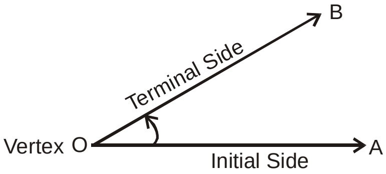
Trigonometry

When writing about transcendental issues, be transcendentally clear....... Descartes, Rene

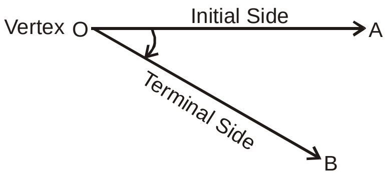
The word 'trigonometry' is derived from the Greek words 'trigon' and ' metron' and it means 'measuring the sides and angles of a triangle'.

Angle :

Angle is a measure of rotation of a given ray about its initial point. The original ray is called the initial side and the final position of the ray after rotation is called the terminal side of the angle. The point of rotation is called the vertex. If the direction of rotation is anticlockwise, the angle is said to be positive and if the direction of rotation is clockwise, then the angle is negative.



(i) Positive angle (anticlockwise measurement)



(ii) Negative angle (clockwise measurement)

Systems For Measurement of Angles :

An angle can be measured in the following systems.
One complete rotation is equal to 360 degree = 400 grade = 2 π radian

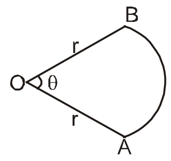
Relation between radian, degree and grade :

From \ To	Sexagesimal System (British system)	Centesimal System (French system)	Circular System (Radian Measurement)
Sexagesimal System (British system)		1 degree = $\frac{400}{360}$ grade	1 degree (1°) = $\frac{\pi}{180}$ radian 1min(1′)= $\frac{1}{60}$ degree (1°=60′) 1 sec(1″) = $\frac{1}{60}$ min (1′ = 60″)
Centesimal System (French system)	1 grade = $\frac{360}{400}$ degree		
Circular System (Radian Measurement)	1 radian = $\frac{180}{\pi}$ degree 1 degree = 60 min (1°=60′) 1 min = 60 sec (1′ = 60″)	1 radian = $\frac{200}{\pi}$ grade 1 grade=100 min(1^g = 100') 1 min = 100 sec(1' =100")	

Note : # The minutes and seconds in the Sexagesimal system are different with the minutes and seconds respectively in the Centesimal System. Symbols in both systems are also different.

\# If no symbol is mentioned while showing measurement of angle, then it is considered to be measured in radians.

e.g. $\theta = 15$ implies 15 radian



Arc length AB = $\ell = r\theta$

Area of circular sector = $\frac{1}{2} r^2\theta$ sq. units

Trigonometric Ratios for Acute Angles :

Let a revolving ray OP starts from OA and revolves into the position OP, thus tracing out the angle AOP.

In the revolving ray take any point P and draw PM perpendicular to the initial ray OA.

In the right angle triangle MOP, OP is the hypotenuse, PM is the perpendicular, and OM is the base.

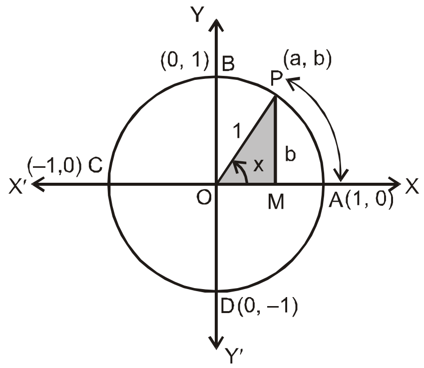
The trigonometrical ratios, or functions, of the angle AOP are defined as follows :

sin(∠AOP)	cos(∠AOP)	tan(∠AOP)	cot(∠AOP)	sec(∠AOP)	cosec(∠AOP)
$\frac{\text{Perp}}{\text{Hyp}} = \frac{\text{MP}}{\text{OP}}$	$\frac{\text{Base}}{\text{Hyp}} = \frac{\text{OM}}{\text{OP}}$	$\frac{\text{Perp}}{\text{Base}} = \frac{\text{MP}}{\text{OM}}$	$\frac{\text{Base}}{\text{Perp}} = \frac{\text{OM}}{\text{MP}}$	$\frac{\text{Hyp}}{\text{Base}} = \frac{\text{OP}}{\text{OM}}$	$\frac{\text{Hyp}}{\text{Prep}} = \frac{\text{OP}}{\text{MP}}$

It can be noted that the trigonometrical ratios are all real numbers.

Trigonometric ratios for angle $\theta \in R$:

We will now extend the definition of trigonometric ratios to any angle in terms of radian measure and study them as trigonometric functions. (also called circular functions) Consider a unit circle (radius 1 unit) with centre at origin of the coordinate axes. Let at origin of the coordinate axes. Let P(a, b) be any point on the circle with angle AOP = x radian, i.e., length of arc AP = x We define cos x = a and sin x = b Since Δ OMP is a right triangle, we have $OM^2 + MP^2 = OP^2$ or $a^2 + b^2 = 1$ Thus, for every point on the unit circle, we have $a^2 + b^2 = 1$ or $\cos^2 x + \sin^2 x = 1$



Since one complete revolution subtends an angle of 2π radian at the centre of the circle, $\angle AOB = \frac{\pi}{2}$, $\angle AOC = \pi$ and $\angle AOD = \frac{3\pi}{2}$. All angles which are integral multiples of $\frac{\pi}{2}$ are called quadrantal angles. The coordinates of the points A, B, C and D are, respectively, (1, 0), (0, 1), (–1, 0) and (0, –1). Therefore, for quadrantal angles, we have

$\cos 0 = 1$ $\quad\quad$ $\sin 0 = 0,$

$\cos \frac{\pi}{2} = 0$ $\quad\quad$ $\sin \frac{\pi}{2} = 1$

$\cos \pi = -1$ $\quad\quad$ $\sin \pi = 0$

$\cos \frac{3\pi}{2} = 0$ $\quad\quad$ $\sin \frac{3\pi}{2} = -1$

$\cos 2\pi = 1$ $\quad\quad$ $\sin 2\pi = 0$

Now if we take one complete revolution from the position OP, we again come back to same position OP. Thus, we also observe that if x increases (or decreases) by any integral multiple of 2π , the values of sine and cosine functions do not change. Thus, $\sin (2n\pi + x) = \sin x$, $n \in Z$, $\cos (2n\pi + x) = \cos x$, $n \in Z$. Further, $\sin x = 0$, if $x = 0, \pm \pi, \pm 2\pi, \pm 3\pi$ ....., i.e., when x is an integral multiple of π and $\cos x = 0$, if $x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}$, ...i.e., cos x vanishes when x is an odd multiple of $\frac{\pi}{2}$. Thus $\sin x = 0$ implies $x = n\pi$, where n is any integer $\cos x = 0$ implies $x = (2n + 1)\frac{\pi}{2}$, where n is any integer.

