - b\beta - c\gamma = 0$

Therefore, locus of foot of perpendicular P (α, β, γ) is $x^2 + y^2 + z^2 - ax - by - cz = 0$ ..... (7)

Self practice problems :

(57) Find the intercepts of the plane $3x + 4y - 7z = 84$ on the axes. Also find the length of perpendicular from origin to this plane and direction cosines of this normal.

(58) Find : (i) perpendicular distance (ii) foot of perpendicular (iii) image of (1, 1, 1) in the plane $3x + 4y - 12z + 13 = 0$

Answers : (57) $a = 28, b = 21, c = -12, p = \frac{84}{\sqrt{74}}; \frac{3}{\sqrt{74}}, \frac{4}{\sqrt{74}}, \frac{-7}{\sqrt{74}}$

(58) (i) $\frac{\sqrt{17}}{2}$ (ii) (–1, 1/2, 1) (iii) (–3, 0,1)

Angle between two planes :

(i) Consider two planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$. Angle between these planes is the angle between their normals. Since direction ratios of their normals are (a, b, c) and (a′, b′, c′) respectively, hence θ, the angle between them, is given by

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{a^2+b^2+c^2}\ \sqrt{a'^2+b'^2+c'^2}}$$

Planes are perpendicular if $aa' + bb' + cc' = 0$ and planes are parallel if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

(ii) The angle θ between the planes $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$ is given by, $\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|\ |\vec{n}_2|}$

Planes are perpendicular if $\vec{n}_1\cdot\vec{n}_2 = 0$ & planes are parallel if $\vec{n}_1 = \lambda\vec{n}_2$.

Distance between parallel planes :

Distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $\frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$

Example # 48 : Find the distance between the parallel planes $2x - y + 2z + 3 = 0$ and $4x - 2y + 4z + 5 = 0$

Solution : Given planes are $2x - y + 2z + 3 = 0$ and $2x - y + 2z + 5/2 = 0$

Required distance between planes = $\frac{|3 - 5/2|}{\sqrt{(2)^2+(-1)^2+(2)^2}} = \frac{1}{6}$

Angle bisectors

(i) The equations of the planes bisecting the angle between two given planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \quad \ldots\ldots(1)$$

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = -\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \quad \ldots\ldots..(2)$$

(ii) If $a_1\alpha + b_1\beta + c_1\gamma + d_1$ and $a_2\alpha + b_2\beta + c_2\gamma + d_2$ are of same/opposite sign then (1)/(2) gives equation of angle bisector of region containing point (α,β,γ)

(iii) If $a_1a_2 + b_1b_2 + c_1c_2 > 0$, then equation (1)/(2) gives obtuse/acute angle bisector and if $a_1a_2 + b_1b_2 + c_1c_2 < 0$, then equation (1)/(2) gives acute/obtuse angle bisector.

Family of planes

(i) Any plane passing through the line of intersection of non–parallel planes or equation of the plane through the given line in non symmetric form.

$a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$ is

$a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0$, where $\lambda \in R$

(ii) The equation of plane passing through the intersection of the planes $\vec{r}\cdot\vec{n}_1 = d_1$ & $\vec{r}\cdot\vec{n}_2 = d_2$ is. $\vec{r}\ (n_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$ where λ is arbitrary scalar

Example # 49 : Find the equation of the plane through the line of intersection of the planes $x + 2y + 3z + 2 = 0$, $2x + 3y - z + 3 = 0$ and perpendicular to the plane $x + y + z = 0$

Solution : The plane is $x + 2y + 3z + 2 + \lambda (2x + 3y - z + 3) = 0$

or $(1 + 2\lambda) x + (2 + 3\lambda) y + (3 - \lambda) z + 2 + 3\lambda = 0$

It is perpendicular to $x + y + z = 0$

$\therefore\ 1 + 2\lambda + 2 + 3\lambda + 3 - \lambda = 0$ or $2\lambda + 3 = 0 \Rightarrow \lambda = -\dfrac{3}{2}$

Substituting we get $4x + 5y - 9z + 5 = 0$

Example # 50 : Find the equation of the plane through the point (1, 1, 1) which passes through the line of intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$.

