Straight Line

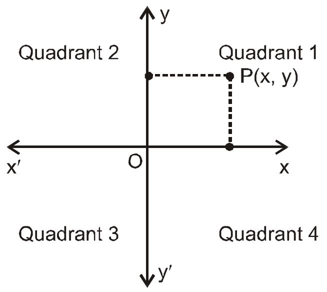
The knowledge of which geometry aims is the knowledge of the eternal...... Plato

A french mathematician and a greatest philosopher named Rene Descartes, pioneered the use of algebra in Geometry. He suggested methods to study geometry by algebraic methods without making direct reference to the actual figures

This geometry was called co-ordinate geometry or analytical geometry and it is the branch of geometry in which algebraic equations are used to denote points, lines and curves.

Rectangular cartesian co-ordinate systems :

We shall right now focus on two-dimensional co-ordinate geometry in which two perpendicular lines called co-ordinate axes (x-axis and y-axis) are used to locate a point in the plane.



O is called origin. Any point P in this plane can be represented by a unique ordered pair (x, y), which are called co-ordinates of that point. x is called x co-ordinate or abscissa and y is called y co-ordinate or ordinate. The two perpendicular lines xox′ and yoy′ divide the plane in four regions which are called quadrants, numbered as shown in the figure.

Let us look at some of the formulae linked with points now.

Distance Formula :

In rectangular Cartesian coordinate system

The distance between the points $A(x_1,y_1)$ and $B(x_2,y_2)$ is = $\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$.

Example # 1 : Find the value of x, if the distance between the points (x, 8) and (4, 3) is 13

Solution : Let P(x ,8) and Q(4, 3) be the given points. Then PQ = 13 (given)

$\sqrt{(x-4)^2+(8-3)^2} = 13 \Rightarrow (x-4)^2 + 25 = 169 \Rightarrow x = 16$ or $x = -8$

Self practice problems :

(1) Show that four points (0, –1), (6, 7) (–2, 3) and (8, 3) are the vertices of a rectangle.

(2) Find the co-ordinates of the circumcentre of the triangle whose vertices are (8, 6), (8, –2) and (2, –2). Also find its circumradius.

Ans. (2) (5, 2), 5

Section Formula :

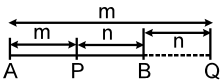
If P(x, y) divides the line joining $A(x_1, y_1)$ & $B(x_2, y_2)$ in the ratio m : n, then;

$x = \dfrac{mx_2 + nx_1}{m + n}$; $y = \dfrac{my_2 + ny_1}{m + n}$.

Notes : (i) If $\frac{m}{n}$ is positive, the division is internal, but if $\frac{m}{n}$ is negative, the division is external.

(ii) If P divides AB internally in the ratio m : n & Q divides AB externally in the ratio m : n then P & Q are said to be harmonic conjugate of each other w.r.t. AB.

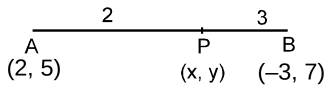
Mathematically, $\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ}$ i.e. AP, AB & AQ are in H.P.



Example # 2 : Find the co-ordinates of the point which divides the line segment joining the points (2, 5) and (– 3, 7) in the ratio 2 : 3 (i) internally and (ii) externally.

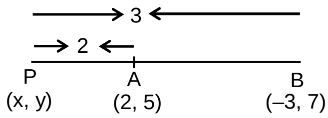
Solution : Let P (x, y) be the required point.

(i) For internal division :



$$(x, y) = \left(\frac{-6+6}{2+3}, \frac{14+15}{2+3}\right) = \left(0, \frac{29}{5}\right)$$

(ii) For external division



$$(x, y) = \left(\frac{-6-6}{2-3}, \frac{14-15}{2-3}\right) = (12, 1)$$

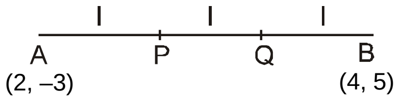
Example # 3 : Find the co-ordinates of points which trisect the line segment joining (2, – 3) and (4, 5).

Solution : Let A (2, –3) and B(4, 5) be the given points. Let the points of trisection be P and Q. Then AP = PQ = QB = λ (say)

$\therefore$ PB = PQ + QB = 2λ and AQ = AP + PQ = 2λ

$\Rightarrow$ AP : PB = λ : 2λ = 1 : 2 and AQ : QB = 2λ : λ = 2 : 1

So P divides AB internally in the ratio 1 : 2 while Q divides internally in the ratio 2 : 1



So P divides AB internally in the ratio 1:2 while Q divides internally in the ratio 2: 1

$\therefore$ the co-ordinates of P are $\left(\frac{4+4}{1+2}, \frac{5-6}{1+2}\right) = \left(\frac{8}{3}, -\frac{1}{3}\right)$

and the co-ordinates of Q are $\left(\frac{8+2}{1+2}, \frac{10-3}{1+2}\right) = \left(\frac{10}{3}, \frac{7}{3}\right)$

Hence, the points of trisection are $\left(\frac{8}{3}, -\frac{1}{3}\right)$ and $\left(\frac{10}{3}, \frac{7}{3}\right)$

Self practice problems :

(3) In what ratio does the point (4, 1) divide the line segment joining the points (1, –2) and (5, 2).

(4) The three vertices of a parallelogram taken in order are (–2, 0), (4, 2) and (5, 3) respectively. Find the co-ordinates of the fourth vertex.

Ans. (3) 3 : 1 internally (4) (–1, 1)

The ratio in which a given line divides the line segment joining two points :

Let the given line ax + by + c = 0 divide the line segment joining A(x_1, y_1) & B(x_2, y_2) in the ratio m : n, then $\frac{m}{n} = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$. If A & B are on the same side of the given line then m/n is negative but if A & B are on opposite sides of the given line, then m/n is positive

Example # 4 : Find the ratio in which the line joining the points A (1, 2) and B(– 3, 4) is divided by the line $x + y - 5 = 0$.

Solution : Let the line $x + y = 5$ divides AB in the ratio k : 1 at P

$\therefore$ co-ordinate of P are $\left(\frac{-3k+1}{k+1}, \frac{4k+2}{k+1}\right)$

Since P lies on $x + y - 5 = 0$

$\therefore$ $\frac{-3k+1}{k+1} + \frac{4k+2}{k+1} - 5 = 0 \quad \Rightarrow \quad k = -\frac{1}{2}$

$\therefore$ Required ratio is 1 : 2 externally. .

Aliter : Let the ratio is m : n

$\therefore$ $\frac{m}{n} = -\frac{(1\times1+1\times2-5)}{1\times(-3)+1\times4-5} = -\frac{1}{2}$ $\quad\therefore$ ratio is 1 : 2 externally.

Self practice problem :

(5) If the line $2x - 3y + \lambda = 0$ divides the line joining the points A (– 1, 2) & B(– 3, – 3) internally in the ratio 2 : 3, find λ.

Ans. $\frac{18}{5}$

Slope Formula :

If θ is the angle at which a straight line is inclined to the positive direction of x–axis, & $0° \le \theta < 180°$, $\theta \ne 90°$, then the slope of the line, denoted by m, is defined by $m = \tan\theta$. If θ is 90°, m does not exist, but the line is parallel to the y–axis. If $\theta = 0$, then m = 0 & the line is parallel to the x-axis. If A (x_1, y_1) & B (x_2, y_2), $x_1 \ne x_2$, are points on a straight line, then the slope m of the line is given by : $m = \left(\frac{y_1 - y_2}{x_1 - x_2}\right)$.