We now define other trigonometric functions in terms of sine and cosine functions :

$\text{cosec } x = \frac{1}{\sin x}$, $x \neq n\pi$, where n is any integer.

$\sec x = \frac{1}{\cos x}$, $x \neq (2n + 1)\frac{\pi}{2}$, where n is any integer.

$\tan x = \frac{\sin x}{\cos x}$, $x \neq (2n + 1)\frac{\pi}{2}$, where n is any integer.

$\cot x = \frac{\cos x}{\sin x}$, $x \neq n\pi$, where n is any integer.

We have shown that for all real x, $\sin^2 x + \cos^2 x = 1$

It follows that $\quad 1 + \tan^2 x = \sec^2 x$ $\quad$ (Think !) $\quad$ $\{x \neq (2n + 1)\frac{\pi}{2}\ ;\ n \in Z\}$

$1 + \cot^2 x = \text{cosec}^2 x$ $\quad$ (Think !) $\quad$ $\{x \neq n\pi\ ;\ n \in Z\}$

Sign of The Trigonometric Functions

(i) If θ is in the first quadrant then P(a, b) lies in the first quadrant. Therefore $a > 0$, $b > 0$ and hence the values of all the trigonometric functions are positive.

(ii) If θ is in the II quadrant then P(a, b) lies in the II quadrant. Therefore $a < 0$, $b > 0$ and hence the values sin, cosec are positive and the remaining are negative.

(iii) If θ is in the III quadrant then P(a, b) lies in the III quadrant. Therefore $a < 0$, $b < 0$ and hence the values of tan, cot are positive and the remaining are negative.

(iv) If θ is in the IV quadrant then P(a, b) lies in the IV quadrant. Therefore $a > 0$, $b < 0$ and hence the values of cos, sec are positive and the remaining are negative.

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\text{cosec}\theta$
Ist Quadrant	+	+	+	+	+	+
IInd Quadrant	+	–	–	–	–	+
IIIrd Quadrant	–	–	+	+	–	–
IVth Quadrant	–	+	–	–	+	–

Values of trigonometric functions of certain popular angles are shown in the following table :

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	$\sqrt{\frac{0}{4}}=0$	$\sqrt{\frac{1}{4}}=\frac{1}{2}$	$\sqrt{\frac{2}{4}}=\frac{1}{\sqrt{2}}$	$\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$	$\sqrt{\frac{4}{4}}=1$
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	N.D.

N.D. implies not defined

The values of cosec x, sec x and cot x are the reciprocal of the values of sin x, cosx and tan x, respectively.

Trigonometric Ratios of allied angles

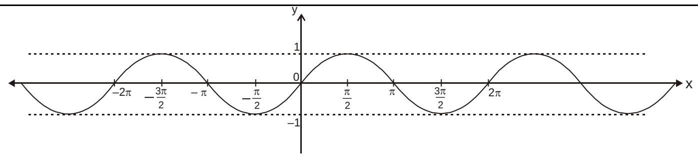
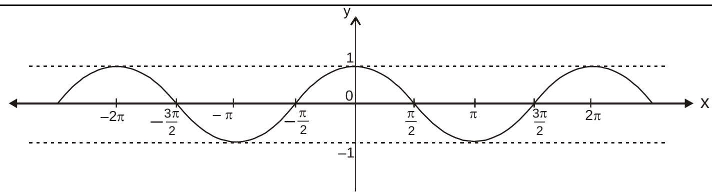
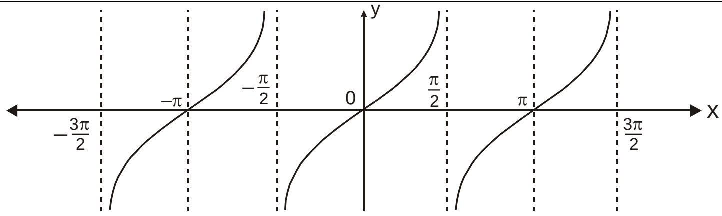
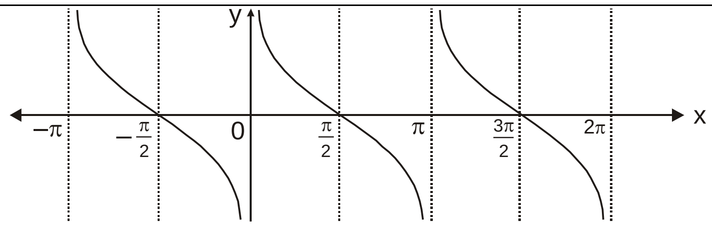
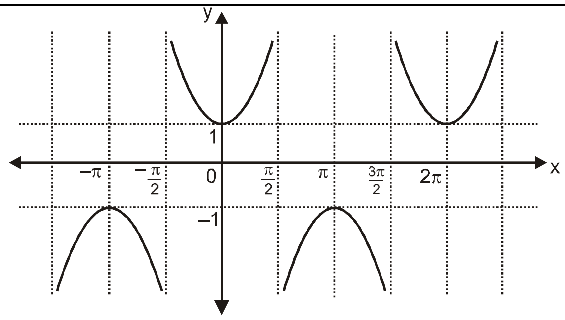
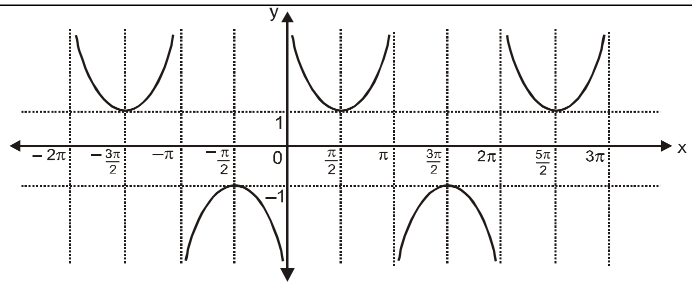
If θ is any angle, then $-\theta$, $\frac{\pi}{2} \pm \theta$, $\pi \pm \theta$, $\frac{3\pi}{2} \pm \theta$, $2\pi \pm \theta$ etc. are called allied angles.