Solution : Given planes are $x + y + z - 6 = 0$ ..... (1)

and $2x + 3y + 4z + 5 = 0$ ..... (2)

Given point is P (1, 1, 1).

Equation of any plane through the line of intersection of planes (1) and (2) is

$x + y + z - 6 + k (2x + 3y + 4z + 5) = 0$ ..... (3)

If plane (3) passes through point P, then

$1 + 1 + 1 - 6 + k (2 + 3 + 4 + 5) = 0$ or, $k = \dfrac{3}{14}$

From (3) required plane is $20x + 23y + 26z - 69 = 0$

Example # 51 Let planes are $2x + y + 2z = 9$ and $3x - 4y + 12z + 13 = 0$. Which of these bisector planes bisects the acute angle between the given planes. Does origin lie in the acute angle or obtuse angle between the given planes ?

Solution : Given planes are $-2x - y - 2z + 9 = 0$ ..... (1)

and $3x - 4y + 12z + 13 = 0$ ..... (2)

Equations of bisecting planes are $\dfrac{-2x - y - 2z + 9}{\sqrt{(-2)^2 + (-1)^2 + (-2)^2}} = \pm \dfrac{3x - 4y + 12z + 13}{\sqrt{3^2 + (-4)^2 + (12)^2}}$

or, $13 [-2x - y - 2z + 9] = \pm 3 (3x - 4y + 12z + 13)$

or, $35x + y + 62z = 78$, ... (3) [Taking +ve sign]

and $17x + 25y - 10z = 156$ ..... (4) [Taking – ve sign]

Now $a_1a_2 + b_1b_2 + c_1c_2 = (-2)(3) + (-1)(-4) + (-2)(12)$

$= -6 + 4 - 24 = -26 < 0$

$\therefore$ Bisector of acute angle is given by $35x + y + 62z = 78$

$\because$ $a_1a_2 + b_1b_2 + c_1c_2 < 0$, origin lies in the acute angle between the planes.

Example # 52 : If the planes $x - cy - bz = 0$, $cx - y + az = 0$ and $bx + ay - z = 0$ pass through a straight line, then find the value of $a^2 + b^2 + c^2 + 2abc$.

Solution : Given planes are $x - cy - bz = 0$ ..... (1)

$cx - y + az = 0$ ..... (2)

$bx + ay - z = 0$ ..... (3)

Equation of any plane passing through the line of intersection of planes (1) and (2) may be taken as $x - cy - bz + \lambda (cx - y + az) = 0$

or, $x (1 + \lambda c) - y (c + \lambda) + z (-b + a\lambda) = 0$ ..... (4)

If planes (3) and (4) are the same, then equations (3) and (4) will be identical.

$$\therefore \quad \underset{\text{(i)}}{\frac{1 + c\lambda}{b}} = \underset{\text{(ii)}}{\frac{-(c + \lambda)}{a}} = \underset{\text{(iii)}}{\frac{-b + a\lambda}{-1}}$$

From (i) and (ii), $a + ac\lambda = -bc - b\lambda$

or, $\lambda = -\dfrac{(a + bc)}{(ac + b)}$ ..... (5)

From (ii) and (iii),

$c + \lambda = -ab + a^2\lambda$ or $\lambda = \dfrac{-(ab + c)}{1 - a^2}$ ..... (6)

From (5) and (6), we have $\dfrac{-(a + bc)}{ac + b} = \dfrac{-(ab + c)}{(1 - a^2)}$.

or, $a - a^3 + bc - a^2bc = a^2bc + ac^2 + ab^2 + bc$

or, $a^2bc + ac^2 + ab^2 + a^3 + a^2bc - a = 0$ or, $a^2 + b^2 + c^2 + 2abc = 1$.

Self practice problems:

(59) Find the equation of plane passing through the line of intersection of the planes $2x - 7y + 4z = 3$ and $3x - 5y + 4z = 11$ and the point $(-2, 1, 3)$.

(60) Find the equations of the planes bisecting the angles between the planes $x + 2y + 2z - 3 = 0$, $3x + 4y + 12z + 1 = 0$ and sepecify the plane which bisects the acute angle between them.