Example # 5 : What is the slope of a line whose inclination with the positive direction of x-axis is :

(i) 30º (ii) 90º (iii) 135º

Solution : (i) Here $\theta = 30º$

Slope = $\tan\theta = \tan 30º = \frac{1}{\sqrt{3}}$.

(ii) Here $\theta = 90º$

$\therefore$ The slope of line is not defined.

(iii) Here $\theta = 135º$

$\therefore$ Slope = $\tan\theta = \tan 135º = \tan(180º - 45º) = -\tan 45º = -1$.

Example # 6 : Find the slope of the line passing through the points :

(i) (2, 7) and (– 3, 4) (ii) (6, 9) and (–2, 7)

Solution : (i) Let A = (2, 7) and B = (– 3, 4)

$\therefore$ Slope of AB = $\frac{4-7}{-3-2} = \frac{3}{5}$ $\left(\text{Using slope} = \frac{y_2 - y_1}{x_2 - x_1}\right)$

(ii) Let A = (6, 9), B = (–2, 7) $\therefore$ Slope of AB = $\frac{7-9}{-2-6} = \frac{1}{4}$

Self practice problems :

(6) Find the value of x, if the slope of the line joining (1, 5) and (x, –7) is 4.

(7) What is the inclination of a line whose slope is

(i) 0 (ii) 1 (iii) –1 (iv) $-1/\sqrt{3}$

Ans. (6) – 2 (7) (i) 0º, (ii) 45º, (iii) 135º, (iv) 150º

Condition of collinearity of three points :

Points A (x_1, y_1), B (x_2, y_2), C(x_3, y_3) are collinear if

(i) $m_{AB} = m_{BC} = m_{CA}$ i.e. $\left(\dfrac{y_1 - y_2}{x_1 - x_2}\right) = \left(\dfrac{y_2 - y_3}{x_2 - x_3}\right)$

(ii) Δ ABC = 0 i.e. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(iii) AC = AB + BC or AB ~ BC

(iv) A divides the line segment BC in some ratio.

Example # 7 : Show that the points (–2, –1), (2, 7) and (5, 13) are collinear.

Solution : Let (–2, –1) (2, 7) and (5, 13) be the co-ordinates of the points A, B and C respectively.

Slope of AB = $\dfrac{7+1}{2+2}$ = 2 and Slope of BC = $\dfrac{13-7}{5-2}$ = 2

$\therefore$ Slope of AB = slope of BC

$\therefore$ AB & BC are parallel

$\therefore$ A, B, C are collinear because B is on both lines AB and BC.

Self practice problem :

(8) Prove that the points (a, 0), (0, b) and (1, 1) are collinear if $\dfrac{1}{a} + \dfrac{1}{b} = 1$

Area of a Triangle :

If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the vertices of triangle ABC, then its area is equal to

Δ ABC = $\dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$, provided the vertices are considered in the counter clockwise sense. The above formula will give a $(-)$ve area if the vertices (x_i, y_i), i = 1, 2, 3 are placed in the clockwise sense.

Note: Area of n-sided polygon formed by points (x_1, y_1) ; (x_2, y_2); ....;(x_n, y_n) is given by

$$\frac{1}{2}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \ldots\ldots\ldots\ldots + \begin{vmatrix} x_{n-1} & x_n \\ y_{n-1} & y_n \end{vmatrix} + \begin{vmatrix} x_n & x_1 \\ y_n & y_1 \end{vmatrix}\right).$$

Here vertices are taken in order.

Example # 8 : If the co-ordinates of two points A and B are (2, 1) and (4, –3)respectively. Find the co-ordinates of any point P if PA = PB and Area of ΔPAB = 6.

Solution : Let the co-ordinates of P be (x, y). Then

PA = PB $\Rightarrow$ $PA^2 = PB^2$

$\Rightarrow$ $(x-2)^2 + (y-1)^2 = (x-4)^2 + (y+3)^2$ $\Rightarrow$ x – 2y = 5 ....(i)

Now, Area of ΔPAB = 6 $\Rightarrow$ $\dfrac{1}{2}\begin{vmatrix} x & y & 1 \\ 2 & 1 & 1 \\ 4 & -3 & 1 \end{vmatrix} = \pm 6$ $\Rightarrow$ 4x + 2y – 10 = ± 12

$\Rightarrow$ 4x + 2y = 22 or 4x + 2y = –2 $\Rightarrow$ 2x + y = 11 or 2x + y = –1

Solving 2x + y = 11 and x – 2y = 5 we get x = $\dfrac{27}{5}$, y = $\dfrac{1}{5}$.

Solving 2x + y = –1 and x – 2y = 5, we get x = $\dfrac{3}{5}$, y = $-\dfrac{11}{5}$.

Thus, the co-ordinates of P are $\left(\dfrac{27}{5}, \dfrac{1}{5}\right)$ or $\left(\dfrac{3}{5}, -\dfrac{11}{5}\right)$

Self practice problems :

(9) The area of a triangle is 5. Two of its vertices are (2, 1) and (3, –2). The third vertex lies on $y = x + 3$. Find the third vertex.

(10) The coordinates of A, B, C are (6, 3), (–3, 5) & (4,–2) respectively and p is any point (x, y).Show that the ratio of the areas of the triangles PBC and ABC is $\left|\frac{x+y-2}{7}\right|$

Ans. (9) $\left(\frac{7}{2}, \frac{13}{2}\right)$ or $\left(-\frac{3}{2}, \frac{3}{2}\right)$

Equation of a Straight Line in various forms :

Now let us understand, how a line can be represented with the help of an algebraic equation. A moving point (point with variable co-ordinates) is assumed on the line and a link is established between its co-ordinates with the help of some given parameters. There are various forms of lines depending on the specified parameter

Point - Slope form :

$y - y_1 = m(x - x_1)$ is the equation of a straight line whose slope is m & which passes through the point (x_1, y_1).

Example # 9 : Find the equation of a line passing through (3, –4) and inclined at an angle of 150º with the positive direction of x-axis.

Solution : Here, m = slope of the line = tan 150º = tan (90º + 60º) =– cot 60º = $-\frac{1}{\sqrt{3}}$, $x_1 = 3$, $y_1 = -4$

So, the equation of the line is $y - y_1 = m(x - x_1)$

i.e. $y + 4 = -\frac{1}{\sqrt{3}}(x - 3)$

$x + \sqrt{3}\,y + 4\sqrt{3} - 3 = 0$

Self practice problem :

(11) Find the equation of s a line passing through P(–3, 5) and whose slope is –2.
Ans. $2x + y + 1 = 0$

Slope-intercept form :

$y = mx + c$ is the equation of a straight line whose slope is m & which makes an intercept c on the y–axis.

Example # 10 : Find the equation of a line with slope –3 and cutting off an intercept of 5 units on negative direction of y-axis.
Solution : Here m = –3 and c = – 5. So, the equation of the line is $y = mx + c$
i.e. $y = -3x - 5$ or $3x + y + 5 = 0$

Self practice problem :

(12) Find the equation of a straight line which cuts off an intercept of length 3 on y- axis and whose slope is – 3.
Ans. $3x + y - 3 = 0$

Two point form :

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ is the equation of a straight line which passes through the points (x_1, y_1) & (x_2, y_2).