	$-\theta$	$\frac{\pi}{2}-\theta$	$\frac{\pi}{2}+\theta$	$\pi-\theta$	$\pi+\theta$	$\frac{3\pi}{2}-\theta$	$\frac{3\pi}{2}+\theta$	$2\pi-\theta$	$2\pi+\theta$
sin	$-\sin\theta$	$\cos\theta$	$\sin\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$
cos	$\cos\theta$	$\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$	$\cos\theta$
tan	$-\tan\theta$	$\cot\theta$	$-\tan\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$
cot	$-\cot\theta$	$\tan\theta$	$-\cot\theta$	$-\cot\theta$	$\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$
sec	$\sec\theta$	$\mathrm{cosec}\,\theta$	$-\sec\theta$	$-\sec\theta$	$-\sec\theta$	$-\mathrm{cosec}\,\theta$	$\mathrm{cosec}\,\theta$	$\sec\theta$	$\sec\theta$
cosec	$-\mathrm{cosec}\theta$	$\sec\theta$	$\mathrm{cosec}\theta$	$\mathrm{cosec}\theta$	$-\mathrm{cosec}\theta$	$-\sec\theta$	$-\sec\theta$	$-\mathrm{cosec}\theta$	$\mathrm{cosec}\theta$

Think, and fill up the blank blocks in following table.

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1									
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0									
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	N.D.									

Trigonometric functions :

	Domain	Range	Graph
y = sinx	R	[–1, 1]	
y = cosx	R	[–1, 1]	
y = tanx	R– $\left\{(2n+1)\frac{\pi}{2},\ n \in I\right\}$	R	
y = cotx	R – {nπ, n ∈ I }	R	
y = secx	R– $\left\{(2n+1)\frac{\pi}{2},\ n \in I\right\}$	(– ∞, – 1] ∪ [1, ∞)	
y=cosecx	R – {nπ, n ∈ I}	(– ∞, – 1] ∪ [1, ∞)	

Trigonometric functions of sum or difference of two angles:

(a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
(b) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
(c) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B).\sin(A-B)$
(d) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B).\cos(A-B)$
(e) $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ (f) $\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$
(g) $\sin(A + B + C) = \sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B - \sin A \sin B \sin C$
(h) $\cos(A + B + C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$
(i) $\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$.
(j) $\tan(\theta_1 + \theta_2 + \theta_3 + \ldots\ldots + \theta_n) = \frac{S_1 - S_3 + S_5 - \ldots\ldots}{1 - S_2 + S_4 - \ldots\ldots}$
where S_i denotes sum of product of tangent of angles taken i at a time

Example # 1 : Prove that
(i) $\sin(45^\circ + A)\cos(45^\circ - B) + \cos(45^\circ + A)\sin(45^\circ - B) = \cos(A - B)$
(ii) $\tan\left(\frac{\pi}{4} + \theta\right)\tan\left(\frac{3\pi}{4} + \theta\right) = -1$

Solution : (i) Clearly $\sin(45^\circ + A)\cos(45^\circ - B) + \cos(45^\circ + A)\sin(45^\circ - B)$
$= \sin(45^\circ + A + 45^\circ - B) = \sin(90^\circ + A - B) = \cos(A - B)$
(ii) $\tan\left(\frac{\pi}{4} + \theta\right) \times \tan\left(\frac{3\pi}{4} + \theta\right) = \frac{1 + \tan\theta}{1 - \tan\theta} \times \frac{-1 + \tan\theta}{1 + \tan\theta} = -1$

Self practice problems :

(1) If $\cos\alpha = \frac{2\sqrt{2}}{3}$, $\sin\beta = \frac{4}{5}$, then find $\cos(\alpha + \beta)$ (2) Find the value of $\cos 375^\circ$

(3) Prove that $1 + \tan A \tan \frac{A}{2} = \tan A \cot \frac{A}{2} - 1 = \sec A$

Answers : (1) $\frac{\pm 6\sqrt{2} \pm 4}{15}$ (2) $\frac{\sqrt{3} + 1}{2\sqrt{2}}$

Transformation formulae :

(i) $\sin(A+B) + \sin(A - B) = 2 \sin A \cos B$
(ii) $\sin(A+B) - \sin(A - B) = 2 \cos A \sin B$
(iii) $\cos(A+B) + \cos(A - B) = 2 \cos A \cos B$
(iv) $\cos(A - B) - \cos(A+B) = 2 \sin A \sin B$

(a) $\sin C + \sin D = 2 \sin \frac{C + D}{2} \cos \frac{C - D}{2}$
(b) $\sin C - \sin D = 2 \cos \frac{C + D}{2} \sin \frac{C - D}{2}$
(c) $\cos C + \cos D = 2 \cos \frac{C + D}{2} \cos \frac{C - D}{2}$
(d) $\cos C - \cos D = 2 \sin \frac{C + D}{2} \sin \frac{D - C}{2}$

Example # 2 : Prove that $\cos 7A + \cos 8A = 2\cos\left(\frac{15A}{2}\right)\cos\left(\frac{A}{2}\right)$

Solution : L.H.S. $\cos 7A + \cos 8A = 2\cos\left(\frac{15A}{2}\right)\cos\left(\frac{A}{2}\right)$
[$\because \cos C + \cos D = 2\cos\frac{C + D}{2}\cos\frac{C - D}{2}$]

Example # 3 : Find the value of $2\sin 3\theta \sin\theta - \cos 2\theta + \cos 4\theta$
Solution : $2\sin 3\theta \sin\theta - \cos 2\theta + \cos 4\theta = 2\sin 3\theta \sin\theta - 2\sin 3\theta \sin\theta = 0$

Example # 4 : Prove that

(i) $\dfrac{\sin 8\theta \cos\theta - \sin 6\theta \cos 3\theta}{\cos 2\theta \cos\theta - \sin 3\theta \sin 4\theta} = \tan 2\theta$

(ii) If A + B = 45º then prove that (1 + tanA) (1 + tanB) = 2

Solution : (i) $\dfrac{2\sin 8\theta \cos\theta - 2\sin 6\theta \cos 3\theta}{2\cos 2\theta \cos\theta - 2\sin 3\theta \sin 4\theta} = \dfrac{\sin 9\theta + \sin 7\theta - \sin 9\theta - \sin 3\theta}{\cos 3\theta + \cos\theta - \cos\theta + \cos 7\theta} = \dfrac{2\sin 2\theta \cos 5\theta}{2\cos 5\theta \cos 2\theta} = \tan 2\theta$

(ii) A + B = 45º

$\tan (A + B) = 1 \Rightarrow \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = 1$

$\tan A + \tan B = 1 - \tan A \tan B \Rightarrow \tan A + \tan B + \tan A \tan B + 1 = 2$