(61) Show that the origin lies in the acute angle between the planes $x + 2y + 2z - 9 = 0$ and $4x - 3y + 12z + 13 = 0$

(62) Prove that the planes $12x - 15y + 16z - 28 = 0$, $6x + 6y - 7z - 8 = 0$ and $2x + 35y - 39z + 12 = 0$ have a common line of intersection.

Answers : (59) $15x - 47y + 28z = 7$

(60) $2x + 7y - 5z = 21$, $11x + 19y + 31z = 18$; $2x + 7y - 5z = 21$

Angle between a plane and a line:

(i) If θ is the angle between line $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ and the plane $ax + by + cz + d = 0$, then

$$\sin\theta = \left[\frac{a\ell + bm + cn}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{\ell^2 + m^2 + n^2}}\right].$$

(ii) Vector form: If θ is the angle between a line $\vec{r} = (\vec{a} + \lambda\vec{b})$ and $\vec{r}\cdot\vec{n} = d$ then $\sin\theta = \left[\frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\,|\vec{n}|}\right]$.

(iii) Condition for perpendicularity $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$ $\quad \vec{b}\times\vec{n} = 0$

(iv) Condition for parallel $a\ell + bm + cn = 0$ $\quad \vec{b}\cdot\vec{n} = 0$

Condition for a line to lie in a plane

(i) **Cartesian form:** Line $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ would lie in a plane $ax + by + cz + d = 0$, if $ax_1 + by_1 + cz_1 + d = 0$ & $a\ell + bm + cn = 0$.

(ii) **Vector form:** Line $\vec{r} = \vec{a} + \lambda\vec{b}$ would lie in the plane $\vec{r}\cdot\vec{n} = d$ if $\vec{b}\cdot\vec{n} = 0$ & $\vec{a}\cdot\vec{n} = d$

Example # 53 : Find the distance of the point $(1, 0, -3)$ from the plane $x - y - z = 9$ measured parallel to the line $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$.

Solution : Given plane is $x - y - z = 9$ ..... (1)

Given line AB is $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$ ..... (2)

Equation of a line passing through the point $Q(1, 0, -3)$ and parallel to line (2) is

$\frac{x-1}{2} = \frac{y}{3} = \frac{z+3}{-6} = r$. ..... (3)

Co-ordinates of any point on line (3) may be taken as

$P(2r + 1, 3r, -6r - 3)$

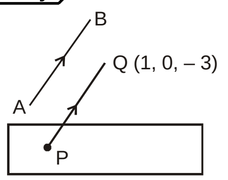
If P is the point of intersection of line (3) and plane (1), then P lies on plane (1),

$\therefore$ $(2r + 1) - (3r) - (-6r - 3) = 9$

$r = 1$

or, $P \equiv (3, 3, -9)$

Distance between points $Q(1, 0, -3)$ and $P(3, 3, -9)$



$PQ = \sqrt{(3-1)^2 + (3-0)^2 + (-9-(-\ \ 3))^2} \ = \sqrt{4+9+36} = 7.$

Example # 54 : Find the equation of the plane passing through (1, 2, 0) which contains the line $\frac{x+3}{3} = \frac{y-1}{4} = \frac{z-2}{-2}$.

Solution : Equation of any plane passing through (1, 2, 0) may be taken as

$a (x - 1) + b (y - 2) + c (z - 0) = 0$ ..... (1)

where a, b, c are the direction ratios of the normal to the plane. Given line is

$\frac{x+3}{3} = \frac{y-1}{4} = \frac{z-2}{-2}$ ..... (2)

If plane (1) contains the given line, then

$3a + 4b - 2c = 0$ ..... (3)

Also point (– 3, 1, 2) on line (2) lies in plane (1)

$\therefore$ $a (-3 - 1) + b (1 - 2) + c (2 - 0) = 0$

or, $-4a - b + 2c = 0$ ..... (4)

Solving equations (3) and (4), we get $\frac{a}{8-2} = \frac{b}{8-6} = \frac{c}{-3+16}$

or, $\frac{a}{6} = \frac{b}{2} = \frac{c}{13}$ = k (say). ..... (5)

Substituting the values of a, b and c in equation (1), we get

$6 (x - 1) + 2 (y - 2) + 13 (z - 0) = 0.$

or, $6x + 2y + 13z - 10 = 0$. This is the required equation.