Example # 11 : Find the equation of the line joining the points (3, 4) and (–2, 5)
Solution : Here the two points are $(x_1, y_1) = (3, 4)$ and $(x_2, y_2) = (-2, 5)$.
So, the equation of the line in two-point form is

$y - 4 = \frac{5-4}{-2-3}(x - 3) \Rightarrow x + 5y = 23$

Self practice problem :

(13) Find the equations of the sides of the triangle whose vertices are (–1, 8), (4, –2) and (–5, –3). Also find the equation of the median through (–1, 8)

Ans. $2x + y – 6 = 0$, $x – 9y – 22 = 0$, $11x – 4y + 43 = 0$, $21x + y + 13 = 0$

Determinant form : Equation of line passing through (x_1, y_1) and (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Example # 12 : Find the equation of line passing through (2, 4) & (– 1, 3).

Solution : $\begin{vmatrix} x & y & 1 \\ 2 & 4 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 0 \quad \Rightarrow \quad x – 3y + 10 = 0$

Self practice problem :

(14) Find the equation of the passing through (– 2, 3) & (– 1, – 1).

Ans. $4x + y + 5 = 0$

Intercept form : $\frac{x}{a} + \frac{y}{b} = 1$ is the equation of a straight line which makes intercepts a & b on OX & OY respectively.

Example # 13 : Find the equation of the line which passes through the point (–3, 8) and the sum of its intercepts on the axes is 7.

Solution : Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$ ....(i)

This passes through (–3, 8), therefore $-\frac{3}{a} + \frac{8}{b} = 1$ ....(ii)

It is given that $a + b = 7 \Rightarrow b = 7 – a$.

Putting $b = 7 – a$ in (ii), we get $-\frac{3}{a} + \frac{8}{7-a} = 1$

$\Rightarrow a^2 + 4a – 21 = 0 \Rightarrow a = 3, – 7$

For a = 3, b = 4 and for a = –7, b = 14

Putting the values of a and b in (i), we get the equations of the lines

$\frac{x}{3} + \frac{y}{4} = 1$ and $\frac{x}{-7} + \frac{y}{14} = 1$ or $4x + 3y = 12$ and $2x – y + 14 = 0$

Self practice problem :

(15) Find the equation of the line through (2, 3) so that the segment of the line intercepted between the axes is bisected at this point.

Ans. $3x + 2y = 12$.

Perpendicular/Normal form :

$x\cos\alpha + y\sin\alpha = p$ (where $p > 0$, $0 \le \alpha < 2\pi$) is the equation of the straight line where the length of the perpendicular from the origin O on the line is p and this perpendicular makes an angle α with positive x–axis.

Example # 14 : Find the equation of the line which is at a distance 3 from the origin and the perpendicular from the origin to the line makes an angle of 30º with the positive direction of the x-axis.

Solution : Here $p = 3$, $\alpha = 30º$

$\therefore$ Equation of the line in the normal form is

$x \cos 30º + y \sin 30º = 3$ or $x\frac{\sqrt{3}}{2} + \frac{y}{2} = 3$ or $\sqrt{3}x + y = 6$

Self practice problem :

(16) The length of the perpendicular from the origin to a line is 7 and the line makes an angle of 150^o with the positive direction (clock-wise) of y-axis. Find the equation of the line.

Ans. $\sqrt{3}\,x - y + 14 = 0$

General Form : $ax + by + c = 0$ is the equation of a straight line in the general form

In this case, slope of line $= -\dfrac{a}{b}$

x - intercept $= -\dfrac{c}{a}$, y - intercept $= -\dfrac{c}{b}$

Example # 15 : Find slope, x-intercept & y-intercept of the line $2x - 3y + 5 = 0$.

Solution : Here, $a = 2$, $b = -3$, $c = 5$

$\therefore$ slope $= -\dfrac{a}{b} = \dfrac{2}{3}$

x-intercept $= -\dfrac{c}{a} = -\dfrac{5}{2}$

y-intercept $= \dfrac{5}{3}$

Self practice problem :

(17) Find the slope, x-intercept & y-intercept of the line $3x - 5y - 8 = 0$.

Ans. (17) $\dfrac{3}{5}, \dfrac{8}{3}, -\dfrac{8}{5}$

Parametric form :

$P(r) = (x, y) = (x_1 + r\cos\theta,\ y_1 + r\sin\theta)$ or $\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta} = r$ is the equation of the line in parametric form, where 'r' is the parameter whose absolute value is the distance of any point (x, y) on the line from the fixed point (x_1, y_1) on the line.

Remark : The above form is derived from point-slope form of line.

$y - y_1 = m(x - x_1)$ where $m = \tan\theta$ $\Rightarrow$ $y - y_1 = \dfrac{\sin\theta}{\cos\theta}(x - x_1)$

Example # 16 : Find the equation of the line through the point A(1, 4) and making an angle of 45^o with the positive direction of x-axis. Also determine the length of intercept on it between A and the line $x + y - 10 = 0$

Solution : The equation of a line through A and making an angle of 45^o with the x-axis is

$$\frac{x-1}{\cos 45^o} = \frac{y-4}{\sin 45^o} \quad \text{or} \quad \frac{x-1}{\frac{1}{\sqrt{2}}} = \frac{y-4}{\frac{1}{\sqrt{2}}} \quad \text{or} \quad x - y + 3 = 0$$

Suppose this line meets the line $x + y - 10 = 0$ at P such that AP = r. Then the co-ordinates of P are given by

$$\frac{x-1}{\cos 45^o} = \frac{y-4}{\sin 45^o} = r \quad \Rightarrow \quad x = 1 + r\cos 45^o,\ y = 4 + r\sin 45^o$$

$$\Rightarrow \quad x = 1 + \frac{r}{\sqrt{2}},\ y = 4 + \frac{r}{\sqrt{2}}$$

Thus, the co-ordinates of P are $\left(1 + \dfrac{r}{\sqrt{2}},\ 4 + \dfrac{r}{\sqrt{2}}\right)$

Since P lies on $x + y - 10 = 0$, so $1 + \dfrac{r}{\sqrt{2}} + 4 + \dfrac{r}{\sqrt{2}} = 10$

$$\Rightarrow \quad 5 + \sqrt{2}\,r = 10 \Rightarrow r = \frac{5}{\sqrt{2}} \Rightarrow \text{length AP} = |r| = \frac{5}{\sqrt{2}}$$

Thus, the length of the intercept = $\frac{5}{\sqrt{2}}$.

Self practice problem :

(18) A straight line is drawn through the point $A\left(\sqrt{3},\ 2\right)$ making an angle of $\pi/6$ with positive direction of the x-axis. If it meets the straight line $\sqrt{3}x - 4y + 8 = 0$ in B, find the distance between A and B.

Ans. 6 units

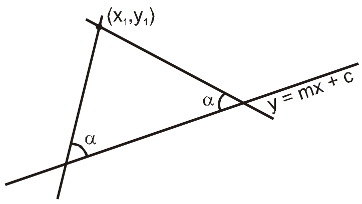
Angle between two straight lines in terms of their slopes:

If m_1 & m_2 are the slopes of two intersecting straight lines ($m_1 m_2 \neq -1$) & θ is the acute angle between them, then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$.