$(1 + \tan A)(1 + \tan B) = 2$

Self practice problems

(4) Prove that

(i) $\cos 8x - \cos 5x = -2 \sin \dfrac{13x}{2} \sin \dfrac{3x}{2}$

(ii) $\dfrac{\cos A - \cos 3A}{\sin A - \sin 3A} = -\tan 2A$

(iii) $\dfrac{\sin 2A + \sin 4A + \sin 6A + \sin 8A}{\cos 2A + \cos 4A + \cos 6A + \cos 8A} = \tan 5A$

(iv) $\dfrac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \dfrac{\sin 3A}{\sin 5A}$

(v) $\dfrac{\sin A - \sin 5A + \sin 9A - \sin 13A}{\cos A - \cos 5A - \cos 9A + \cos 13A} = \cot 4A$

(5) Prove that $\sin \dfrac{\theta}{2} \sin \dfrac{7\theta}{2} + \sin \dfrac{3\theta}{2} \sin \dfrac{11\theta}{2} = \sin 2\theta \sin 5\theta$

(6) Prove that cos A sin (B – C) + cos B sin (C – A) + cos C sin (A – B) = 0

(7) Prove that $2 \cos \dfrac{\pi}{13} \cos \dfrac{9\pi}{13} + \cos \dfrac{3\pi}{13} + \cos \dfrac{5\pi}{13} = 0$

Multiple and sub-multiple angles :

(a) $\sin 2A = 2 \sin A \cos A$ Note : $\sin\theta = 2 \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2}$ etc.

(b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2 \sin^2 A$

Note : $2 \cos^2 \dfrac{\theta}{2} = 1 + \cos\theta$, $2 \sin^2 \dfrac{\theta}{2} = 1 - \cos\theta$.

(c) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$ Note : $\tan\theta = \dfrac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$

(d) $\sin 2A = \dfrac{2 \tan A}{1 + \tan^2 A}$, $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

(e) $\sin 3A = 3 \sin A - 4 \sin^3 A$

(f) $\cos 3A = 4 \cos^3 A - 3 \cos A$

(g) $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

Example # 5 : Prove that

(i) $\frac{\sin 2A}{1+\cos 2A}$ = tan A (ii) tan A + cot A = 2 cosec 2 A

(iii) $\frac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)} = \tan\frac{A}{2}\cot\frac{B}{2}$

Solution : (i) L.H.S. $\frac{\sin 2A}{1+\cos 2A} = \frac{2\sin A\cos A}{2\cos^2 A}$ = tan A

(ii) L.H.S. tan A + cot A = $\frac{1+\tan^2 A}{\tan A} = 2\left(\frac{1+\tan^2 A}{2\tan A}\right) = \frac{2}{\sin 2A}$ = 2 cosec 2 A

(iii) L.H.S. $\frac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)} = \frac{2\sin^2\frac{A}{2}+2\sin\frac{A}{2}\sin\left(\frac{A}{2}+B\right)}{2\cos^2\frac{A}{2}-2\cos\frac{A}{2}\cos\left(\frac{A}{2}+B\right)}$

$= \tan\frac{A}{2}\left[\frac{\sin\frac{A}{2}+\sin\left(\frac{A}{2}+B\right)}{\cos\frac{A}{2}-\cos\left(\frac{A}{2}+B\right)}\right] = \tan\frac{A}{2}\left[\frac{2\sin\frac{A+B}{2}\cos\left(\frac{B}{2}\right)}{2\sin\frac{A+B}{2}\sin\left(\frac{B}{2}\right)}\right] = \tan\frac{A}{2}\cot\frac{B}{2}$

Self practice problems

(8) Prove that $\frac{\sin 4\theta+\sin 2\theta}{1+\cos 4\theta+\cos 2\theta} = \frac{2\tan\theta}{1-\tan^2\theta}$

(9) Prove that $\sin\frac{\pi}{18}\sin\frac{3\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18} = \frac{1}{16}$

(10) Prove that tan 3A tan 2A tan A = tan 3A – tan 2A – tan A

(11) Prove that $\tan\left(45^o+\frac{A}{2}\right)$ = sec A + tan A

Important trigonometric ratios of standard angles :

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$, where $n \in I$

(b) $\sin 15°$ or $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75°$ or $\cos\frac{5\pi}{12}$;

$\cos 15°$ or $\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75°$ or $\sin\frac{5\pi}{12}$;

$\tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75°$; $\tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15°$

(c) $\sin\frac{\pi}{10}$ or $\sin 18° = \frac{\sqrt{5}-1}{4} = \cos 72°$

$\cos 36°$ or $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4} = \sin 54°$

Conditional identities:

If $A + B + C = \pi$ then :

(i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(ii) $\sin A + \sin B + \sin C = 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$

(iii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(iv) $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

(v) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(vi) $\tan\frac{A}{2} \tan\frac{B}{2} + \tan\frac{B}{2} \tan\frac{C}{2} + \tan\frac{C}{2} \tan\frac{A}{2} = 1$

(vii) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2} . \cot\frac{B}{2} . \cot\frac{C}{2}$

(viii) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

Example # 6 : If $A + B + C = 90°$, Prove that, $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$

Solution : $A + B = 90^\circ - C$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \cot C$$

$\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$

Example # 7 : If $x + y + z = xyz$, Prove that $\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2} . \frac{2y}{1-y^2} . \frac{2z}{1-z^2}$.

Solution : Put $x = \tan A$, $y = \tan B$ and $z = \tan C$,

so that we have

$\tan A + \tan B + \tan C = \tan A \tan B \tan C \Rightarrow A + B + C = n\pi$, where $n \in I$

Hence L.H.S.

$$\therefore \frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2\tan A}{1-\tan^2 A} + \frac{2\tan B}{1-\tan^2 B} + \frac{2\tan C}{1-\tan^2 C} .$$

$= \tan 2A + \tan 2B + \tan 2C$ $[\because A + B + C = n\pi]$

$= \tan 2A \tan 2B \tan 2C = \frac{2x}{1-x^2} . \frac{2y}{1-y^2} . \frac{2z}{1-z^2}$

Self practice problem

(12) If $A + B + C = 180°$, prove that

(i) $\sin(B + 2C) + \sin(C + 2A) + \sin(A + 2B) = 4\sin\frac{B-C}{2} \sin\frac{C-A}{2} \sin\frac{A-B}{2}$

(ii) $\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$.

(13) If $A + B + C = 2S$, prove that

(i) $\sin(S - A) \sin(S - B) + \sin S \sin (S - C) = \sin A \sin B$.

(ii) $\sin(S - A) + \sin (S - B) + \sin(S - C) - \sin S = 4\sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$.