Example # 55 : Find the equation of the projection of the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ on the plane x + 2y + z = 9.

Solution : Let the given line AB be $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ ..... (1)

Given plane is $x + 2y + z = 9$ ..... (2)

Let DC be the projection of AB on plane (2)

Clearly plane ABCD is perpendicular to plane (2).

Equation of any plane through AB may be taken as (this plane passes through the point (1, – 1, 3) on line AB)

$a (x - 1) + b (y + 1) + c (z - 3) = 0$ ..... (3)

where $2a - b + 4c = 0$ ..... (4)

[$\because$ normal to plane (3) is perpendicular to line (1)]

Since plane (3) is perpendicular to plane (2),

$\therefore$ $a + 2b + c = 0$ ..... (5)

Solving equations (4) & (5), we get $\frac{a}{-9} = \frac{b}{2} = \frac{c}{5}$.

Substituting these values of a, b and c in equation (3), we get

$9 (x - 1) - 2 (y + 1) - 5 (z - 3) = 0$

or, $9x - 2y - 5z + 4 = 0$ ...... (6)

Since projection DC of AB on plane (2) is the line of intersection of plane ABCD and plane (2), therefore equation of DC will be

and $\left.\begin{matrix} 9x - 2y - 5z + 4 = 0 & .....(i) \\ x + 2y + z - 9 = 0 & .....(ii) \end{matrix}\right\}$ ..... (7)

Let ℓ, m, n be the direction ratios of the line of intersection of planes (i) and (ii)

$\therefore$ $9\ell - 2m - 5n = 0$ ..... (8) and $\ell + 2m + n = 0$ ..... (9)

$\therefore$ $\frac{\ell}{-2+10} = \frac{m}{-5-9} = \frac{n}{18+2}$ $\Rightarrow$ $\frac{\ell}{4} = \frac{m}{-7} = \frac{n}{10}$

Let any point on line (7) is $(\alpha, \beta, 0)$ $\Rightarrow$ $9\alpha - 2\beta + 4 = 0$

$\alpha + 2\beta - 9 = 0 \quad \Rightarrow \quad \alpha = \frac{1}{2}, \ \beta = \frac{17}{4}$

So equation of line is $\frac{x-\frac{1}{2}}{4} = \frac{y-\frac{17}{4}}{-7} = \frac{z-0}{10}$

Self practice problems :

(63) Find the values of a and b for which the line $\frac{x-2}{a} = \frac{y+3}{4} = \frac{z-6}{-2}$ is perpendicular to the plane $3x - 2y + bz + 10 = 0$.

(64) Find the equation of the plane containing the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{3}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$.

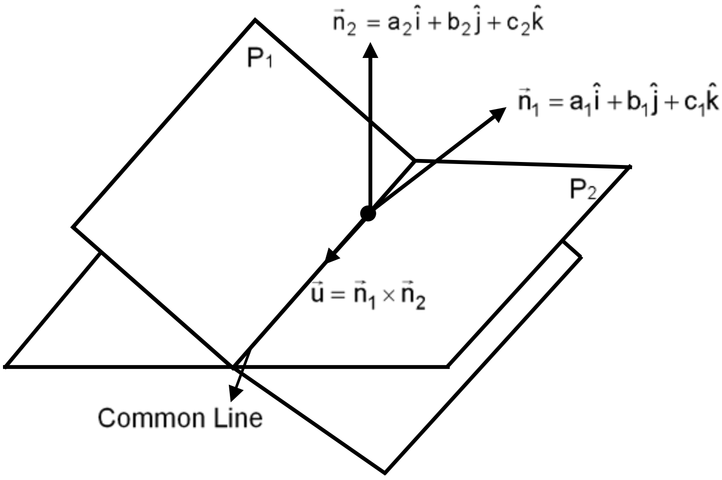
(65) Find the plane containing the line $\frac{x-2}{2} = \frac{y-3}{3} = \frac{z-4}{5}$ and parallel to the line $\frac{x+1}{1} = \frac{y-1}{-2} = \frac{-z+1}{1}$

(66) Show that the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ & $\frac{x-4}{5} = \frac{y-1}{2} = z$ are intersecting each other. Find their intersection point and the plane containing the line.