Notes : (i) Let m_1, m_2, m_3 are the slopes of three lines $L_1 = 0; L_2 = 0; L_3 = 0$ where $m_1 > m_2 > m_3$, then the tangent of interior angles of the Δ ABC formed by these lines are given by, $\tan A = \frac{m_1 - m_2}{1 + m_1 m_2}$; $\tan B = \frac{m_2 - m_3}{1 + m_2 m_3}$ & $\tan C = \frac{m_3 - m_1}{1 + m_3 m_1}$

(ii) The equation of lines passing through point (x_1, y_1) and making angle α with the line $y = mx + c$ are given by :

$(y - y_1) = \tan(\theta - \alpha)(x - x_1)$ & $(y - y_1) = \tan(\theta + \alpha)(x - x_1)$, where $\tan\theta = m$.



Example # 17 : The acute angle between two lines is $\pi/4$ and slope of one of them is $-1/3$. Find the slope of the other line.

Solution : If θ be the acute angle between the lines with slopes m_1 and m_2, then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

Let $\theta = \frac{\pi}{4}$ and m_1 $-1/3$

$\therefore \quad \tan\frac{\pi}{4} = \left|\frac{-\frac{1}{3} - m_2}{1 - \frac{1}{3}m_2}\right| \quad \Rightarrow \quad 1 = \left|\frac{3m_2 + 1}{3 - m_2}\right| \quad \Rightarrow \quad \frac{3m_2 + 1}{3 - m_2} = 1 \text{ or } -1$

Now $\frac{3m_2 + 1}{3 - m_2} = 1 \quad \Rightarrow \quad m_2 = \frac{1}{2}$ and $\frac{3m_2 + 1}{3 - m_2} = -1 \quad \Rightarrow m_2 = -2.$

$\therefore$ The slope of the other line is either 1/2 or –2

Example # 18 : Find the equation of the straight line which passes through the point (3,–2) and making angle 60º with the line $\sqrt{3}x + y = 1$.

Solution : Given line is $\sqrt{3}x + y = 1$..

$\Rightarrow$ Slope of (1) = $-\sqrt{3}$.

Let slope of the required line be m. Also between these lines is given to be 60º.

$\Rightarrow \quad \tan 60^\circ = \left|\dfrac{m-(-\sqrt{3})}{1+m\ (-\sqrt{3})}\right| \quad \Rightarrow \quad \sqrt{3} = \left|\dfrac{m+\sqrt{3}}{1-\sqrt{3}m}\right| \quad \Rightarrow \quad \dfrac{m+\sqrt{3}}{1-\sqrt{3}m} = \pm\sqrt{3}$

$\dfrac{m+\sqrt{3}}{1-\sqrt{3}m} = \sqrt{3} \quad \Rightarrow \quad m + \sqrt{3} = \sqrt{3} - 3m \quad \Rightarrow \quad m = 0$

the equation of the required line is $y + 2 = 0$

$\dfrac{m+\sqrt{3}}{1-\sqrt{3}m} = -\sqrt{3} \quad \Rightarrow \quad m = \sqrt{3}$

$\therefore$ The equation of the required line is $y + 2 = \sqrt{3}\ (x-3) \quad \Rightarrow \quad y\sqrt{3} = x - 2 - 3\sqrt{3}$

Self practice problem :

(19) A vertex of an equilateral triangle is (2, 3) and the equation of the opposite side is $x + y = 2$. Find the equation of the other sides of the triangle.

Ans. (19) $(2-\sqrt{3})x - y + 2\sqrt{3} - 1 = 0$ and $(2+\sqrt{3})x - y - 2\sqrt{3} - 1 = 0$.

Parallel Lines :

(i) When two straight lines are parallel their slopes are equal. Thus any line parallel to $y = mx + c$ is of the type $y = mx + d$, where 'd' is a parameter.

(ii) Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are parallel if $\dfrac{a}{a'} = \dfrac{b}{b'} \neq \dfrac{c}{c'}$.

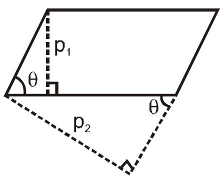
Thus any line parallel to $ax + by + c = 0$ is of the type $ax + by + k = 0$, where k is a parameter.

(iii) The distance between two parallel lines with equations $ax + by + c_1 = 0$ &

$ax + by + c_2 = 0$ is $= \left|\dfrac{c_1 - c_2}{\sqrt{a^2+b^2}}\right|$.

Note that coefficients of x & y in both the equations must be same.

(iv) The area of the parallelogram $= \dfrac{p_1 p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides. Note that area of the parallelogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given by $\left|\dfrac{(c_1-c_2)\ (d_1-d_2)}{m_1-m_2}\right|$.



Example # 19 : Find the equation of the straight line that has y-intercept 5 and is parallel to the straight line $3x - 7y = 8$.

Solution : Given line is $3x - 7y = 8$

$\therefore$ Slope of (1) is $\dfrac{3}{7}$

The required line is parallel to (1), so its slope is also 3/7, y-intercept of required line = 5

$\therefore$ By using y = mx + c form, the equation of the required line is

$y = \frac{3}{7}x + 5$ or $3x - 7y + 35 = 0$

Example # 20 : Two sides of a square lie on the lines $5x - 12y + 6 = 0$ and $5x - 12y = 20$. What is its area ?

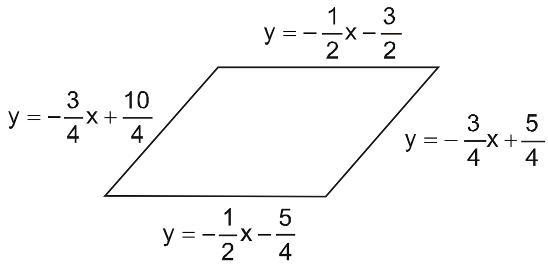
Solution : Clearly the length of the side of the square is equal to the distance between the parallel lines
$5x - 12y + 6 = 0$........(i) and $5x - 12y = 20$ ....(ii)
Now, Distance between the parallel lines

$$= \frac{|6+20|}{\sqrt{5^2 + (-12)^2}} = 2$$

Thus, the length of the side of the square is 2 and hence its area = 4

Example # 21 : Find the area of the parallelogram whose sides are $x + 2y + 3 = 0$, $3x + 4y - 5 = 0$,
$2x + 4y + 5 = 0$ and $3x + 4y - 10 = 0$

Solution :



Here, $c_1 = -\frac{3}{2}$, $c_2 = -\frac{5}{4}$, $d_1 = \frac{10}{4}$, $d_2 = \frac{5}{4}$, $m_1 = -\frac{1}{2}$, $m_2 = -\frac{3}{4}$

$$\therefore \quad \text{Area} = \left| \frac{\left(-\frac{3}{2}+\frac{5}{4}\right)\left(\frac{10}{4}-\frac{5}{4}\right)}{\left(-\frac{1}{2}+\frac{3}{4}\right)} \right| = \frac{5}{4} \text{ sq. units}$$

Self practice problem :

(20) Find the area of parallelogram whose sides are given by $4x - 5y + 1 = 0$, $x - 3y - 6 = 0$,
$4x - 5y - 2 = 0$ and $2x - 6y + 5 = 0$

Ans. (20) $\frac{51}{14}$ sq. units

Perpendicular Lines:

(i) When two lines of slopes m_1 & m_2 are at right angles, the product of their slopes is -1,
i.e. $m_1 m_2 = -1$. Thus any line perpendicular to y = mx + c is of the form
$y = -\frac{1}{m}x + d$, where 'd' is any parameter.