Sine and Cosine series:

(i) $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + ... + \sin\{\alpha + (n-1)\beta\} = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \sin\left(\alpha + \frac{n-1}{2}\beta\right)$

(ii) $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + ... + \cos\{\alpha + (n-1)\beta\} = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cos\left(\alpha + \frac{n-1}{2}\beta\right)$

where : $\beta \neq 2m\pi$, $m \in I$

Example # 8 : (i) Prove that $\sin\theta + \sin3\theta + \sin5\theta + ... + \sin(2n-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}$

(ii) Find the average of sin2º, sin4º, sin6º , ... , sin180º

(iii) Prove that $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11} = \frac{1}{2}$

Solution : (i) $\sin\theta + \sin3\theta + \sin5\theta + ... + \sin(2n-1)\theta = \frac{\sin n\left(\frac{2\theta}{2}\right)\sin\left(\frac{\theta+(2n-1)\theta}{2}\right)}{\sin\left(\frac{2\theta}{2}\right)} = \frac{\sin^2 n\theta}{\sin\theta}$

(ii) $= \frac{\sin2^o + \sin4^o + ... + \sin180^o}{90} = \frac{\sin90^o(\sin91^o)}{90\sin1^o} = \frac{\cos1^o}{90\sin1^o} = \frac{\cot1^o}{90}$

(iii) $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11} = \frac{\cos\frac{10\pi}{22}\sin\frac{5\pi}{11}}{\sin\frac{\pi}{11}} = \frac{\sin\frac{10\pi}{11}}{2\sin\frac{\pi}{11}} = \frac{1}{2}$

Self practice problem

Find sum of the following series :

(14) $\cos\frac{\pi}{2n+1} + \cos\frac{3\pi}{2n+1} + \cos\frac{5\pi}{2n+1}$ + ... up to n terms.

(15) $\sin2\alpha + \sin3\alpha + \sin4\alpha + ... + \sin n\alpha$, where $(n + 2)\alpha = 2\pi$

Answers : (14) $-\frac{1}{2}$ (15) 0.

Product series of cosine angles

$\cos\theta \cdot \cos2\theta \cdot \cos2^2\theta \cdot \cos2^3\theta ...... \cos2^{n-1}\theta = \frac{\sin2^n\theta}{2^n\sin\theta}$

Range of trigonometric expression:

$E = a\sin\theta + b\cos\theta$

$\Rightarrow$ $E = \sqrt{a^2+b^2}\left\{\frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta\right\}$

Let $\frac{b}{\sqrt{a^2+b^2}} = \sin\alpha$ & $\frac{a}{\sqrt{a^2+b^2}} = \cos\alpha$

$\Rightarrow$ $E = \sqrt{a^2+b^2}\sin(\theta+\alpha)$, where $\tan\alpha = \frac{b}{a}$

Hence for any real value of θ,

$-\sqrt{a^2+b^2} \le \mathbf{E} \le \sqrt{a^2+b^2}$

Example # 9 : (i) If $\alpha + \beta = 90^o$ then find the maximum value of $\sin\alpha\sin\beta$

(ii) Find maximum and minimum value of $1 + 2\sin x + 3\cos^2 x$

Solution : (i) $\sin\alpha\sin(90^o - \alpha) = \sin\alpha\cos\alpha = \frac{1}{2}\times\sin2\alpha$

maximum value $= \frac{1}{2}$

(ii) $1 + 2\sin x + 3\cos^2 x = -3\sin^2 x + 2\sin x + 4 = -3\left(\sin^2 x - \frac{2\sin x}{3}\right) + 4 = -3\left(\sin x - \frac{1}{3}\right)^2 + \frac{13}{3}$

Now $0 \le \left(\sin x - \frac{1}{3}\right)^2 \le \frac{16}{9} \Rightarrow -\frac{16}{9} \le -3\left(\sin x - \frac{1}{3}\right)^2 \le 0$

$-1 \le -3\left(\sin x - \frac{1}{3}\right)^2 + \frac{13}{3} \le \frac{13}{3}$

Self practice problems

(16) Find maximum and minimum values of following

(i) $3 + (\sin x - 2)^2$

(ii) $9\cos^2 x + 48\sin x \cos x - 5\sin^2 x - 2$

(iii) $2\sin\left(\theta + \frac{\pi}{6}\right) + \sqrt{3}\cos\left(\theta - \frac{\pi}{6}\right)$

Answers : (i) max = 12, min = 4. (ii) max = 25, min = –25

(iii) max = $\sqrt{13}$, min = $-\sqrt{13}$

Trigonometric Equation :

An equation involving one or more trigonometric ratios of an unknown angle is called a trigonometric equation.

Solution of Trigonometric Equation :

A solution of trigonometric equation is the value of the unknown angle that satisfies the equation.

e.g. if $\sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$, ...

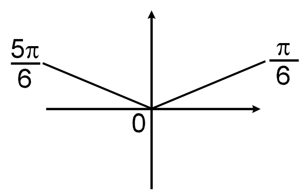
Thus, the trigonometric equation may have infinite number of solutions (because of their periodic nature) and can be classified as :

(i) Principal solution (ii) General solution.

Principal solutions :

The solutions of a trigonometric equation which lie in the interval $[0, 2\pi)$ are called Principal solutions.

e.g. Find the Principal solutions of the equation $\sin x = \frac{1}{2}$.



Solution **:**

$\because$ $\sin x = \frac{1}{2}$

$\because$ there exists two values

i.e. $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ which lie in $[0, 2\pi)$ and whose sine is $\frac{1}{2}$

$\therefore$ Principal solutions of the equation $\sin x = \frac{1}{2}$ are $\frac{\pi}{6}, \frac{5\pi}{6}$

General Solution :

The expression involving an integer 'n' which gives all solutions of a trigonometric equation is called General solution. General solution of some standard trigonometric equations are given below.

General Solution of Some Standard Trigonometric Equations :

(i) If $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$.

(ii) If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$, $n \in I$.

(iii) If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$.

(iv) If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$.

(v) If $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$.