Answers : (63) $a = -6, b = 1$ (64) $3x - y - z + 2 = 0$
(65) $13x + 3y - 7z - 7 = 0$ (66) $(-1, -1, -1)$ & $5x - 18y + 11z - 2 = 0$

Non-symmetrical form of line :

A straight line in space is characterised by the intersection of two planes which are not parallel and therefore, the equation of a straight line is a solution of the system constituted by the equations of the two planes, $P_1 \equiv a_1x + b_1y + c_1z + d_1 = 0$ and $P_2 \equiv a_2x + b_2y + c_2z + d_2 = 0$. This form is also known as non-symmetrical form.



To find the equation of the line in symmetrical form, we must know (i) its direction ratios (ii) coordinate of any point on it.

(i) Direction ratios: Let ℓ, m, n be the direction ratios of the line. Since the line lies in both the planes, it must be perpendicular to normals of both planes.
So $a_1\ell + b_1m + c_1n = 0$, $a_2\ell + b_2m + c_2n = 0$. From these equations, proportional values of ℓ, m, n can be found by cross-multiplication as $\frac{\ell}{b_1c_2 - b_2c_1} = \frac{m}{c_1a_2 - c_2a_1} = \frac{n}{a_1b_2 - a_2b_1}$

(ii) Point on the line – Note that as ℓ, m, n cannot be zero simultaneously, so at least one must be non-zero. Let $a_1b_2 - a_2b_1 \neq 0$, then the line cannot be parallel to xy plane, so it intersect it.

Let it intersect xy-plane in $(x_1, y_1, 0)$. Then $a_1x_1 + b_1y_1 + d_1 = 0$ and $a_2x_1 + b_2y_1 + d_2 = 0$. Solving these, we get a point on the line.

Note : If $\ell \neq 0$, then we can take a point on yz-plane as $(0, y_1, z_1)$ and if $m \neq 0$, then we can take a point on xz-plane as $(x_1, 0, z_1)$.

Coplanar lines : Condition of coplanarity if both the lines are in general form Let the lines be

$$ax + by + cz + d = 0 = a'x + b'y + c'z + d' \ \ \& \ \ \alpha x + \beta y + \gamma z + \delta = 0 = \alpha' x + \beta' y + \gamma' z + \delta'$$

They are coplanar if $\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \end{vmatrix} = 0$

Example # 56 : Find the equation of the line of intersection of planes $4x + 4y - 5z = 12$, $8x + 12y - 13z = 32$ in the symmetric form.

Solution : Given planes are $4x + 4y - 5z - 12 = 0$ ..... (1)

and $2x - 3y + 4z = 5$ ..... (2)

Let ℓ, m, n be the direction ratios of the line of intersection :

then $4\ell - 4m - 3n = 0$ ..... (3)

and $42\ell - 12m + 13n = 0 \ \therefore \ \frac{\ell}{-8+9} = \frac{m}{6-4} = \frac{n}{-3+4}$ or, $\frac{\ell}{1} = \frac{m}{2} = \frac{n}{1}$

Hence direction ratios of line of intersection are 1, 2, 1.

Let the line of intersection meet the xy-plane at P $(\alpha, \beta, 0)$.

Then P lies on planes (1) and (2)

$\therefore$ $\alpha - 2\beta = 4$ or, $2\alpha - 3\beta = 5$ ..... (5)

$\alpha = -2$, $\beta = -3$

Hence equation of line of intersection in symmetrical form is $\frac{x+2}{1} = \frac{y+3}{2} = \frac{z}{1}$.

Example # 57 : Find the angle between the lines $x - 3y - 4 = 0$, $4y - z + 5 = 0$ and $x + 3y - 11 = 0$, $2y - z + 6 = 0$.