(ii) Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are perpendicular if $aa' + bb' = 0$. Thus any line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$, where 'k' is any parameter.

Example # 22 : Find the equation of the straight line that passes through the point (3, 4) and perpendicular to the line $3x + 2y + 5 = 0$

Solution : The equation of a line perpendicular to $3x + 2y + 5 = 0$ is
$2x - 3y + \lambda = 0$ ....(i)
This passes through the point (3, 4)
$\therefore$ $3 \times 2 - 3 \times 4 + \lambda = 0 \Rightarrow \lambda = 6$
Putting $\lambda = 6$ in (i), we get $2x - 3y + 6 = 0$, which is the required equation.

Aliter The slope of the given line is –3/2. Since the required line is perpendicular to the given line. So, the slope of the required line is 2/3. As it passes through (3, 4). So, its equation is $y - 4 = \frac{2}{3}(x - 3)$

or $2x - 3y + 6 = 0$

Self practice problem :

(21) The vertices of a triangle are A(10, 4), B (–4, 9) and C(–2, –1). Find the equation of its altitudes. Also find its orthocentre.

Ans. (21) $x - 5y + 10 = 0$, $12x + 5y + 3 = 0$, $14x - 5y + 23 = 0$, $\left(-1, \frac{9}{5}\right)$

Position of the point (x_1, y_1) relative of the line $ax + by + c = 0$:

If $ax_1 + by_1 + c$ is of the same sign as c, then the point (x_1, y_1) lie on the origin side of $ax + by + c = 0$. But if the sign of $ax_1 + by_1 + c$ is opposite to that of c, the point (x_1, y_1) will lie on the non–origin side of $ax + by + c = 0$.

In general two points (x_1, y_1) and (x_2, y_2) will lie on same side or opposite side of $ax + by + c = 0$ according as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of same or opposite sign respectively.

Example # 23 : Show that (2, –1) and (–3, 3) lie on the opposite sides of the line $2x - 3y + 5 = 0$.

Solution : At (2, –1), the value of $2x - 3y + 5 = 4 + 3 + 5 = 12 > 0$.

At (–3, 3), the value of $2x - 3y + 5 = -6 - 9 + 5 = -10 < 0$

$\therefore$ The points (2, –1) and (–3, 3) are on the opposite sides of the given line.

Self practice problems :

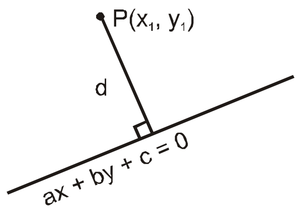
(22) Are the points (3, – 4) and (2, 6) on the same or opposite side of the line $3x - 4y = 8$?

(23) Which one of the points (1, 1), (–1, 2) and (2, 3) lies on the side of the line $4x + 3y - 5 = 0$ on which the origin lies?

Ans. (22) Opposite sides (23) (–1, 2)

Length of perpendicular from a point on a line :

The length of perpendicular from $P(x_1, y_1)$ on $ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$.



Example # 24 : Find the distance between the line $4x - 3y + 8 = 0$ and the point (–2, 3)

Solution : The required distance = $\left|\frac{(-2) \times 4 - 3 \times 3 + 8}{\sqrt{4^2 + (-3)^2}}\right| = \frac{9}{5}$

Example # 25 : Find all points on $x - y + 2 = 0$ that lie at a unit distance from the line $12x - 5y + 9 = 0$.

Solution : Note that the co-ordinates of an arbitrary point on $x - y + 2 = 0$ can be obtained by putting $x = t$ (or $y = t$) and then obtaining y (or x) from the equation of the line, where t is a parameter. Putting $x = t$ in the equation $x - y + 2 = 0$ of the given line, we obtain $y = 2 + t$.
So, co-ordinates of an arbitrary point on the given line are P(t, 2 + t). Let P(t, 2 + t) be the required point. Then, distance of P from the line $12x - 5y + 9 = 0$ is unity i.e.

$\Rightarrow \left|\dfrac{12t-5(t+2)+9}{\sqrt{12^2+5^2}}\right| = 1 \Rightarrow |7t-1| = 13$

$\Rightarrow 7t - 1 = \pm 13 \Rightarrow t = 2$ or $t = -12/7$

Hence, required points are (2,4) or (–12/7,2/7)

Self practice problem :

(24) Find the length of the altitudes from the vertices of the triangle with vertices : (–1, 1), (5, 2) and (3, –1).

Ans. (24) $\dfrac{16}{\sqrt{13}}, \dfrac{8}{\sqrt{5}}, \dfrac{16}{\sqrt{37}}$

Reflection of a point about a line :

(i) Foot of the perpendicular from a point (x_1, y_1) on the line $ax + by + c = 0$ is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = -\left(\frac{ax_1+by_1+c}{a^2+b^2}\right)$$

(ii) The image of a point (x_1, y_1) about the line $ax + by + c = 0$ is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = -2\left(\frac{ax_1+by_1+c}{a^2+b^2}\right)$$

Example # 26 : Find the foot of perpendicular of the line drawn from P (2, –3) on the line $x - 2y + 5 = 0$.

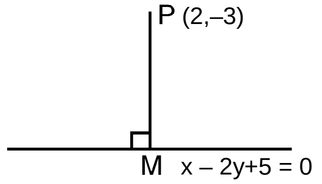
Solution : Slope of PM = – 2

$\therefore$ Equation of PM is

$2x + y = 1$ ....(i)

solving equation (i) with $x - 2y + 5 = 0$, we get

co-ordinates of M (–3/5, 11/5)



Aliter Here, $\dfrac{x-2}{1} = \dfrac{y+3}{-2} = -\dfrac{2+6+5}{1+4} \Rightarrow \dfrac{x-2}{1} = \dfrac{y+3}{-2} = -\dfrac{13}{5} \Rightarrow x = -3/5$, $y = 11/5$

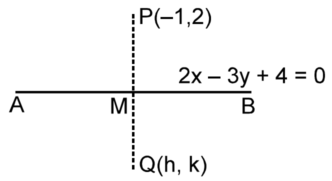
Example # 27 : Find the image of the point P(–1, 2) in the line mirror $2x - 3y + 4 = 0$.

Solution : Let image of P is Q.

$\therefore$ PM = MQ & PQ $\perp$ AB

Let Q is (h, k)

$\therefore$ M is $\left(\dfrac{h-1}{2}, \dfrac{k+2}{2}\right)$



It lies on $2x - 3y + 4 = 0$.

$\therefore$ $2\left(\dfrac{h-1}{2}\right) - 3\left(\dfrac{k+2}{2}\right) + 4 = 0.$

or $2h - 3k = 0$ ....(i)

slope of PQ = $\dfrac{k-2}{h+1}$

PQ $\perp$ AB

$\therefore$ $\dfrac{k-2}{h+1} \times \dfrac{2}{3} = -1. \Rightarrow 3h + 2k - 1 = 0.$ ....(ii)

soving (i) & (ii), we get $h = \dfrac{3}{13}$, $k = \dfrac{2}{13}$

$\therefore$ Image of P(– 1, 2) is $Q\left(\dfrac{3}{13}, \dfrac{2}{13}\right)$

Aliter The image of P (– 1, 2) about the line

$2x - 3y + 4 = 0$ is $\frac{x+1}{2} = \frac{y-2}{-3} = -2\ \frac{[2(-1)-3(2)+4]}{2^2+(-3)^2}$

$\frac{x+1}{2} = \frac{y-2}{-3} = \frac{8}{13}$

$\Rightarrow$ $13x + 13 = 16 \Rightarrow x = \frac{3}{13}$ & $13y - 26 = -24 \Rightarrow y = \frac{2}{13}$

$\therefore$ image is $\left(\frac{3}{13}, \frac{2}{13}\right)$

Self practice problems :

(25) Find the foot of perpendicular of the line drawn from (– 2, – 3) on the line 3x – 2y – 1 = 0.