(vi) If $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$. [**Note:** α is called the principal angle]

Some Important deductions :

(i) $\sin\theta = 0 \Rightarrow \theta = n\pi, \quad n \in I$

(ii) $\sin\theta = 1 \Rightarrow \theta = (4n + 1)\frac{\pi}{2}, n \in I$

(iii) $\sin\theta = -1 \Rightarrow \theta = (4n - 1)\frac{\pi}{2}, n \in I$

(iv) $\cos\theta = 0 \Rightarrow \theta = (2n + 1)\frac{\pi}{2}, n \in I$

(v) $\cos\theta = 1 \Rightarrow \theta = 2n\pi, n \in I$

(vi) $\cos\theta = -1 \Rightarrow \theta = (2n + 1)\pi, \quad n \in I$

(vii) $\tan\theta = 0 \Rightarrow \theta = n\pi, \quad n \in I$

Example # 10: Solve $\cos\theta = \frac{1}{2}$

Solution : $\because \cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \cos\frac{\pi}{3} \quad \therefore \quad \theta = 2n\pi \pm \frac{\pi}{3}, n \in I$

Example # 11 : Solve : $\sec 2\theta = -\frac{2}{\sqrt{3}}$

Solution : $\because \sec 2\theta = -\frac{2}{\sqrt{3}} \Rightarrow \cos 2\theta = -\frac{\sqrt{3}}{2}$

$\Rightarrow \cos 2\theta = \cos\frac{5\pi}{6} \Rightarrow 2\theta = 2n\pi \pm \frac{5\pi}{6}, n \in I \Rightarrow \theta = n\pi \pm \frac{5\pi}{12}, n \in I$

Example # 12 : Solve $\tan\theta = \frac{3}{4}$

Solution : $\because \tan\theta = \frac{3}{4}$ ....(i)

Let $\frac{3}{4} = \tan\alpha \Rightarrow \tan\theta = \tan\alpha$

$\Rightarrow \theta = n\pi + \alpha$, where $\alpha = \tan^{-1}\left(\frac{3}{4}\right), n \in I$

Self Practice Problems :

(17) Solve $\cot\theta = -1$

(18) Solve $\cos 4\theta = -\frac{\sqrt{3}}{2}$

Answers : (17) $\theta = n\pi - \frac{\pi}{4}, n \in I$ (18) $\frac{n\pi}{2} \pm \frac{\pi}{24}, n \in I$

Example # 13 : Solve $\tan^2\theta = 1$

Solution : $\because \tan^2\theta = 1 \Rightarrow \tan^2\theta = (1)^2$

$\Rightarrow \tan^2\theta = \tan^2\frac{\pi}{4} \Rightarrow \theta = n\pi \pm \frac{\pi}{4}, n \in I$

Example # 14 : Solve $4\sec^2\theta = 5 + \tan^2\theta$

Solution : $\because$ $4\sec^2\theta = 5 + \tan^2\theta$ ....(i)

For equation (i) to be defined $\theta \neq (2n + 1)\frac{\pi}{2}$, $n \in I$

$\because$ equation (i) can be written as:

$4(1 + \tan^2\theta) = 5 + \tan^2\theta$

$3\tan^2\theta = 1$

$\tan^2\theta = \tan^2\pi/6$

$\theta = n\pi \pm \frac{\pi}{6}$, $n \in I$

Self Practice Problems :

(19) Solve $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \ \tan 2x} = 1$

(20) Solve $2\cos^2 x + \sin^2 2x = 2$

Answers : (19) no Solution (20) $n\pi$, $n \in I$ or $n\pi \pm \frac{\pi}{4}$, $n \in I$

Types of Trigonometric Equations :

Type -1

Trigonometric equations which can be solved by use of factorization.

Example # 15 : $\frac{\sin^3\frac{x}{2} - \cos^3\frac{x}{2}}{2 + \sin x} = \frac{\cos x}{3}$

Solution :

$$\frac{\sin^3\frac{x}{2} - \cos^3\frac{x}{2}}{2 + \sin x} = \frac{\cos x}{3} \Rightarrow \frac{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + \sin\frac{x}{2}\cos\frac{x}{2}\right)}{2 + \sin x} = \frac{\cos x}{3}$$

$$\frac{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)(2 + \sin x)}{2(2 + \sin x)} = \frac{\cos x}{3} \Rightarrow 3\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) - 2\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) = 0$$

$$\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)\left(3 + 2\sin\frac{x}{2} + 2\cos\frac{x}{2}\right) = 0 \Rightarrow \sin\frac{x}{2} = \cos\frac{x}{2} \Rightarrow \tan\frac{x}{2} = 1$$

$$\frac{x}{2} = n\pi + \frac{\pi}{4}, n \in I \quad \Rightarrow \quad x = 2n\pi + \frac{\pi}{2}, n \in I$$

Self Practice Problems :

(21) Solve $\cos^3 x + \cos^2 x - 4\cos^2\frac{x}{2} = 0$

(22) Solve $\tan^2\theta + 3\sec\theta + 3 = 0$

Answers : (21) $(2n + 1)\pi$, $n \in I$

(22) $2n\pi \pm \frac{2\pi}{3}$, $n \in I$ or $(2n + 1)\pi$, $n \in I$

Type - 2

Trigonometric equations which can be solved by reducing them in quadratic equations.

Example # 16 : Solve $\sin^2 x - \frac{\cos x}{4} = \frac{1}{4}$

Solution : $\sin^2 x - \frac{\cos x}{4} = \frac{1}{4}$

$4(1 - \cos^2 x) - \cos x = 1$

$4\cos^2 x + \cos x - 3 = 0$

(cosx + 1) (4cosx –3) = 0

cosx = – 1 , cosx = $\frac{3}{4}$

x = (2n+1)π , x = (2m$\pi\pm\alpha$) where $\alpha = \cos^{-1}\frac{3}{4}$, m, n $\in$ I

Self Practice Problems :

(23) Solve $4\sin^2\theta + 2\sin\theta\left(\sqrt{3}-1\right)-\sqrt{3} = 0$

(24) Solve $4\cos\theta - 3\sec\theta = \tan\theta$

Answers : (23) $n\pi + (-1)^n\frac{\pi}{6}$, n $\in$ I or $n\pi + (-1)^n\left(\frac{-\pi}{3}\right)$, n $\in$ I

(24) $n\pi + (-1)^n\alpha$ where $\alpha = \sin^{-1}\left(\frac{-1-\sqrt{17}}{8}\right)$, n $\in$ I

or $n\pi + (-1)^n\beta$ where $\beta = \sin^{-1}\left(\frac{-1+\sqrt{17}}{8}\right)$, n $\in$ I

Type - 3

Trigonometric equations which can be solved by transforming a sum or difference of trigonometric ratios into their product.