Solution : Given lines are $\left.\begin{matrix} x - 3y - 4 = 0 \\ 4y - z + 5 = 0 \end{matrix}\right\}$ ..... (1)

and $\left.\begin{matrix} x + 3y - 11 = 0 \\ 2y - z + 6 = 0 \end{matrix}\right\}$ ..... (2)

Let ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 be the direction cosines of lines (1) and (2) respectively

$\because$ line (1) is perpendicular to the normals of each of the planes

$x - 3y - 4 = 0$ and $4y - z + 5 = 0$

$\therefore$ $\ell_1 - 3m_1 + 0.n_1 = 0$ ..... (3) and $0\ell_1 + 4m_1 - n_1 = 0$ ..... (4)

Solving equations (3) and (4), we get $\frac{\ell_1}{3-0} = \frac{m_1}{0-(-1)} = \frac{n_1}{4-0}$

or, $\frac{\ell_1}{3} = \frac{m_1}{1} = \frac{n_1}{4} = k$ (let).

Since line (2) is perpendicular to the normals of each of the planes

$x + 3y - 11 = 0$ and $2y - z + 6 = 0$,

$\therefore$ $\ell_2 + 3m_2 = 0$ ..... (5) and $2m_2 - n_2 = 0$ ..... (6)

$\therefore$ $\ell_2 = -3m_2$ or, $\frac{\ell_2}{-3} = m_2$ and $n_2 = 2m_2$ or, $\frac{n_2}{2} = m_2$.

$\therefore$ $\frac{\ell_2}{-3} = \frac{m_2}{1} = \frac{n_2}{2} = t$ (let).

If θ be the angle between lines (1) and (2), then $\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$

$= (3k)(-3t) + (k)(t) + (4k)(2t) = -9kt + kt + 8kt = 0 \ \therefore \ \theta = 90°$.

Example # 58 : Show that the lines $\frac{x-3}{2}=\frac{y+1}{-3}=\frac{z+2}{1}$ and $\frac{x-7}{-3}=\frac{y}{1}=\frac{z+7}{2}$ are coplanar. Also find the equation of the plane containing them.

Solution : Given lines are $\frac{x-3}{2}=\frac{y+1}{-3}=\frac{z+2}{1}$ = r (say)..... (1) and $\frac{x-7}{-3}=\frac{y}{1}=\frac{z+7}{2}$ = R (say) ..... (2)

If possible, let lines (1) and (2) intersect at P.

Any point on line (1) may be taken as (2r + 3, – 3r – 1, r – 2) = P (let).

Any point on line (2) may be taken as (– 3R + 7, R, 2R – 7) = P (let).

$\therefore$ 2r + 3 = – 3R + 7 or, 2r + 3R = 4 ..... (3)

Also – 3r – 1 = R or, – 3r – R = 1 ..... (4)

and r – 2 = 2R – 7 or, r – 2R = – 5. ..... (5)

Solving equations (3) and (4), we get, r = – 1, R = 2

Clearly r = – 1, R = 2 satisfies equation (5).

Hence lines (1) and (2) intersect.

$\therefore$ lines (1) and (2) are coplanar.

Equation of the plane containing lines (1) and (2) is $\begin{vmatrix} x-3 & y+1 & z+2 \\ 2 & -3 & 1 \\ -3 & 1 & 2 \end{vmatrix} = 0$

or, (x – 3) (– 6 – 1) – (y + 1) (4 + 3) + (z + 2) (2 – 9) = 0

or, – 7 (x – 3) – 7 (y + 1) – 7 (z + 2) = 0 or, x – 3 + y + 1 + z + 2 = 0 or, x + y + z = 0.

Self practice problems:

(67) Find the equation of the line of intersection of the plane
x – y + 2z = 5, 3x + y + z = 6.

(68) Prove that the three planes 2x + y – 4z – 17 = 0, 3x + 2y – 2z – 25 = 0, 2x – 4y + 3z + 25 = 0 intersect at a point and find its co-ordinates.

Answers : (67) $\frac{4x-11}{-33}=\frac{4y+9}{5}=\frac{z}{1}$ (68) (3, 7, –1)