(26) Find the image of the point (1, 2) in y-axis.

Ans. (25) $\left(\frac{-23}{13}, \frac{-41}{13}\right)$ (26) (– 1, 2)

Centroid, Incentre & Excentre :

If A (x_1, y_1), B(x_2, y_2), C(x_3, y_3) are the vertices of triangle ABC, whose sides BC, CA, AB are of lengths a, b, c respectively, then the co-ordinates of the special points of triangle ABC are as follows :

Centroid G $\equiv \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

Incentre I $\equiv \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$, and

Excentre (to A) $I_1 \equiv \left(\frac{-ax_1+bx_2+cx_3}{-a+b+c}, \frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$ and so on.

Notes :
(i) Incentre divides the angle bisectors in the ratio, $(b+c) : a$; $(c+a) : b$ & $(a+b) : c$.
(ii) Incentre and excentre are harmonic conjugate of each other w.r.t. the angle bisector on which they lie.
(iii) Orthocentre, Centroid & Circumcentre are always collinear & centroid divides the line joining orthocentre & circumcentre in the ratio 2 : 1.
(iv) In an isosceles triangle G, O, I & C lie on the same line and in an equilateral triangle, all these four points coincide.
(v) In a right angled triangle orthocentre is at right angled vertex and circumcentre is mid point of hypotenuse
(vi) In case of an obtuse angled triangle circumcentre and orthocentre both are out side the triangle.

Example # 28: Find the co-ordinates of (i) centroid (ii) in-centre of the triangle whose vertices are (0, 0), (5, 0) and (0, 12).

Solution : (i) We know that the co-ordinates of the centroid of a triangle whose angular points are (x_1, y_1), (x_2, y_2) (x_3, y_3) are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

So the co-ordinates of the centroid of a triangle whose vertices are (0, 0), (5, 0) and (0, 12) are $\left(\frac{0+5+0}{3}, \frac{0+0+12}{3}\right)$ or $\left(\frac{5}{3}, 4\right)$.

(ii) Let A (0, 0), B (5, 0) and C(0, 12) be the vertices of triangle ABC.
Then c = AB = 5, b = CA =12
and a = BC = 13.

The co-ordinates of the in-centre are $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

or $\left(\frac{13\times0+12\times5+5\times0}{5+12+13}, \frac{13\times0+12\times0+5\times12}{5+12+13}\right)$ or (2, 2)

Self practice problems :

(27) Two vertices of a triangle are (3, –5) and (–7, 4). If the centroid is (2, –1), find the third vertex.

(28) Find the co-ordinates of the centre of the circle inscribed in a triangle whose vertices are (– 36, 7), (20, 7) and (0, – 8)

Ans. (27) (10, – 2) (28) (–1, 0)

Bisectors of the angles between two lines:

Equations of the bisectors of angles between the lines ax + by + c = 0 & $a'x + b'y + c' = 0$ ($ab' \neq a'b$) are : $\dfrac{ax + by + c}{\sqrt{a^2 + b^2}} = \pm \dfrac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$

Note : Equation of straight lines passing through $P(x_1, y_1)$ & equally inclined with the lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ are those which are parallel to the bisectors between these two lines & passing through the point P.

Example # 29 : Find the equations of the bisectors of the angle between the straight lines 3x + y + 1 = 0 and x + 3y + 1 = 0.

Solution : The equations of the bisectors of the angles between 3x+y+1= 0 and x+3y+1=0 are $\dfrac{3x + y + 1}{\sqrt{3^2 + 1^2}}$

$= \pm \dfrac{x + 3y + 1}{\sqrt{1^2 + 3^2}}$ or 3x + y + 1 = ± (x + 3y + 1)

Taking the positive sign, we get x = y as one bisector
Taking the negative sign, we get 2x + 2y + 1 = 0 as the other bisector.

Self practice problem :

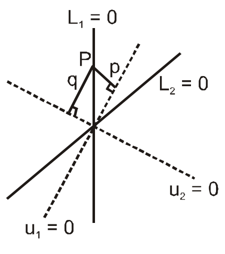
(29) Find the equations of the bisectors of the angles between the following pairs of straight lines 3x + 4y + 13 = 0 and 12x – 5y + 32 = 0

Ans. (29) 21x – 77y – 9 = 0 and 99x + 27y + 329 = 0

Methods to discriminate between the acute angle bisector & the obtuse angle bisector:

(i) If θ be the angle between one of the lines & one of the bisectors, find $\tan\theta$.
If $|\tan\theta| < 1$, then $2\theta < 90°$ so that this bisector is the acute angle bisector.
If $|\tan\theta| > 1$, then we get the bisector to be the obtuse angle bisector.

(ii) Let $L_1 = 0$ & $L_2 = 0$ are the given lines & $u_1 = 0$ and $u_2 = 0$ are the bisectors between $L_1 = 0$ & $L_2 = 0$. Take a point P on any one of the lines $L_1 = 0$ or $L_2 = 0$ and drop perpendicular on $u_1 = 0$ & $u_2 = 0$ as shown in figure. If,
$|p| < |q| \Rightarrow u_1$ is the acute angle bisector.
$|p| > |q| \Rightarrow u_1$ is the obtuse angle bisector.
$|p| = |q| \Rightarrow$ the lines L_1 & L_2 are perpendicular.



(iii) If $aa' + bb' < 0$, then the equation of the bisector of this acute angle is
$$\frac{ax + by + c}{\sqrt{a^2 + b^2}} = + \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$$

If, however, $aa' + bb' > 0$, the equation of the bisector of the obtuse angle is :

$$\frac{ax + by + c}{\sqrt{a^2 + b^2}} = + \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$$

Example # 30 : For the straight lines $2x - y + 1 = 0$ and $x - 2y - 2 = 0$, find the equation of the
(i) bisector of the obtuse angle between them;
(ii) bisector of the acute angle between them;

Solution : $2x - y + 1 = 0$ ....(1)

and $x - 2y - 2 = 0$ ....(2)

Here $a_1 = 2$, $a_2 = 1$, $b_1 = -1$, $b_2 = -2$

Now $a_1a_2 + b_1b_2 = 4 > 0$

$\therefore$ bisector of the obtuse angle between lines (1) and (2) will be

$$\frac{2x - y + 1}{\sqrt{2^2 + (-1)^2}} = \frac{x - 2y - 2}{\sqrt{1^2 + (-2)^2}}$$

or $x + y + 3 = 0$

and the equation of the bisector of the acute angle will be

$$\frac{2x - y + 1}{\sqrt{2^2 + (-1)^2}} = -\frac{x - 2y - 2}{\sqrt{1^2 + (-2)^2}}$$

or $3x - 3y = 1$

Self practice problem :

(30) Find the equations of the bisectors of the angles between the lines $x + y - 3 = 0$ and $7x - y + 5 = 0$ and state which of them bisects the acute angle between the lines.