Example # 17 : Solve cosx + cos3x – 2cos2x = 0

Solution : cosx + cos3x – 2cos2x = 0

2cos2x cosx – 2cos2x = 0

2cos2x (cosx–1) = 0

cos2x = 0, cosx = 1

x = (2n + 1), $\frac{\pi}{2}$ x = 2mπ , m, n $\in$ I

Self Practice Problems :

(25) Solve $\sin 7\theta = \sin 3\theta + \sin\theta$ (26) Solve 1 + cos3x = 2cos 2x

(27) Solve $8\cos x\cos 2x\cos 4x = \frac{\sin 6x}{\sin x}$

Answers : (25) $\frac{n\pi}{3}$, n $\in$ I or $\frac{n\pi}{2}\pm\frac{\pi}{12}$, n $\in$ I

(26) $n\pi\pm\frac{\pi}{6}$, n $\in$ I or 2nπ, n $\in$ I (27) $\frac{n\pi}{7}+\frac{\pi}{14}$, n $\in$ I

Type - 4

Trigonometric equations which can be solved by transforming a product of trigonometric ratios into their sum or difference.

Example # 18 : Solve sec4θ – sec2θ = 2

Solution : $\frac{1}{\cos 4\theta}-\frac{1}{\cos 2\theta} = 2$

cos2θ – cos4θ = 2 cos4θ cos2θ

cos2θ – cos4θ = cos6θ + cos2θ

cos6θ + cos4θ = 0

2cos5θ cosθ = 0

cos5θ = 0 or cosθ = 0

$5\theta = (2n+1)\frac{\pi}{2}$ $\theta = (2m+1)\frac{\pi}{2}$ m, n $\in$ I

Type - 5

Trigonometric Equations of the form a sinx + b cosx = c, where a, b, c $\in$ R, can be solved by dividing both sides of the equation by $\sqrt{a^2+b^2}$.

Example # 19 : Solve sinx + 2cosx = $\sqrt{5}$

Solution : $\because$ sinx + 2cosx = $\sqrt{5}$ ....(i)

Here a = 1, b = 2.

$\therefore$ divide both sides of equation (i) by $\sqrt{5}$, we get

$\sin x \cdot \frac{1}{\sqrt{5}} + 2\cos x \cdot \frac{2}{\sqrt{5}} = 1 \Rightarrow \sin x.\sin\alpha + \cos x.\cos\alpha = 1 \Rightarrow \cos(x-\alpha) = 1$

$\Rightarrow$ $x - \alpha = 2n\pi, n \in I$ $\Rightarrow$ $x = 2n\pi + \alpha, n \in I$

$\therefore$ Solution of given equation is $2n\pi + \alpha, n \in I$ where $\alpha = \tan^{-1}\left(\frac{1}{2}\right)$

Note : Trigonometric equation of the form a sinx + b cosx = c can also be solved by changing sinx and cosx into their corresponding tangent of half the angle.

Example # 20 : Solve 3cosx + 4sinx = 5

Solution : $\because$ 3cosx + 4sinx = 5 ....(i)

$\because$ $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ & $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$\therefore$ equation (i) becomes

$\Rightarrow$ $3\left(\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) + 4\left(\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) = 5$ ....(ii)

Let $\tan\frac{x}{2} = t$

$\therefore$ equation (ii) becomes $3\left(\dfrac{1-t^2}{1+t^2}\right) + 4\left(\dfrac{2t}{1+t^2}\right) = 5$

$\Rightarrow$ $4t^2 - 4t + 1 = 0$ $\Rightarrow$ $(2t-1)^2 = 0$

$\Rightarrow$ $t = \frac{1}{2}$ $\because$ $t = \tan\frac{x}{2}$

$\Rightarrow$ $\tan\frac{x}{2} = \frac{1}{2}$ $\Rightarrow$ $\tan\frac{x}{2} = \tan\alpha$, where $\tan\alpha = \frac{1}{2}$

$\Rightarrow$ $\frac{x}{2} = n\pi + \alpha$ $\Rightarrow$ $x = 2n\pi + 2\alpha$ where $\alpha = \tan^{-1}\left(\frac{1}{2}\right)$, $n \in I$

Self Practice Problems :

(28) Solve $2\sqrt{2}\cos x + \sin x = 3$

(29) Solve $\sin x + \tan\frac{x}{2} = 0$

Answers : (28) $2n\pi + \alpha, n \in I$ where $\alpha = \tan^{-1}\left(\frac{1}{2\sqrt{2}}\right)$

(29) $x = 2n\pi, n \in I$

Type - 6

Trigonometric equations of the form P(sinx ± cosx, sinx cosx) = 0, where p(y, z) is a polynomial, can be solved by using the substitution sinx ± cosx = t.

Example # 21 : Solve $\sin 2x + 3\sin x = 1 + 3\cos x$

Solution : $\sin 2x + 3\sin x = 1 + 3\cos x$

$\sin 2x + 3(\sin x - \cos x) = 1$ .... (i)

Let $\sin x - \cos x = t$

$\Rightarrow$ $\sin^2 x + \cos^2 x - 2\sin x.\cos x = t^2$ $\Rightarrow$ $\sin 2x = 1 - t^2$

Now put $\sin x - \cos x = t$ and $\sin 2x = 1 - t^2$ in (i)

$1 - t^2 + 3t = 1$

$t^2 - 3t = 0$

$t = 0$ or $t = 3$ (not possible)

$\sin x - \cos x = 0$

$\tan x = 1$ $\Rightarrow$ $x = n\pi + \frac{\pi}{4}$, $n \in I$

Self Practice Problems:

(30) Solve $1 - \sin 2x + 2\sin x - 2\cos x = 0$

(31) Solve $2\cos x + 2\sin x + \sin 3x - \cos 3x = 0$

(32) Solve $(1 - \sin 2x)(\cos x - \sin x) = 1 - 2\sin^2 x$.

Answers : (30) $n\pi + \frac{\pi}{4}$, $n \in I$ (31) $n\pi - \frac{\pi}{4}$ or $\frac{n\pi}{2} + (-1)^n \frac{\pi}{12}$, $n \in I$

(32) $2n\pi + \frac{\pi}{2}$, $n \in I$ or $2n\pi$, $n \in I$ or $n\pi + \frac{\pi}{4}$, $n \in I$

Type - 7

Trigonometric equations which can be solved by the use of boundness of the trigonometric ratios sinx and cosx.

Example # 22 : Solve $\sin 2x + \cos 4x = 2$

Solution : $\sin 2x + \cos 4x = 2$

Now equation will be true if $\sin 2x = 1$ and $\cos 4x = 1$

$\Rightarrow$ $2x = (4n + 1)\frac{\pi}{2}$, $n \in I$ and $4x = 2m\pi$, $m \in I$

$\Rightarrow x = (4n + 1)\frac{\pi}{4}$, $n \in I$ and $x = \frac{m\pi}{2}$, $m \in I \Rightarrow (4n + 1)\frac{\pi}{4} = \frac{m\pi}{2} \Rightarrow m = \frac{4n+1}{2}$

Which is not possible for $m, n \in I$

Self Practice Problems :

(33) Solve $\cos^{50}x - \sin^{50}x = 1$

(34) Solve $12\sin x + 5\cos x = 2y^2 - 8y + 21$ for x & y

Answers : (33) $n\pi$, $n \in I$ (34) $x = 2n\pi + \alpha$ where $\alpha = \cos^{-1}\left(\frac{5}{13}\right)$, $n \in I$ $y = 2$

IMPORTANT POINTS :

1. Many trigonometrical equations can be solved by different methods. The form of solution obtained in different methods may be different. From these different forms of solutions, the students should not think that the answer obtained by one method are wrong and those obtained by another method are correct. The solutions obtained by different methods may be shown to be equivalent by some supplementary transformations.

To test the equivalence of two solutions obtained from two methods, the simplest way is to put values of
n = ...–2, –1, 0, 1, 2, 3....... etc. and then to find the angles in [0, 2π]. If all the angles in both solutions are same, the solutions are equivalent.

2. While manipulating the trigonometrical equation, avoid the danger of losing roots. Generally, some roots are lost by cancelling a common factor from the two sides of an equation. For example, suppose we have the equation tanx = 2 sinx. Here by dividing both sides by sinx, we get cosx = $\frac{1}{2}$. This is not equivalent to the original equation. Here the roots obtained by sinx = 0, are lost. Thus in place of dividing an equation by a common factor, the students are advised to take this factor out as a common factor from all terms of the equation.

3. While equating one of the factors to zero, take care of the other factor that it should not become infinite. For example, if we have the equation sinx = 0, which can be written as cos x tan x = 0. Here we cannot put
cosx = 0, since for cos x = 0, tanx = sinx/ cosx is infinite.

4. Avoid squaring : When we square both sides of an equation, some extraneous roots appear. Hence it is necessary to check all the solutions found by substituting them in the given equation and omit the solutions not satisfying the given equation.
For example : Consider the equation,
$\sin\theta + \cos\theta = 1$ .....(1)
Squaring we get
$1 + \sin 2\theta = 1$ or $\sin 2\theta = 0$ .....(2)
i.e. $2\theta = n\pi$ or $\theta = n\pi/2$,
This gives $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, ...$
Verification shows that π and $\frac{3\pi}{2}$ do not satisfy the equation as $\sin\pi + \cos\pi = -1, \neq 1$
and $\sin\frac{3\pi}{2} + \cos\frac{3\pi}{2} = -1, \neq 1.$
The reason for this is simple.
The equation (2) is not equivalent to (1) and (2) contains two equations : $\sin\theta + \cos\theta = 1$ and $\sin\theta + \cos\theta = -1$. Therefore we get extra solutions.
Thus if squaring is must, verify each of the solution.

5. Some necessary restrictions :
If the equation involves tanx, secx, take cosx ≠ 0. If cot x or cosec x appear, take sinx ≠ 0.
If log appear in the equation, i.e. log [f(θ)] appear in the equation, use f(θ) > 0 and base of log > 0, ≠ 1.
Also note that $\sqrt{[f(\theta)]}$ is always positive, for example $\sqrt{\sin^2\theta} = |\sin\theta|$, not $\pm \sin\theta$.

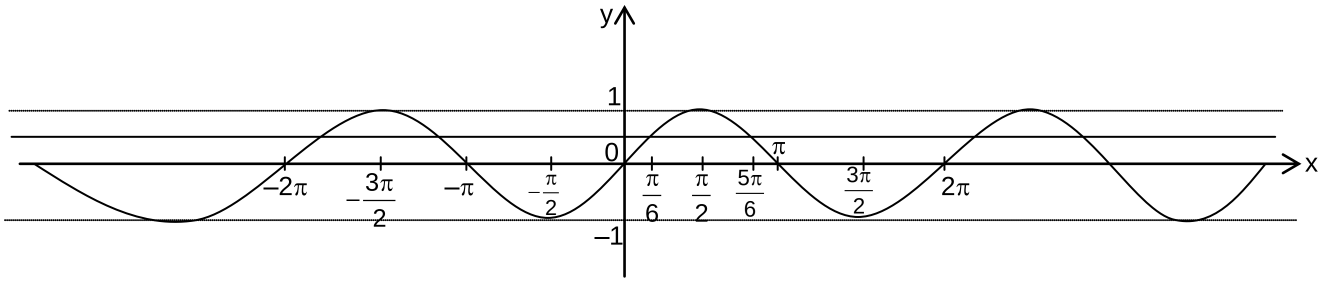
6. Verification : Student are advice to check whether all the roots obtained by them satisfy the equation and lie in the domain of the variable of the given equation.

Trigonometric Inequalities :

To solve a trigonometric inequality, transform it into many basic trigonometric inequalities. The transformation process proceeds exactly the same as in solving trigonometric equations. The common period of a trigonometric inequality is the least common multiple of all periods of the trigonometric functions presented in the inequality. **For example :** the trigonometric inequality sinx + sin2x + cos x/2 < 1 has 4π as common period. Unless specified, the solution set of a trigonometric inequality must be solved, at least, within one whole common period.

Example : Find the solution set of inequality sinx > 1/2.

Solution : When sinx = 1/2, the two values of x between 0 and 2π are π/6 and 5π/6.



y = sinx

From, the graph of y = sinx, it is obvious that, between 0 and 2π, sinx > 1/2 $\Rightarrow$ π/6 < x < 5π/6.

Hence sinx >1/2 $\Rightarrow$ 2nπ + π/6 < x < 2nπ + 5π/6, n $\in$ I .

The required solution set is $\bigcup_{n\in I}$ (2nπ + π/6, 2nπ + 5π/6)

Self practice problems

(35) Solve the following inequations

(i) (sinx – 2) (2sinx–1) < 0 (ii) sinx + $\sqrt{3}$ cosx $\geq$ 1

Ans. (i) $x \in \bigcup_{n\in I}\left(\frac{\pi}{6}+2n\pi, \frac{5\pi}{6}+2n\pi\right)$ (ii) $x \in \bigcup_{n\in I}\left[-\frac{\pi}{6}+2n\pi, 2n\pi+\frac{\pi}{2}\right]$

Heights and distances :

Angle of elevation and depression :

Let OX be a horizontal line and P be a point which is above point O. If an observer (eye of observer) is at point O and an object is lying at point P then ∠XOP is called angle of elevation as shown in figure. If an observer (eye of observer) is at point P and object is at point O then ∠QPO is called angle of depression.