Ans. (30) $x - 3y + 10 = 0$ (bisector of the obtuse angle);
$6x + 2y - 5 = 0$ (bisector of the acute angle)

Condition of Concurrency :

Three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ & $a_3x + b_3y + c_3 = 0$ are concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

Alternatively : If three constants A, B & C (not all zero) can be found such that $A(a_1x + b_1y + c_1) + B(a_2x + b_2y + c_2) + C(a_3x + b_3y + c_3) \equiv 0$, then the three straight lines are concurrent.

Example # 31 : If the straight lines $x + 2y = 9$, $3x + 5y = 5$ and $ax - by = 1$ are concurrent. Then find the value of $35a + 22b$

Solution : Given lines are

$x + 2y = 9$ ....(1)

$3x + 5y = 5$ ....(2)

and $ax - by = 1$ ....(3)

Lines will be concurrent if $\Delta = 0$

$$\Delta = \begin{vmatrix} 1 & 2 & -9 \\ 3 & 5 & -5 \\ a & -b & -1 \end{vmatrix} = 0 \Rightarrow 35a + 22b = -1$$

Self practice problem :

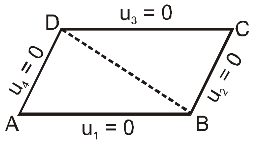
(31) Find the value of m so that the lines $4x - 3y + 2 = 0$, $3x + 4y - 4 = 0$ and $x + my + 6 = 0$ may be concurrent.

Ans. (31) -7

Family of Straight Lines :

The equation of a family of straight lines passing through the point of intersection of the lines, $L_1 \equiv a_1x + b_1y + c_1 = 0$ & $L_2 \equiv a_2x + b_2y + c_2 = 0$ is given by $L_1 + k\,L_2 = 0$ i.e. $(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0$, where k is an arbitrary real number.

Note : (i) If $u_1 = ax + by + c$, $u_2 = a'x + b'y + d$, $u_3 = ax + by + c'$, $u_4 = a'x + b'y + d'$ then $u_1=0$; $u_2= 0$; $u_3= 0$; $u_4= 0$ form a parallelogram. The diagonal BD can be given by $u_2u_3 - u_1u_4= 0$.



(ii) The diagonal AC is also given by $u_1 + \lambda u_4 = 0$ and $u_2 + \mu u_3 = 0$, if the two equations are identical for some real λ and μ.
[For getting the values of λ & μ compare the coefficients of x, y & the constant terms].

Example # 32 : If $3a + 2b + 5c = 0$ and the set of lines $ax + by + c = 0$ passes through a fixed point . Find co-ordinates of that point.

Solution : $3a + 2b + 5c = 0$ .... (i)

$ax + by + c = 0$ .... (ii)

Eliminating c, we get.

$$ax + by - \frac{1}{5}(3a + 2b) = 0 \Rightarrow a\left(x - \frac{3}{5}\right) + b\left(y - \frac{2}{5}\right) = 0 \Rightarrow \left(x - \frac{3}{5}\right) + \frac{b}{a}\left(y - \frac{2}{5}\right) = 0$$

It is of the form $L_1 + \lambda L_2 = 0$

Which passes through the point of intersection $\left(\frac{3}{5}, \frac{2}{5}\right)$ of $L_1= 0$ & $L_2= 0$ for all real values of a & b

Aliter : $3a + 2b + 5c = 0 \quad \Rightarrow \quad \frac{3}{5}a + \frac{2}{5}b + c = 0$

$\left(\frac{3}{5}, \frac{2}{5}\right)$ lies on the line $ax + by + c = 0$ Hence fixed point $\left(\frac{3}{5}, \frac{2}{5}\right)$

Example # 33 : Obtain the equations of the lines passing through the intersection of lines $3x + 7y = 17$ and $x + 2y = 5$ and is prependicular to the straight line $3x + 4y = 10$.

Solution : The equation of any line through the intersection of the given lines is

$(x + 2y - 5) + \lambda\,(3x + 7y - 17) = 0$

or $x\,(3\lambda + 1) + y\,(7\lambda + 2) - 17\lambda - 5 = 0$ ....(i)

This is prependicular to the line $3x + 4y = 10$

$$\therefore \left(-\frac{3\lambda + 1}{7\lambda + 2}\right)\left(-\frac{3}{4}\right) = -1 \Rightarrow \lambda = -\frac{11}{37}$$

Putting this value of λ in (i), the equation of required line $4x - 3y + 2 = 0$

Self practice problem :

(32) Find the equation of the lines through the point of intersection of the lines $x - 3y + 1 = 0$ and $2x + 5y - 9 = 0$ and whose distance from the origin is $\sqrt{5}$

Ans. (32) $2x + y - 5 = 0$

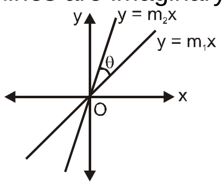
A Pair of straight lines through origin:

(i) A homogeneous equation of degree two, "$ax^2 + 2hxy + by^2 = 0$" always represents a pair of straight lines passing through the origin if :

(a) $h^2 > ab \quad \Rightarrow$ lines are real & distinct .

(b) $h^2 = ab \quad \Rightarrow$ lines are coincident .

(c) $h^2 < ab$ $\Rightarrow$ lines are imaginary with real point of intersection i.e. (0, 0)



This equation is obtained by multiplying the two equations of lines

$(m_1x - y)(m_2x - y) = 0$

$\Rightarrow m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0$

(ii) If $y = m_1x$ & $y = m_2x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then;

$m_1 + m_2 = -\frac{2h}{b}$ & $m_1 m_2 = \frac{a}{b}$.

(iii) If θ is the acute angle between the pair of straight lines represented by,

$ax^2 + 2hxy + by^2 = 0$, then $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$.

(iv) The condition that these lines are :

(a) at right angles to each other is $a + b = 0$. i.e. co–efficient of x^2 +co–efficient of $y^2 = 0$.

(b) coincident is $h^2 = ab$.

(c) equally inclined to the axis of x is $h = 0$.i.e. coeff. of $xy = 0$.

Note that a homogeneous equation of degree n represents n straight lines passing through origin.

(v) The equation to the pair of straight lines bisecting the angles between the straight lines

$ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$.

Example # 34 : Show that the equation $18x^2 - 9xy + y^2 = 0$ represents a pair of distinct straight lines, each passing through the origin. Find the separate equations of these lines.

Solution : The given equation is a homogeneous equation of second degree. So, it represents a pair of straight lines passing through the origin. Comparing the given equation with $ax^2 + 2hxy + by^2 = 0$, we obtain $a = 18$, $b = 1$ and $2h = -9$.

$\therefore$ $h^2 - ab = \frac{81}{4} - 18 = \frac{9}{4} > 0$ $\Rightarrow$ $h^2 > ab$

Hence, the given equation represents a pair of distinct lines passing through the origin.

Now, $18x^2 - 9xy + y^2 = 0 \Rightarrow \left(\frac{y}{x}\right)^2 - 9\left(\frac{y}{x}\right) + 18 = 0 \Rightarrow \left(\frac{y}{x} - 6\right)\left(\frac{y}{x} - 3\right) = 0$

$\Rightarrow$ $- 6 = 0$ or $- 3 = 0 \Rightarrow y - 6x = 0$ or $y - 3x = 0$

So the given equation represents the straight lines $y - 3x = 0$ and $y - 6x = 0$.

Example # 35 : Find the equations to the pair of lines through the origin which are perpendicular to the lines represented by $6x^2 - xy - 12y^2 = 0$.

Solution : We have $6x^2 - xy - 12y^2 = 0$.

$\Rightarrow$ $6x^2 - 9xy + 8xy - 12y^2 = 0$ $\Rightarrow$ $(3x + 4y)(2x - 3y) = 0$

$\Rightarrow$ $3x + 4y = 0$ & $2x - 3y = 0$

Thus the given equation represents the lines $3x + 4y = 0$ and $2x - 3y = 0$. The equations of the lines passing through the origin and perpendicular to the given lines are $4x - 3y = 0$ & $3x + 2y = 0$ there combined equations $12x^2 - xy - 6y^2 = 0$

Example # 36 : Find the angle between the pair of straight lines $x^2 - 3xy + 2y^2 = 0$

Solution : Given equation is $x^2 - 3xy + 2y^2 = 0$

Here a = coeff. of $x^2 = 1$, b = coeff. of $y^2 = 2$

and 2h = coeff. of $xy = -3$ $\therefore$ $h = -\frac{3}{2}$

Now $\tan\theta = \left|\dfrac{2\sqrt{h^2-ab}}{a+b}\right| = \left|\dfrac{2\sqrt{\frac{9}{4}-2}}{1+2}\right| = \dfrac{1}{3}$

Where θ is the acute angle between the lines.

$\therefore$ acute angle between the lines is $\tan^{-1}\left(\dfrac{1}{3}\right)$ and obtuse angle between them is

$\pi - \tan^{-1}\left(\dfrac{1}{3}\right)$

Example # 37 : Find the equation of the bisectors of the angle between the lines represented by $3x^2 - 5xy + 4y^2 = 0$

Solution : Given equation is $3x^2 - 5xy + 4y^2 = 0$ ....(1)
comparing it with the equation $ax^2 + 2hxy + by^2 = 0$ ....(2)
we have a = 3, 2h = –5; and b = 4
Now the equation of the bisectors of the angle between the pair of lines (1) is

$\dfrac{x^2-y^2}{a-b} = \dfrac{xy}{h}$ or $\dfrac{x^2-y^2}{3-4} = \dfrac{xy}{-\frac{5}{2}}$; or $\dfrac{x^2-y^2}{-1} = \dfrac{2xy}{-5}$ or $5x^2 - 2xy - 5y^2 = 0$

Self practice problems :

(33) Find the area of the triangle formed by the lines $y^2 - 9xy + 18x^2 = 0$ and y = 9.

(34) If the pairs of straight lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, prove that pq = –1.

Ans. (33) $\dfrac{27}{4}$ sq. units

General equation of second degree representing a pair of Straight lines :

(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if :

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, i.e. if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

Such an equation is obtained again by multiplying the two equation of lines
$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$

(ii) The angle θ between the two lines representing by a general equation is the same as that between the two lines represented by its homogeneous part only.

Example # 38 : Prove that the equation $6x^2 + 13xy + 6y^2 + 8x + 7y + 2 = 0$ represents a pair of straight lines. Find the co-ordinates of their point of intersection.

Solution : Given equation is $6x^2 + 13xy + 6y^2 + 8x + 7y + 2 = 0$
Writing the equation (1) as a quadratic equation in x we have
$6x^2 + (13y + 8)x + 6y^2 + 7y + 2 = 0$

$\therefore$ $x = \dfrac{-(13y+8) \pm \sqrt{(13y+8)^2 - 4.6(6y^2+7y+2)}}{12}$

$= \dfrac{-(13y+8) \pm \sqrt{169y^2 + 208y + 64 - 144y^2 - 168y - 48}}{12}$

$= \dfrac{-(13y+8) \pm \sqrt{25y^2 + 40y + 16}}{12} = \dfrac{-(13y+8) \pm (5y+4)}{12}$

$\Rightarrow$ 12x = –13y – 8 + 5y + 4 or 12x = –13y – 8 – 5y – 4

$\Rightarrow$ 3x + 2y + 1 = 0 or 2x + 3y + 2 = 0

Hence equation (1) represents a pair of straight lines whose equation are
3x + 2y + 1 = 0 .....(1) and 2x + 3y + 2 = 0 .....(2)
Solving these two equations, the required point of intersection is

Self practice problem :

(35) Find the combined equation of the straight lines passing through the point (1, 1) and parallel to the lines represented by the equation $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$ and find the angle between them.

Ans. (35) $x^2 - 5xy + 4y^2 + 3x - 3y = 0$, $\tan^{-1}\left(\frac{3}{5}\right)$

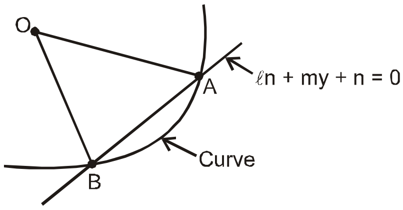
Homogenization :

This method is used to write the joint equation of two lines connecting origin to the points of intersection of a given line and a given second degree curve. The equation of a pair of straight lines joining origin to the points of intersection of the line $L \equiv \ell x + my + n = 0$ and a second degree curve

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

is $ax^2 + 2hxy + by^2 + 2gx\left(\frac{\ell x + my}{-n}\right) + 2fy\left(\frac{\ell x + my}{-n}\right) + c\left(\frac{\ell x + my}{-n}\right)^2 = 0.$

The equation is obtained by homogenizing the equation of curve with the help of equation of line.



Notes : (i) Here we have written 1 as $\frac{\ell x + my}{-n}$ and converted all terms of the curve to second degree expressions

(ii) Equation of any curve passing through the points of intersection of two curves $C_1 = 0$ and $C_2 = 0$ is given by $\lambda C_1 + \mu C_2 = 0$, where λ & μ are parameters.

Example # 39 : All chords of the curve $2x^2 - 3y^2 + 4x - 2y = 0$ which subtend a right angle at the origin passes through a fixed point. Find that point

Solution : Let $Ax + By = 1$ be a chord of the curve $2x^2 - 3y^2 + 4x - 2y = 0$

Equation of the given curve is $2x^2 - 3y^2 + 4x - 2y = 0$

and equation of the chord is $Ax + By = 1$ ....(i)

Making equation (1) homogeneous equation of the second degree in x any y with the help of (1), we have $2x^2 - 3y^2 + 4x(Ax + By) - 2y(Ax + By) = 0$

$x^2(2 + 4A) + y^2(-3 - 2B) + xy(4B - 2A) = 0$ ....(ii)

since line represented by (ii) are at right angle

$\Rightarrow$ coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow$ $2 + 4A + (-3 - 2B) = 0$

$4A - 2B = 1$

chord $Ax + By = 1$

$x = 4 \quad y = -2$

fixed point $(4, -2)$

Self practice problems :

(36) Find the equation of the straight lines joining the origin to the points of intersection of the line $3x + 4y - 5 = 0$ and the curve $2x^2 + 3y^2 = 5$.

(37) Find the equation of the straight lines joining the origin to the points of intersection of the line $lx + my + n = 0$ and the curve $y^2 = 4ax$. Also, find the condition of their perpendicularity.

Ans. (36) $x^2 - y^2 - 24xy = 0$ (37) $4alx^2 + 4amxy + ny^2 = 0$; $4al + n = 0$